11. PLANE WAVE SPECTRA vs. 3.1

GLOBAL PLANE WAVE SPECTRUM

For general fields, each crossection E(x,y,z) can be considered to be formed by the interference of a continuum of plane and evanescent waves S6 $\tilde{E}(k_x,k_y)\exp(jk_xx+jk_yy+jk_zz)$ propagating in different directions. (In communication theory each arbitrary signal similarly consists of a continuum of sinusoidal signals with different frequencies.)

Mathematically this may be expressed as

$$E(x,y,z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{E}(k_x,k_y)\exp(+jk_xx+jk_yy+jk_zz)dk_xdk_y \tag{11.1}$$

where the factor $\frac{1}{(2\pi)^2}$ is introduced for convenience, and $k_x^2+k_y^2+k_z^2= k^2$.

Note that in (11.1) the integration limits are -∞ and +∞. It will be clear that this range of k_x and k_y includes evanescent waves S6 These occur whenever $k_x^2 + k_y^2 > k^2$. Such waves decay quickly in the direction of propagation, and we do not take them into account in our treatment. The corresponding mathematical assumption is that A be zero outside the region $k_x^2 + k_y^2 = k^2$.

The quantity $\tilde{E}(k_x,k_y)$ will be called the virtual plane wave spectrum or global plane wave spectrum .

LOCAL PLANE WAVE SPECTRUM

More commonly used is the local plane wave spectrum :

$$A(k_x,k_{y};z) = \tilde{E}(k_x,k_y)\exp(jk_z z) \qquad (11.2)$$

in terms of which E(x,y,z) may be written as

$$E(x,y,z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(k_x,k_y;z)\exp(jk_x x+jk_y y)dk_x dk_y$$

(11.3)

The variable z in $A(k_x,k_y;z)$ serves as a parameter that locates the cross section E. This is is indicated by the semi-colon.

Eq. (11.3) will be recognized as a Fourier transform S4 and may be written in symbolic form as:

$$E(x,y,z) = F^{-1} A(k_x,k_y;z) \qquad (11.4)$$

where F^{-1} denotes the inverse Fourier transform operator.

The inverse relation to eq. (11.4) is:

$$A(k_x,k_y;z) = F\ E(x,y,z) \qquad (11.5)$$

where F is the Fourier transform operator. Hence

$$A(k_x,k_y;z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x,y,z)\exp(-jk_x x-jk_y y)dxdy \qquad (11.6)$$

PROPAGATION OF THE SPECTRUM

From (11.2) it follows that

$$A(k_x,k_y;z) = H(k_x,k_y)A(k_x,k_y;0) \qquad (11.7)$$

where the propagator H is a simple multiplier, given by

$$H(k_x,k_y) = \exp(jk_z) = \exp(jz\sqrt{k^2-k_x^2-k_y^2}) \qquad (11.8)$$

In the paraxial approximation S6, $|k_x|,|k_y| << k$, H becomes

$$H(k_x,k_y) \approx \exp(jkz)\exp[-j(\frac{k_x^2}{2k} + \frac{k_y^2}{2k})z] \qquad (11.9)$$

ANGULAR FORMULATION

In order to bring out explicitly the fact that (11.3) represents a spectrum of plane waves propagating in different directions, we may write the Fourier transform in terms of azimuth and elevation angles φ and φ' as shown in Fig. 11.1

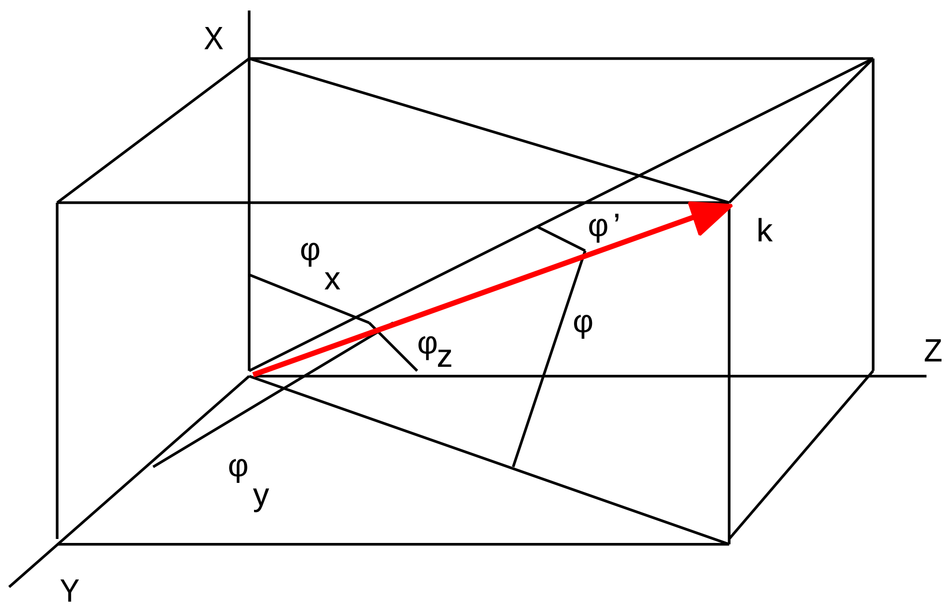

Fig.11.1 Direction cosine angles φ_x, φ_y, φ_z, azimuth angle φ and elevation angle φ'.

Substituting the relations $k_x = k\sin\varphi$, $k_y = k\sin\varphi'$ into (11.3), we find readily that

$$E(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\varphi,\varphi';z)\exp\left(j2\pi\frac{\sin\varphi}{\lambda} + j2\pi\frac{\sin\varphi'}{\lambda}y\right)d\left(\frac{\sin\varphi}{\lambda}\right)d\left(\frac{\sin\varphi'}{\lambda}\right) \quad (11.10)$$

Its inverse is given by

$$A(\varphi,\varphi';z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x,y,z)\exp\left(-j2\pi\frac{\sin\varphi}{\lambda}x - j2\pi\frac{\sin\varphi'}{\lambda}y\right)dxdy \quad (11.11)$$

Assuming that A is limited to small angles and applying the paraxial approximation $\sin\varphi \approx \varphi$, $\sin\varphi' \approx \varphi'$, we may write (11.10) as:

$$E(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\varphi,\varphi';z)\exp\left(j2\pi\frac{\varphi}{\lambda}x + j2\pi\frac{\varphi'}{\lambda}y\right)d\left(\frac{\varphi}{\lambda}\right)d\left(\frac{\varphi'}{\lambda}\right) \quad (11.12)$$

with its inverse:

$$A(\varphi,\varphi';z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x,y,z)\exp\left(-j2\pi\frac{\varphi}{\lambda}x - j2\pi\frac{\varphi'}{\lambda}y\right)dxdy \quad (11.13)$$

In (11.3), (11.10)-(11.13), the A's all denote the same expression, written in different, related variables. If A is expressed in angles, it is often called the angular plane wave spectrum .

SPATIAL FREQUENCY FORMULATION

Finally, we may introduce the spatial frequencies $f_x = k_x/2\pi = \sin\varphi/\lambda$ and $f_y = k_y/2\pi = \sin\varphi'/\lambda$. Then (11.11) becomes:

$$E(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(f_x,f_y;z)\exp(j2\pi f_x x+j2\pi f_y y)\,df_x\,df_y \tag{11.14}$$

with its inverse:

$$A(f_x,f_y;z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x,y,z)\exp(-j2\pi f_x x-j2\pi f_y y)\,dx\,dy \tag{11.15}$$

Expression (11.14) may be seen as a self-evident mathematical Fourier decomposition, without any obvious physical meaning. As such it is used quite often in numerical image processing. The parameter z is then left out, because there is no propagation of any physical fields; we are just considering one image. In this book, we will mostly use the formulations (11.3) and (11.6), unless we want to bring out the systems aspect of the situation rather than the physics.

In many books, (11.14) is used as a starting point, and then (11.3) is interpreted as a continuum of plane waves REF.1. In this book it is postulated ad hoc that any field (assuming no evanescent waves) may be so decomposed. This may, however, be derived from basic physical laws REF.2.

Of particular interest are the plane wave spectra of an elliptical Gaussian laser beam EX. 11.1:

$$E(x,y,0) = E_0 \exp(-x^2/w_1^2 - y^2/w_2^2) \tag{11.16}$$

and of a rectangular uniform beam Ex 11.2.

CIRCULAR SYMMETRY

In many cases the field is circularly symmetrical, i.e. it is a function of $r = \sqrt{x^2 + y^2}$ only. This is for example the case for the Gaussian beam of (11.16) if $w_1 = w_2$:

$$E(r, z=0) = E_0 \exp(-r^2/w_0{}^2) \qquad (11.17)$$

or the uniform beam of radius a/2:

$$E(r) = E_0 \mathrm{circ}(2r/a) \qquad (11.18)$$

In cases of circular symmetry we expect the plane wave spectrum to be also circularly symmetrical, i.e to be a function of $k_t = \sqrt{k_x{}^2 + k_y{}^2}$ only. The variable k_t we shall call the transverse propagation constant.

The calculation of the plane wave spectrum can now be simplified by introducing polar coordinates: r and θ for the spatial coordinates in the XY plane, and k_t and α for those in the $k_x k_y$ plane. This is shown below.

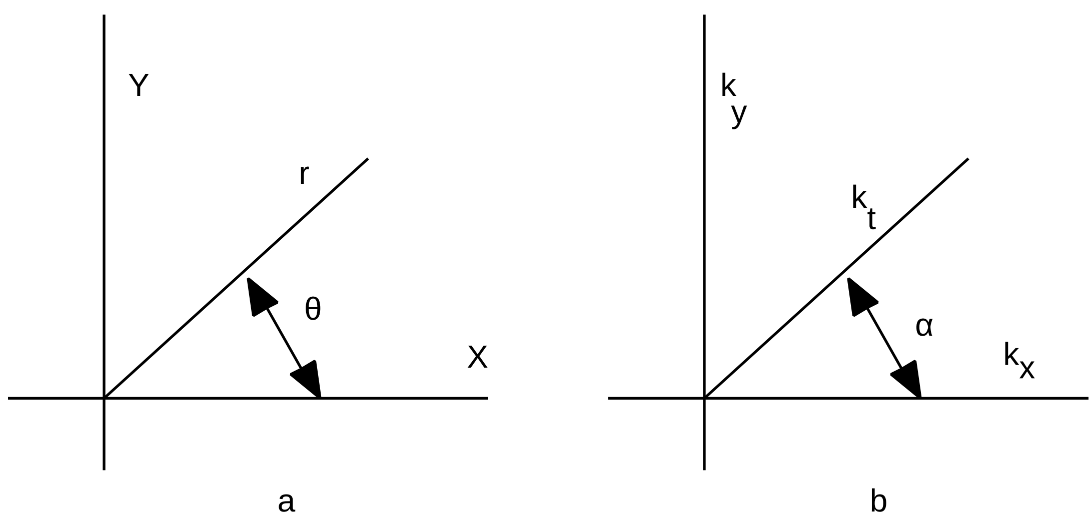

Fig. 11.2 a) Polar coordinates in XY plane, b) polar coordinates in k-plane

In eq. (11.3) the plane wave

$$\exp(jk_x x + jk_y y)$$

may be written as

$$\exp[jk_t r(\cos\alpha\cos\theta + \sin\alpha\sin\theta)] = \exp[jk_t r\cos(\alpha - \theta)] \quad (11.19)$$

The elementary area $dk_x dk_y$ becomes

$$dk_x dk_y = k_t d\alpha dk_t \quad (11.20)$$

and the plane wave spectrum $A(k_x,k_y;z)$ may be written as

$$A(k_x,k_y;z) = A(k_t;z) \quad (11.21)$$

With eq. (11.19)-(11.21), the integral (11.3) becomes:

$$E(x,y,z) = E(r_t,z) = \left(\frac{1}{2\pi}\right)^2 \int_0^\infty A(k_t;z)k_t\left(\int_0^{2\pi}\exp[jk_t r\cos(\alpha-\theta)]d\alpha\right)dk_t \quad (11.22)$$

but, according to one of the definitions of the Bessel function J_0 REF. 3,

$$\int_0^{2\pi}\exp[jk_t r\cos(\alpha-\theta)]d\alpha = 2\pi J_0(k_t r_t) \quad (11.23)$$

Substituting (11.23) into (11.22), we find finally:

$$E(r_t;z) = \frac{1}{2\pi}\int_0^\infty k_t A(k_t;z)J_0(k_t r)dk_t \quad (11.24)$$

FOURIER-BESSEL TRANSFORM

The transform (11.24) is a form of the Fourier-Bessel transform. We shall denote it by the operator B^{-1} The forward transform is given by

$$A(k_t;z) = 2\pi\int_0^\infty rE(r,z)J_o(k_t r)dr \qquad (11.25)$$

In shorthand operator notation we may write:

$$E(r,z) = B\,A(k_t;z) \qquad (11.26)$$

$$A(k_t;z) = B^{-1}\,E(r,z) \qquad (11.27)$$

Useful examples of the application of the Fourier-Bessel transform are the calculation of the angular spectrum of the uniform circular beam. Ex 11.3 and the circular Gaussian beam Ex 11.4

EXAMPLES

Ex. 11.1

Angular spectrum of an elliptical Gaussian beam

An elliptical Gaussian laser beam, say at the output mirror of a laser, is given by

$$E(x,y,0) = E_0 \exp(-x^2/w_1^2 - y^2/w_2^2) =$$

$$E_0 \exp(-x^2/w_0^2)\exp(-y^2/w_0^2)$$

(11.1.1)

Fig. 11.1.1 shows a cross section of the Gaussian beam along X.

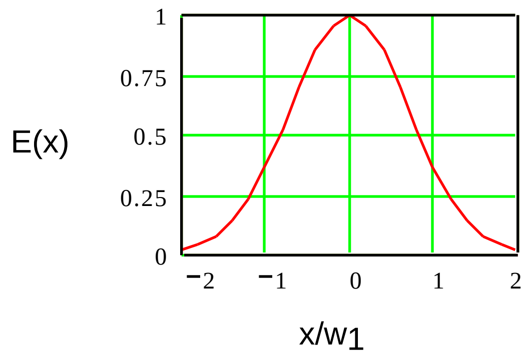

Fig. 11.1.1 Crossection of a Gaussian beam

The widths of the beam in the X and Y direction -- the beam radii -- are defined as the distances from the center to the points where the amplitude equals E_0/e. From (11.1.1) we see that these distances equal w_1 and w_2 respectively. We calculate the plane wave spectrum using (11.6):

$$A(k_x,k_y;0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x,y,0)\exp(-jk_x x - jk_y y)dxdy =$$

$$E_0 \int_{-\infty}^{\infty} \exp(-x^2/w_1^2 - jk_x x)dx \int_{-\infty}^{\infty} \exp(-y^2/w_2^2 - jk_y y)dy \quad (11.1.2)$$

In order to work out integrals like the above, the following formula REF.3 is of great help:

$$\int_{-\infty}^{\infty} \exp(-p^2u^2 \pm qu)\, du = \frac{\sqrt{\pi}}{p}\exp\left(\frac{q^2}{4p^2}\right) \quad (11.1.3)$$

provided $p \neq 0$ and $\mathrm{Re}(p^2) \geq 0$.

Each of the two integrals in (11.1.2) is of the kind of (11.1.3), with u standing for x and y respectively, $p^2 = 1/w_1^2$ and $1/w_2^2$ respectively, and $q = -jk_x$ and $-jk_y$ respectively. Using (11.1.3) we find readily that

$$A(k_x,k_y;0) = E_0\pi w_1 w_2 \exp(-k_x^2 w_1^2/4 - k_y^2 w_2^2/4) \qquad (11.1.4)$$

Fig. 11.1.2 shows a cross section of the spectrum for $k_y = 0$, normalized to unity at the origin.

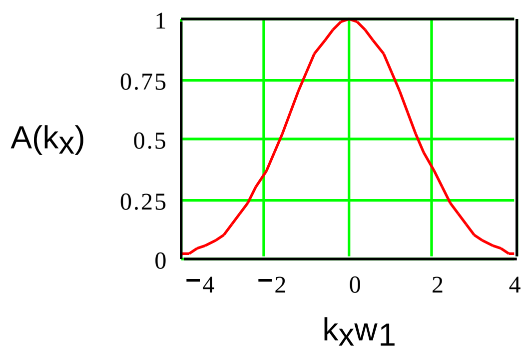

Fig. 11.1.2 Plane wave spectrum of Gaussian beam

Similar to the procedure in the space domain, we can define spectral widths in k_x and k_y of this spectrum by calculating where the amplitude becomes 1/e, and find that this occurs when $k_x = 2/w_1$ and $k_y = 2/w_2$. With $\varphi \approx k_x/k$ and $\varphi' \approx k_y/k$ it is readily seen that the angular widths of this elliptical spectrum are given by $\varphi \approx \lambda/\pi w_1$ and $f' \approx \lambda/\pi\omega_2$. Sometimes S28 the Gaussian beam is written for convenience as $\exp(-\pi r^2/a^2)$.This has the advantage that its angular width equals simply λ/a, as in the case of the rectangular beam.

Ex 11.2

Plane wave spectrum of a rectangular beam

A uniform laser beam, passing through a rectangular aperture is defined by :

$$E(x,y,0) = E_0 \mathrm{rect}(x/a)\mathrm{rect}(y/b) \qquad (11.2.1)$$

According to 11.6) we find:

$$A(k_x,k_y;0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x,y,0)\exp(+jk_x x+jk_y y)dxdy \ =$$

$$E_0 \int_{-a/2}^{a/2} \exp\,(+jk_x x)dx \int_{-b/2}^{b/2} \exp(+jk_y y)dy \qquad (11.2.2)$$

Eq. (11.2.2 is readily evaluated and gives:

$$A(k_x,k_y;0) = E_0\, ab\cdot \mathrm{sinc}\frac{k_x a}{2\pi}\mathrm{sinc}\frac{k_y b}{2\pi} \qquad (11.2.3)$$

A normalized cross section of the spectrum is shown in Fig. 11.2.1

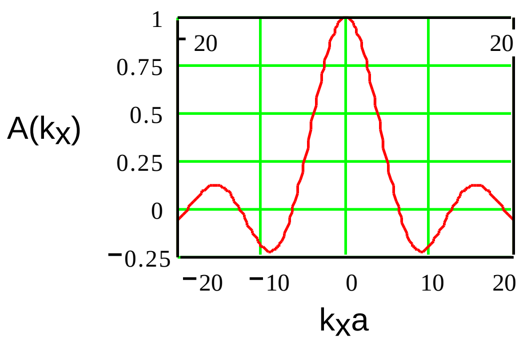

Fig. 11.2.1 Plane wave spectrum of a rectangular beam

The spectral width of this spectrum is usually defined as the distance from the center to the first zero. In the X direction this occurs when $k_x = 2\pi/a$, in the Y direction when $k_y = 2\pi/b$. It is readily seen that the angular widths are given by $\sin\varphi = \lambda/a$ and $\sin\varphi' = \lambda/b$ or, in the paraxial approximation, $\varphi \approx \lambda/a$ and $\varphi' \approx \lambda/b$.

Ex 11.3

Plane wave spectrum of a circular uniform beam

A uniform field limited by a circular aperture of radius a/2 may be written as

$$E(r) = E_0 \mathrm{circ}(2r/a)$$

With (11.25) we find:

$$A(k_t;z) = 2\pi \cdot E_o \int_0^{a/2} r J_o(k_t r)\,dr \qquad (11.3.1)$$

Using a formula from REF.3 :

$$\int_0^x u J_0(u)\,du = xJ_1(x),$$

we find readily that

$$A(k_t;z) = E_0 \pi \left(\frac{a}{2}\right)^2 \frac{J_1(k_t a/2)}{k_t a/4} \qquad (11.3.2)$$

The function $\frac{J_1(k_t a/2)}{k_t a/4}$ is called the Airy pattern. It is shown in Fig. 11.3.1)

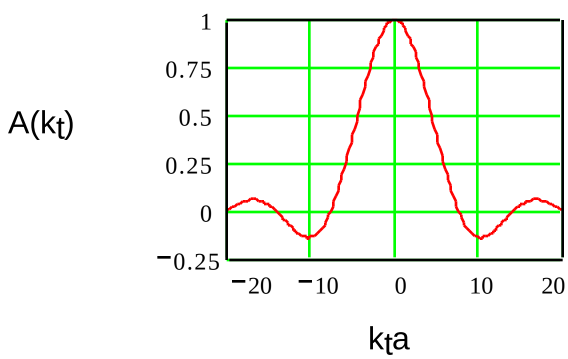

Fig. 11.3.1 Plane wave spectrum of a uniform circular beam

The width of the plane wave spectrum (11.3.2), again defined as the distance from the center to the first zero, is given by $k_t a = 7.66$ corresponding to an angular width of $\sin\varphi_z = k_t/k = 1.22\lambda/a$ or, in the paraxial approximation, $\varphi_z \approx 1.22\lambda/a$ where $a >> \lambda$. Note that the Airy pattern looks very much like the sinc pattern of Fig.11.2.1, but has a 22% larger angular width

Ex 11.4

Plane wave spectrum of a circular Gaussian beam

The field of Ex. 11.1 exhibits circular symmetry when $w_1 = w_2$. It may then be written as $E_0 \exp(-r^2/w_0^2)$ where $r^2 = x^2 + y^2$, and may be treated treated using the Bessel Fourier transform. We find with (11.22)

$$A(k_t;z) = 2\pi \int_0^\infty r_t E_0 \exp(-r^2/w_0^2) J_o(k_t r)\, dr \qquad (11.4.1)$$

A formula in Ref. 3 states:

$$\int_0^\infty x \exp(-ax^2) J_0(bx) = \frac{1}{2a} \exp\left(\frac{-b^2}{4a}\right)$$

Applying this to (11.4.1) with $x=r_t$, $a = 1/w_0^2$ and $b = k_t$, we find

$$A(k_x,k_y;0) = E_0 \pi w_0^2 \exp(-k_t^2 w_0^2/4) \qquad (11.4.2)$$

which is identical to (11.1.4) upon substituting $k_t = \sqrt{k_x^2 + k_y^2}$ and setting $w_1 = w_2 = w_0$.

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