7. PROPAGATION OF THE OPTICAL FIELD vs.3.1

PROPAGATION BY PLANE WAVES

The fundamental problem is the following:

Given the field E_1 at z_1, calculate the field E_2 at z_2.

Ignoring evanescent waves S6 ,the solution of the problem is along the following lines:

- Decompose the field E_1 into a local plane wave spectrum S11 A_1 according to

$$A_1(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_1(x,y)\exp(-jk_xx-jk_yy)dxdy \qquad (7.1)$$

- Propagate all plane waves from z_1 to z_2 by means of the propagator S11 H according to

$$A_2(k_x,k_y) = H(k_x,k_y)A_1(k_x,k_y) \qquad (7.2)$$

where, in the paraxial approximation S6 , H is given by

$$H(k_x,k_y) \approx \exp[jk(z_2-z_1)]\exp[-j(\frac{k_x^2}{2k}+\frac{k_y^2}{2k})(z_2-z_1)] \qquad (7.3)$$

- Sum all plane waves in the propagated spectrum A_2 to give the field E_2 according to

$$E_2(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_2(k_x,k_y)\exp(jk_xx+jk_yy)dk_xdk_y \qquad (7.4)$$

If desired we may use the notation

$$E_1(x,y) = E(x,y,z_1), \quad E_2(x,y) = E(x,y,z_2) \qquad (7.5)$$

$$A_1(k_x,k_y) = A(K_x,k_y;z_1), \quad A_2(k_x,k_y) = A(K_x,k_y;z_2) \qquad (7.6)$$

The sequence (7.1), (7.2) and (7.4) may be represented in shorthand operator S1 notation:

$$E_2 = F^{-1} H F E_1 \qquad (7.7)$$

Fig. 7.1 is a symbolic representation of the operation (7.7). For simplicity only the X and Z axes are shown.

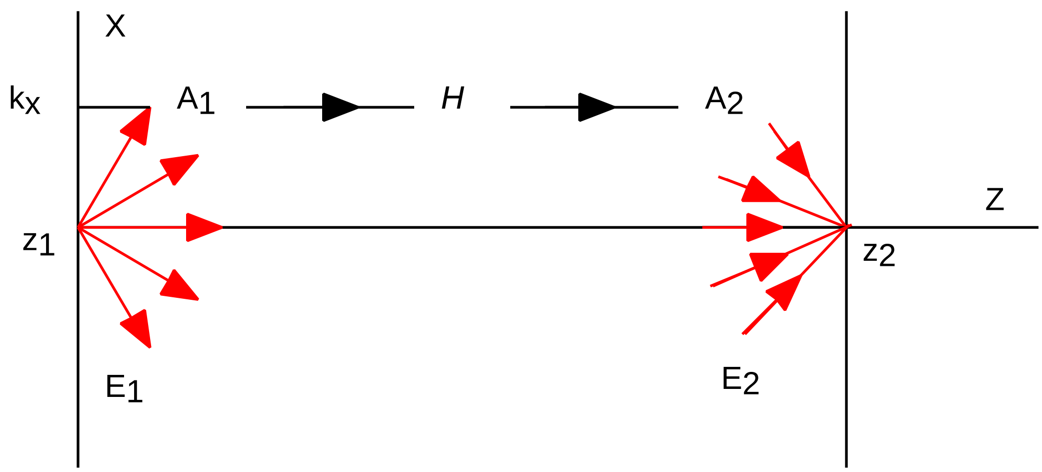

Fig. 7.1 Schematic representation in two dimensions of the propagation of an optical field by plane waves.

EX. 7.1 shows how to calculate the evolution of a Gaussian beam according to (7.7). Examples of numerical calculation using (7.7) may be found in S28.

Circular symmetry

In this case (7.7) should be written

$$E_2 = B^{-1} H \; B \; E_1 \tag{7.8}$$

where B represents the Fourier-Bessel transform. S11

PROPAGATION BY SPHERICAL WAVES

Starting from (7.7), we apply the convolution property S4 of the Fourier transform:

$$E_2 = F^{-1} H F E_1 = F^{-1} \{H F E_1\} = F^{-1} H * F^{-1} F E_1 = F^{-1} H * E_1 \tag{7.9}$$

The propagator H being a simple multiplier, we define

$$h = F^{-1} H \tag{7.10}$$

so that 7.9 becomes

$$E_2 = h * E_1 \tag{7.11}$$

where, with (7.3) and (7.10)

$$h = F^{-1} \{ \exp[jk(z_2-z_1)]\exp[-j(\frac{k_x^2}{2k} + \frac{k_y^2}{2k})(z_2-z_1)]\} =$$

$$\exp[jk(z_2-z_1)] \; (1/2\pi)^2 \times$$

$$\int_{-\infty}^{\infty} \exp[-j(\frac{k_x^2}{2k} + \frac{k_y^2}{2k})(z_2-z_1) + jk_x x + jk_y y] \, dk_x dk_y \tag{7.12}$$

With the relation REF.1

$$\int_{-\infty}^{\infty} \exp(-p^2u^2 \pm qu) \, du = \frac{\sqrt{\pi}}{p}\exp(\frac{q^2}{4p^2})$$

provided $p \neq 0$ and $\mathrm{Re}(p^2) \geq 0$. (7.13) where $u = k_x$ or k_y , $p^2 = j(z_2 - z_1)/2k$, and $q = jx$ or jy, we find readily from (7.12) that

$$h = \frac{k}{2\pi}\cdot\frac{1}{j(z_2-z_1)}\cdot\exp[jk(z_2-z_1)]\cdot\exp[\frac{jk(x^2+y^2)}{2(z_2-z_1)}] \qquad (7.14)$$

With $z_1 = 0$, $z_2 = z$ and $k/2\pi = \lambda$ we may write (7.14) as

$$h = \frac{1}{j\lambda z}\cdot\exp[jkz]\cdot\exp[\frac{jk(x^2+y^2)}{2z}] \qquad (7.15)$$

which is the form most often encountered in the literature. The function h is sometimes called the impulse response S5 of free space.

Applying (7.11) to (7.15) we find

$$E_2(x,z) = \frac{1}{j\lambda z}\cdot\exp[jkz] \times$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_1(x',y')\exp[\frac{jk[(x-x')^2+(y-y')^2]}{2z}]\, dx'dy' \qquad (7.16)$$

EX. 7.2 shows an application of (7.16) to the evolution of an elliptical Gaussian beam. Examples of numerical calculation using (7.11) may be found in S28 .

With reference to Fig. 7.2 we shall now show that the interpretation of (7.16) is that of the propagation of the field by spherical waves S6 .

Let point P_1 of field E_1, located at x'y', emit a spherical wave with amplitude $E_1(x',y')dx'dy'$. The contribution in point P_2, at x,y, to field E_2 is then given by

$$dE_2 \quad E_1(x',y')\frac{\exp jkr_{12}}{r_{12}}\, dx'dy' \qquad (7.17)$$

and hence the total field E_2 is

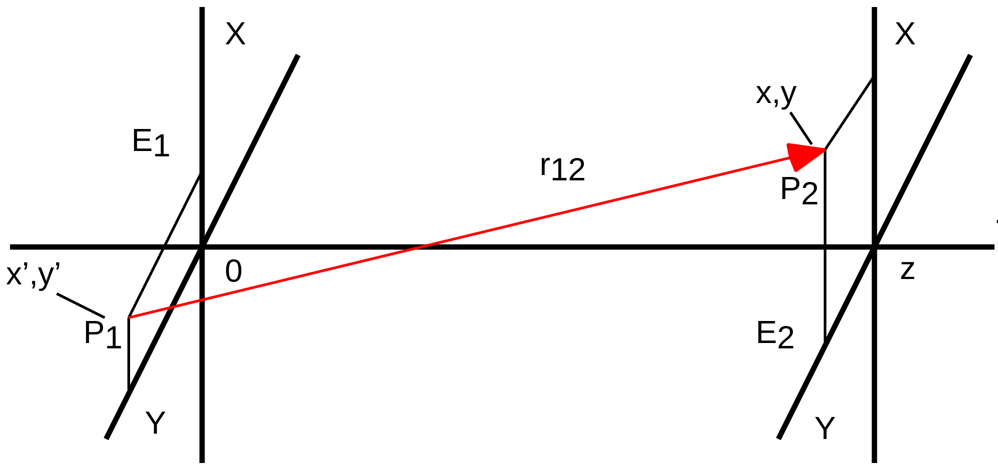

Fig. 7.2 Propagation of the field by spherical waves.

$$E_2(x,y) \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_1(x',y')\frac{\exp(jkr_{12})}{r_{12}}\,dx'dy' \tag{7.18}$$

The radius r_{12} is given by

$$r_{12} = \sqrt{z^2 + (x-x')^2 + (y-y')^2} \tag{7.19}$$

Under conditions of paraxial propagation S6 we assume that the "ray" from P_1 to P_2 lies close to the Z-axis in direcion. Under these conditions $|(x-x')| << |z|$ and $|(y-y')| << |z|$. Eq. (7.19) may then be written as

$$r_{12} \approx z + (x-x')^2/2z + (y-y')^2/2z \tag{7.20}$$

We now substitute (7.20) into the exponential phase term $\exp(jkr_{12})$ of (7.18), but the amplitude term $1/r_{12}$, not being as sensitive as the phase term, we approximate by $1/z$. Eq, (7.18) now becomes

$$E_2(x,z) \quad \frac{1}{z}\cdot\exp[jkz] \times$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_1(x',y')\exp[\frac{jk[(x-x')^2+(y-y')^2]}{2z}]\,dx'dy' \qquad (7.21)$$

It is clear that, but for a constant multiplier $1/j\lambda$, (7.21) is identical to (7.16). Thus the propagation of the field may be interpreted as a propagation by spherical waves. These are emitted by virtual sources –– so-called Huyghens sources –– P_1, located upstream to cause the complete field downstream. The Huyghens sources are 90^0 out of phase with the field, as indicated by the factor $1/j$.

The basic idea of propagation by spherical waves is due to Huyghens REF. 7.2, who originated the wave theory of light. Mathematical formulations similar to (7.21) were first derived by Kirchhoff and later by Rayleigh- Sommerfeld. Some of these formalisms have an additional term, called the obliquity factor, multiplying the integrand in (7.21). For the paraxial approximation used here such factors are equal to unity. More information about the Kirchoff and Rayleigh-Sommerfeld theories may be found in REF. 7.3.

Circular symmetry

We start from (7.8):

$$E_2 = B^{-1} H\, B\; E_1 \qquad (7.22)$$

where $E_1 = E_1(r)$, $E_2 = E_2(r)$, $r = \sqrt{x^2+y^2}$, B is the Fourier - Bessel transform S11, and H is the propagator given by

$$H = \exp(jkz)\exp(jk_t^2 z/2k) \qquad (7.23)$$

where we have written $k_t^2 = k_x + k_y^2$

Eq. (7.22) becomes

$$E_2(r) = \exp(jkz) \times$$

$$\int_0^{\infty}\int_0^{\infty} E_1(r')k_t r' J_0(k_t r')J_0(k_t r)\exp(-jk_t^2 z/2k)dk_t dr' \qquad (7.24)$$

We now integrate over k_t by using the relation REF.1

$$\int_0^\infty J_p(ax)J_p(bx)\exp(-q^2x^2)x\,dx =$$

$$\frac{1}{2q^2}\exp\left(-\frac{a^2+b^2}{4q^2}\right)I_p\left(\frac{ab}{2q^2}\right)$$

where

$$I_p(z) = j^{-p}J_p(jz),\text{ if } p \text{ is an integer.} \qquad (7.25)$$

In our case $a = r$, $b = r'$, $q^2 = jz/2k$, $p = 0$.

We find

$$E_2(r) = \frac{2\pi}{j\lambda z}\exp\left(jkz + \frac{jkr^2}{2z}\right) \times$$

$$\int_0^\infty E_1(r')r'J_o\left(\frac{krr'}{z}\right)\exp\left(\frac{jkr'^2}{2z}\right)dr' \qquad (7.26)$$

EX. 7.3 uses (7.26) to show that the field $E_1(r) = J_o(Kr)$ does not spread by diffraction; it forms a diffraction-free beam S13 .

The term multiplying the integral in (7.26) may, to within the paraxial approximation, be written as

$$\frac{2\pi}{j\lambda}\frac{\exp(jkR)}{R}$$ where R is the distance from the origin to a point

at r in the plane of E_2, as shown in Fig. 7.3.

The exponential term may be interpreted as a spherical wave originating at the origin. In this interpretation the field E_2 is thus essentially a spherical wave, with a (circularly symmetric) complex amplitude distribution across its wavefront, given by the integral following the spherical wave term.

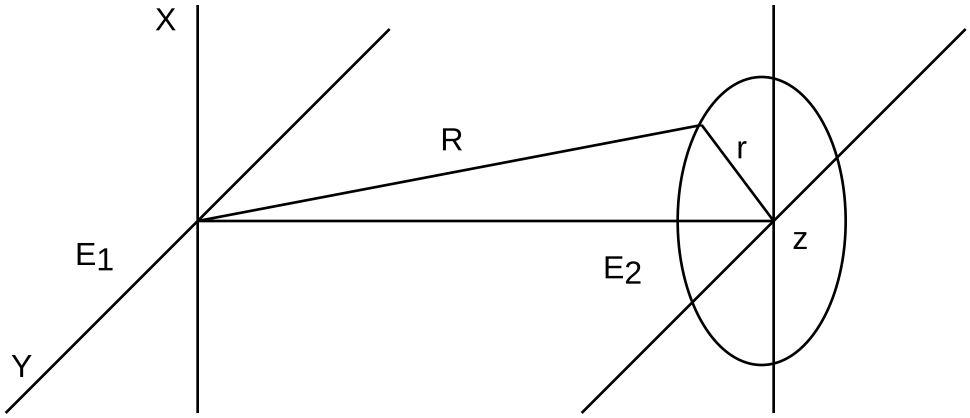

Fig. 7.3 Illustrating $R \approx \sqrt{r^2+z^2} \approx z + r^2/2z$

THE FAR FIELD

When $z \to \infty$ we speak of the far field or Fraunhofer field (Fraunhofer diffraction). The expression for this field is much simplified as we will now show. We start from (7.7):

$$E_2 = F^{-1}\, H\, F\;\; E_1 = F^{-1}\, H\;\; A_1 =$$

$$\frac{\exp(jkz)}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_1(k_x,k_y)\; \exp(-j\frac{k_x{}^2}{2z}+jkx-j\frac{k_y{}^2}{2z}+jky)dk_xdk_y \tag{7.27}$$

Completing the squares in k_x and k_y in (7.27), we write

$$E_2 = \frac{\exp(jkz+jk\frac{x^2+y^2}{2z})}{(2\pi)^2} \times$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_1(k_x,k_y)\; \exp[-j\frac{z}{2k}(k_x-\frac{kx}{z})^2]\exp[-j\frac{z}{2k}(k_y-\frac{ky}{z})^2]dk_xdk_y \tag{7.28}$$

Now, when $z \to \infty$, the exponential terms in the argument vary infinitely fast everywhere but in the points $k_x = kx/z$ and $k_y = ky/z$. Thus nowhere but in the infinitesimal neighborhood of these points is there a net contribution from A_1 to the integral S10 . This contribution is given by

$$A_1(k_x = kx/z, k_y = ky/z) \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-j\frac{z}{2k}(k_x - \frac{kx}{z})^2]\exp[-j\frac{z}{2k}(k_y - \frac{ky}{z})^2]dk_x dk_y =$$

$$A_1(k_x = kx/z, k_y = ky/z) \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-j\frac{z}{2k}k_x^2]\exp[-j\frac{z}{2k}k_y^2]dk_x dk_y = \frac{2k\pi}{jz}$$

(7.29)

by using the relation REF.1

$$\int_{-\infty}^{\infty} \exp(-p^2u^2 \pm qu)\, du = \frac{\sqrt{\pi}}{p}\exp(\frac{q^2}{4p^2})$$

provided $p \neq 0$ and $\mathrm{Re}(p^2) \geq 0$.

Substituting (7.29) into (7.28), we find finally, for $z \to \infty$

$$E_2 = \frac{1}{j\lambda z}\exp(jkz + jk\frac{x^2+y^2}{2z})A_1(k_x = kx/z, k_y = ky/z)$$

(7.30)

The first two terms in (7.30), as in (7.26), represent a spherical wave centered on the origin. To find the field E_2, the wavefront of this wave must be multiplied by the complex amplitude distribution A_1. In other words, the point x,y has an amplitude proportional to the amplitude of the plane wave at $k_x = kx/z$ and $k_y = ky/z$. This is the plane wave propagating at angles $\varphi \approx k_x/k = x/z$ and $\varphi' \approx k_y/k = y/z$, i.e the plane wave whose wave vector is directed along the line from the origin to the point P at x,y. Thus, when compensated for by the spherical wave phase term, the far field E_2 at P measures directly the amplitude of the angular plane wave spectrum in the direction OP, where O is the origin and P the observation point. Fig. 7.4 illustrates this interpretation.

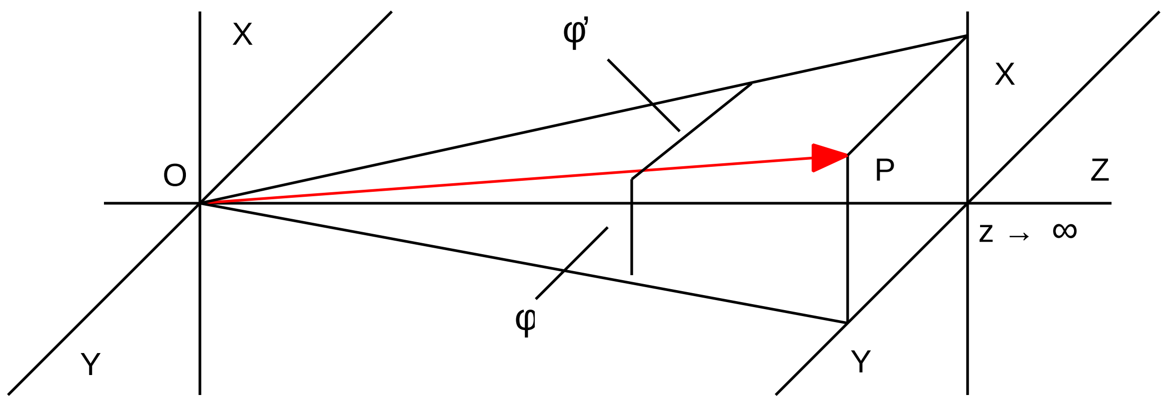

FIG. 7.4 The far field as angular spectrum

Because $A_1 = F E_1$, eq. (7.30) may also be written as

$$E_2 = \frac{1}{j\lambda z}\exp\left(jkz + jk\frac{x^2+y^2}{2z}\right)S\ (kx/z, ky/z)F\ E_1 \qquad (7.31)$$

where S is the symbol exchange operator S1 which indicates that in the Fourier transform $F E_1$, k_x must be replaced by kx/z and k_y by ky/z. Equation (7.31) is the defining equation of Fraunhofer diffraction S16 . The mathematical analysis we have performed to calculate the limiting value of E_2 for $z \to \infty$ is essentially what is called the stationary phase method S10

EX. 7.1

Propagation of an elliptical Gaussian beam. Plane wave method

For simplicity we assume that $z_1 = 0$ and $z_2 = z$. For an elliptical Gaussian beam the field E_1 at z=0 is given by

$$E(x,y,0) = E_0 \exp(-x^2/w_1^2 - y^2/w_2^2) =$$

$$E_0 \exp(-x^2/w_1^2) \exp(-y^2/w_2^2) \qquad (7.1.1)$$

The local plane wave spectrum A_1 at z=0 of this field has been calculated in EX. 11.1.

$$A(k_x,k_y;0) = E_0 \pi w_1 w_2 \exp(-k_x^2 w_1^2/4 - k_y^2 w_2^2/4) \qquad (7.1.2)$$

The local plane wave spectrum A_2 at z is found by multiplying A_1 with

$$H(k_x,k_y) \approx \exp[jkz]\exp[-j(\frac{k_x^2}{2k} + \frac{k_y^2}{2k})z] \qquad (7.1.3)$$

so that A_2 =

$$A(k_x.k_y;z) = E_0 \pi w_1 w_2 \exp(jkz) \times$$

$$\exp(-k_x^2 w_1^2/4 - j\frac{k_x^2}{2k}z) \times$$

$$\exp(-k_y^2 w_2^2/4 - j\frac{k_y^2}{2k}z) \qquad (7.1.4)$$

The field E_2 at z may now be found by taking the Fourier transform of (7.1.4):

$$E(x,y,z) = E_0 \pi w_1 w_2 \exp(jkz) \times$$

$$F_x\{\exp(-k_x^2 w_1^2/4 - j\frac{k_x^2}{2k}z)\} \times$$

$$F_y\{\exp(-k_y^2 w_2^2/4 - j\frac{k_y^2}{2k}z)\} \qquad (7.1.5)$$

where F_x and F_y are one dimensional inverse Fourier transforms in x and y respectively. We may write

$$E(x,y,z) = E_0 \pi w_1 w_2 \exp(jkz) \times$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-k_x^2 w_1^2/4 - j\frac{k_x^2}{2k}z + jk_x x)dk_x \times$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-k_y^2 w_2^2/4 - j\frac{k_y^2}{2k}z + jk_y y)dk_y \qquad (7.1.6)$$

The integrals in (7.1.6) may be solved by application of the following formula REF.1

$$\int_{-\infty}^{\infty} \exp(-p^2u^2 \pm qu)\, du = \frac{\sqrt{\pi}}{p}\exp(\frac{q^2}{4p^2})$$

provided $p \neq 0$ and $Re(p^2) \geq 0$. (7.1.7)

In our case $p^2 = w_1^2/4 + jz/2k$ or $w_2^2/4 + jz/2k$ respectively, and $q = jx$ or jy respectively. Applying (7.1.7) to (7.1.6) we find readily

$$E(x,y,z) = E_0 \exp(jkz) \times$$

$$\frac{1}{\sqrt{1+j2z/kw_1^2}} \exp[\frac{-x^2}{w_1^2(1+ j2z/kw_1^2)}] \times$$

$$\frac{1}{\sqrt{1+j2z/kw_2^2}} \exp[\frac{-y^2}{w_2^2(1 + j2z/kw_2^2)}] \qquad (7.1.8)$$

Introducing characteristic lengths,

$$z_{0x} = kw_1^2/2 \text{ and } z_{0y} = kw_2^2/2 \qquad (7.1.9)$$

we may write for (7.1.8)

$$E(x,y,z) = E_0 \exp(jkz) \times$$

$$\frac{1}{\sqrt{1+jz/z_{0x}}}\exp\left[\frac{-x^2}{w_1^2(1+jz/z_{0x})}\right]\times$$

$$\frac{1}{\sqrt{1+jz/z_{0y}}}\exp\left[\frac{-y^2}{w_2^2(1+jz/z_{0y})}\right] \qquad (7.1.10)$$

An alternative expression in terms of z-dependent waists, radii of phase curvature and phase angles is given in S6.

Fig. 7.1.1 shows a one dimensional ($w_2 = \infty$, $z_{0y} = \infty$) Gaussian in the XZ plane with the amplitude normalized to unity at z=0

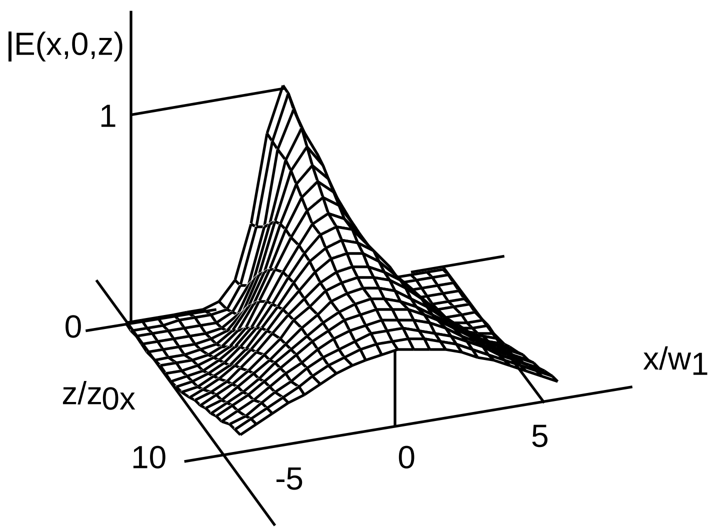

Fig. 7.1.1

Ex. 7.2

Propagation of an elliptical Gaussian beam. Spherical waves method

As in Ex. 7.1 the field E_1 is given by

$$E_1(x,y) = E_0 \exp(-x^2/w_0^2)\exp(-y^2/w_0^2) \qquad (7.2.1)$$

Application of (7.16) gives

$$E_2(x,z) = \frac{E_0}{j\lambda z} \cdot \exp[jkz] \times$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-x'^2/w_1^2 - y'^2/w_2^2)\exp[\frac{jk[(x-x')^2+(y-y')^2]}{2z}]\, dx'dy'$$

or

$$E_2(x,y) = \frac{E_0}{j\lambda z} \cdot \exp[jkz + jk(x^2+y^2)/2z] \times$$

$$\int_{-\infty}^{\infty} \exp[-x'^2(1/w_1^2 - jk/2z) - jkxx'/2z]dx' \times$$

$$\int_{-\infty}^{\infty} \exp[-y'^2(1/w_2^2 - jk/2z) - jkyy''/2z]dy' \qquad (7.2.2)$$

The two integrals may again be solved by application of the relation REF.1

$$\int_{-\infty}^{\infty} \exp(-p^2u^2 \pm qu)\, du = \frac{\sqrt{\pi}}{p}\exp(\frac{q^2}{4p^2})$$

provided $p \neq 0$ and $Re(p^2) \geq 0$. (7.2.3)

with $u = x'$ or y', $p^2 = (1/w_1^2 - jk/2z)$ or $(1/w_2^2 - jk/2z)$ and $q = jkx/2z$ or $jky/2z$

After some algebra we find

$$E_2(x,y) = E_0\exp(jkz) \times$$

$$\frac{1}{\sqrt{1+j2z/kw_1^2}} \exp[\frac{-x^2}{w_1^2(1+j2z/kw_1^2)}] \times$$

$$\frac{1}{\sqrt{1+j2z/kw_2^2}} \exp[\frac{-y^2}{w_2^2(1+j2z/kw_2^2)}] \qquad (7.2.4)$$

which is identical to (7.1.8) which was calculated by the plane wave propagation method.

EX. 7.3

Propagation of a diffraction-free beam

If in (7.26) we substitute $E_2(r') = J_0(Kr')$ where K is a real constant, we find

$$E_2(r) = \frac{2\pi}{j\lambda z} \exp(jkz + \frac{jkr^2}{2z}) \times$$

$$\int_0^\infty J_0(Kr')r'J_0(\frac{krr'}{z})\exp(\frac{jkr'^2}{2z})dr' \qquad (7.3.1)$$

We now use the relation Ref.1

$$\int_0^\infty J_p(ax)J_p(bx)\exp(-q^2x^2)xdx =$$

$$\frac{1}{2q^2}\exp(-\frac{a^2+b^2}{4q^2})I_p(\frac{ab}{2q^2}) \qquad (7.3.2)$$

where

$$I_p(z) = j^{-p}J_p(jz), \text{ if p is an integer.} \qquad (7.3.3)$$

In our case $x = r'$, $p = 0$, $a = K$, $b = kr/z$, and $q^2 = -jk/2z$

By substituting these values into (7.3.2) we find for the integral the value

$$\frac{jz\lambda}{2\pi}\exp[-j\frac{z}{2k}(K^2 + \frac{k^2r^2}{z^2})]J_0(Kr)$$

and hence for E_2:

$$E_2(r) = \exp[j(k - \frac{K^2}{2k})z]\ J_o(Kr) \qquad (7.3.4)$$

Hence $|E_2(r)| = |E_1(r)| = |J_o(Kr)|$, and the beam is diffraction-free.

REFERENCES

7.1 I. S. Gradshteyn and Ryzhik, Table of Integrals, Series and Products , Academic Press, New York, 1965.

7.2 C Huygens, Treatise on Light , Dover Publications, New York, 1962.

7.3 J. W. Goodman, Introduction to Fourier Optics 2nd ed. , McGraw-Hill, New York, 1996.