

Chapters 1 Preliminary Concepts & 2 Fundamental Equations of Compressible Viscous Flow

(6.1) Reference Frames and Coordinate Systems

Reference frame → state of motion of the observer.

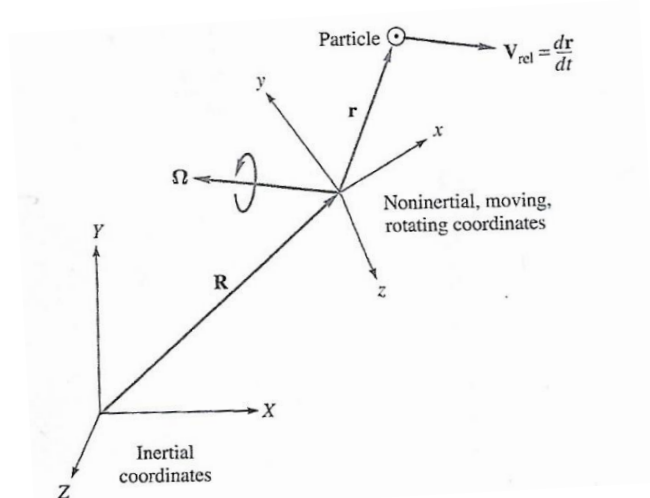
Coordinate system → set of numbers used to map the space points within a reference frame.

For any given reference frame, multiple coordinate systems are possible (e.g. Cartesian, spherical, etc.)

In classical physics and special relativity, an inertial frame of reference is a frame of reference that is not undergoing acceleration. In an inertial frame of reference, a physical object with zero net force acting on it moves with a constant velocity (which might be zero)—or, equivalently, it is a frame of reference in which Newton's first law of motion holds.

Non-inertial Reference Frame: A non-inertial reference frame is a frame of reference that undergoes acceleration with respect to an inertial frame.

Thus far we have assumed use of an inertial reference frame (i.e. fixed with respect to the distant stars in deriving the CV and differential form of the momentum equation). However, in many cases non-inertial reference frames are useful (e.g. rotational machinery, vehicle dynamics, geophysical applications, etc.).



Geometry of fixed (inertial) vs. accelerating (non-inertial) coordinates

$$\underline{a}_i = \frac{D\underline{V}}{Dt} + \underline{a}_{rel}$$

$$\sum \underline{F} = m\underline{a}_i = m \left(\frac{D\underline{V}}{Dt} + \underline{a}_{rel} \right)$$

$$\sum \underline{F} - m\underline{a}_{rel} = m \frac{D\underline{V}}{Dt}$$

i.e Newton's law applies to non-inertial frame with addition of known inertial force terms

\underline{V} = relative velocity in non-inertial reference frame (x,y,z)

$\frac{D\underline{V}}{Dt}$ = non-inertial \underline{a} that must be added to \underline{a}_{rel}

The absolute fluid particle position vector is

$$\underline{S}_j = \underline{R} + \underline{r}$$

Such that

$$\underline{V}_i = \underline{V} + \frac{d\underline{R}}{dt} + \underline{\Omega} \times \underline{r}$$

3rd term from fact that (x,y,z) rotating at $\Omega(t)$.

$$\underline{a}_i = \frac{D\underline{V}}{Dt} + \frac{d^2\underline{R}}{dt^2} + \frac{d\underline{\Omega}}{dt} \times \underline{r} + 2\underline{\Omega} \times \underline{V} + \underline{\Omega} \times (\underline{\Omega} \times \underline{r})$$

$$= \frac{D\underline{V}}{Dt} + \underline{a}_{rel}$$

$$\frac{d^2 \underline{R}}{dt^2} = \text{acceleration } (x,y,z)$$

Bodies in non-inertial reference frames are subject to so-called fictitious forces (pseudo-forces); that is, forces that result from the acceleration of the reference frame itself and not from any physical force acting on the body. Examples of fictitious forces are the centrifugal force and the Coriolis force in rotating reference frames.

$$-m \frac{d\underline{\Omega}}{dt} \times \underline{r} = \text{Euler force due to angular acceleration } (x,y,z)$$

$$-m 2\underline{\Omega} \times \underline{V} = \text{Coriolis force}$$

$$-m \underline{\Omega} \times (\underline{\Omega} \times \underline{r}) = \text{centrifugal force (directed away from the particle normal distance to the axis of rotation with magnitude } -\Omega^2 L, \text{ where } L = \text{normal distance from } \underline{r} \text{ to axis of rotation } \underline{\Omega}).$$

Since \underline{R} and $\underline{\Omega}$ assumed known, although more complicated, we are simply adding known inhomogeneities to the momentum equation.

CV form of Momentum equation for non-inertial reference frame:

$$\sum \underline{F} - \int_{CV} \underline{a}_{rel} \rho d\forall = \frac{d}{dt} \int_{CV} \underline{V} \rho d\forall + \int_{CS} \underline{V} \rho \underline{V}_R \cdot \underline{n} dA$$

where \underline{V}_R is the velocity of the CV relative to the non-inertial reference frame (x,y,z) .

Differential form of momentum equation for non-inertial reference frame:

$$\rho \left[\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right] = \underbrace{-\rho \underline{a}_{rel}}_{\substack{\text{body} \\ \text{force}}} - \nabla (p + \gamma z) + \mu \nabla^2 \underline{V}$$

where

$$\underline{a}_{rel} = \underline{\ddot{R}} + 2\underline{\Omega} \times \underline{V} + \underline{\Omega} \times (\underline{\Omega} \times \underline{r}) + \underline{\dot{\Omega}} \times \underline{r}$$

All terms in \underline{a}_{rel} seldom act in unison (e.g. geophysical flows):

$$\underline{\ddot{R}} \sim 0 \quad \text{earth not accelerating relative to distant stars}$$

$$\underline{\dot{\Omega}} \sim 0 \quad \text{for earth}$$

$$\underline{\Omega} \times (\underline{\Omega} \times \underline{r}) \sim 0 \quad \text{g nearly constant with latitude}$$

$$\therefore 2\underline{\Omega} \times \underline{V} \quad \text{most important!}$$

$$\underline{a}_i = \frac{D\underline{V}}{Dt} + R_0^{-1} (2\underline{\Omega} \times \underline{V}) \quad \underline{V} = \frac{V}{V_0}, t = \frac{tV_0}{L}$$

$$R_0 = \text{Rossby \#} = \frac{V_0^2/L}{\Omega V_0} = \frac{V_0}{\Omega L}$$

if L is large, i.e., comparable to the order of magnitude of the earth radius, $R_0 < 1$, then Coriolis term is larger than the inertia terms and is important.

Examples of Non-inertial Reference Frames

1. Geophysical fluids dynamics: Geophysics is a subject of natural science concerned with the physical processes and physical properties of the Earth and its surrounding space environment, and the use of quantitative methods for their analysis.

Atmosphere and oceans are naturally studied using non-inertial coordinate system rotating with the earth. Two primary forces are Coriolis force and buoyancy force due to density stratification $\rho = \rho(T)$. Both are studied using Boussinesq approximations ($\rho = \text{constant}$, except $-\rho(T)g\hat{k}$ term; and $\mu, k, C_p = \text{constant}$) and thin layer on rotating surface assumption $\left(\frac{W}{U} \sim \frac{H}{L}\right)$.

Differences between atmosphere and oceans: lateral boundaries (continents) in oceans; currents in ocean (gulf and Kuroshio stream) along western boundaries; clouds and latent heat release in atmosphere due to moisture condensation; $V_{\text{ocean}} = 0.1 \sim 1$ or 2 m/s and $V_{\text{atmosphere}} 10 \sim 20$ m/s

$H \ll L = R$ (radius of earth = 6371 km)

Therefore, one can neglect curvature of earth and replace spherical coordinates by local Cartesian tangent plane coordinates.

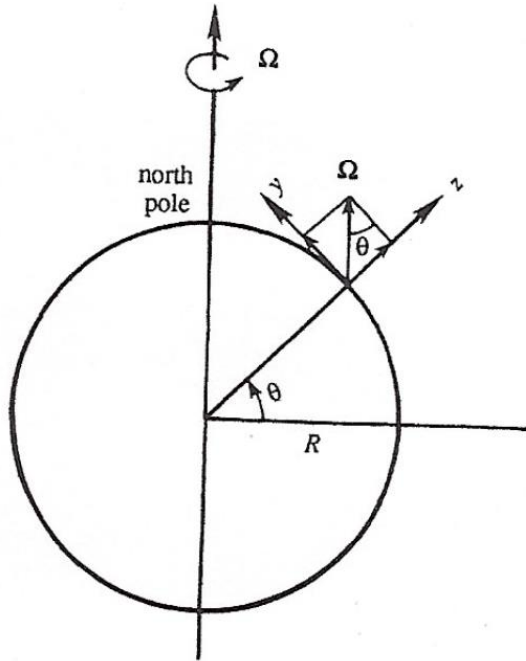


Fig. 13.3 Local Cartesian coordinates. The x -axis is into the plane of the paper.

x = eastward

y = northward

z = upward

$$\Omega = 2\pi\text{rad/day} = 0.73 \times 10^{-4} \text{s}^{-1}$$

$$\Omega_x = 0$$

$$\Omega_y = \Omega \cos\theta$$

$$\Omega_z = \Omega \sin\theta$$

θ = latitude > 0 northern hemisphere and < 0 southern hemisphere

Coriolis force = $2\Omega \times V$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Omega_x & \Omega_y & \Omega_z \\ u & v & w \end{vmatrix}$$

0 since $w \ll v$

$$= 2\Omega \left[\hat{i}(w \cos \theta - v \sin \theta) + \hat{j}u \sin \theta - \hat{k}u \cos \theta \right]$$

$$= -fv \hat{i} + fu \hat{j} - 2\Omega \cos \theta u \hat{k} \qquad f = 2\Omega \sin \theta$$

= planetary vorticity
= 2 * vertical component Ω

Person spins at Ω

- $f > 0$ northern hemisphere
- $f < 0$ southern hemisphere
- $f = \pm \Omega$ at poles
- $f = 0$ at equator

Person translates with inertial period $T_i = \frac{2\pi}{f}$

The Coriolis force acts in a direction perpendicular to the rotation axis and to the velocity of the body in the rotating frame and is proportional to the object's speed in the rotating frame (more precisely, to the component of its velocity that is perpendicular to the axis of rotation).

Equations of Motion

$$\nabla \cdot \underline{V} = 0$$

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} + \nu \nabla^2 v$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} - \frac{\rho g}{\rho_0} + \nu \nabla^2 w$$

vertical component Coriolis force negligible due to thin layer assumption, i.e., magnitude of $2\Omega \cos \theta u \ll$ other terms

$$\rho = \rho_0 [1 - \alpha(T - T_0)]$$

p, ρ = perturbation from hydrostatic condition

Z momentum $\rightarrow \frac{\partial p}{\partial z} = -\rho g$ baroclinic fluid (such as the atmosphere) in which surfaces of constant pressure intersect those of constant density. $p = p(T)$ since $\rho = \rho(T)$ and can be used to eliminate p in above equations whereby $(u, v) = f(T(z))$, which is called thermal wind but not considered here.

Geostrophic Flow: quasi-steady, large-scale motions in atmosphere or ocean far from boundaries = wind/current due balance of $\text{grad}(p)$ and Coriolis force

$$-fv = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} \qquad fu = -\frac{1}{\rho_0} \frac{\partial p}{\partial y}$$

$$\frac{DV}{Dt} \sim 0 \left(\frac{U^2}{L} \right) \quad f\underline{V} \sim 0 (fU) \quad U, L = \text{horizontal scales}$$

$$\text{Rossby number} = \frac{U}{fL}$$

Atmosphere: $U \sim 10 \text{ m/s}$; $f = 10^{-4} \text{ Hz}$; $L \sim 1000 \text{ km}$;
and $R_0 = 0.1$

Ocean: $U \sim 0.1 \text{ m/s}$; $f = 10^{-4} \text{ Hz}$; $L \sim 1000 \text{ km}$;
and $R_0 = 0.01$

Therefore, neglect $\frac{DV}{Dt}$ and since there are no boundaries, neglect $\nu \nabla^2 \underline{V}$.

If we neglect $\rho = \rho(T)$ effects, $(u, v) = f(p)$ and can be determined from measured $p(x, y)$. Not valid near the equator ($\pm 3^\circ$) where f is small.

$$\begin{aligned} (u \hat{i} + v \hat{j}) \cdot \nabla p &= \frac{1}{\rho_0 f} \left(-\frac{\partial p}{\partial y} \hat{i} + \frac{\partial p}{\partial x} \hat{j} \right) \cdot \left(\frac{\partial p}{\partial x} \hat{i} + \frac{\partial p}{\partial y} \hat{j} \right) \\ &= 0 \end{aligned}$$

i.e \underline{V} is perpendicular to $\nabla p \rightarrow$ horizontal velocity is along (and not across) lines of constant horizontal pressure, which is reason isobars and stream lines coincide on a weather map! If $f = \text{constant}$, then $\frac{p}{\rho_0 f}$ can be regarded as a stream function.

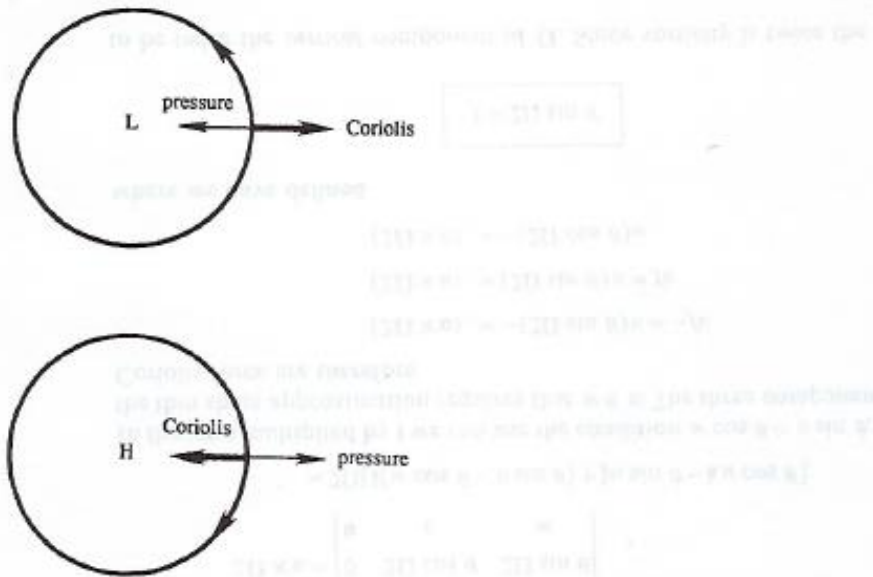


Fig. 13.4 Geostrophic flow around low and high pressure centers. The pressure force ($-\nabla p$) is indicated by a thin arrow, and the Coriolis force is indicated by a thick arrow.

Geostrophic flow around low- and high-pressure centers in the northern hemisphere. Coriolis force acts to the right of the velocity vector. Thus, the flow is counterclockwise (viewed from above) around low pressure and clockwise around high-pressure regions. The sense of circulation is opposite in the southern in the southern hemisphere where the Coriolis forces acts to the left of the velocity vector.

2. Wind-Driven Flows: Impulsive start up and Ekman layers for free surface viscous flows

Viscous layers:

Sudden acceleration flat plate: $u_t = \nu u_{yy}$ $u(y,0) = 0$
 $\delta = 3.64\sqrt{\nu t}$ $u(0,t) = U$
 $u(\infty,t) = 0$

Layer grows in time due viscous diffusion

Oscillating flat plate: $u_t = \nu u_{yy}$ $u(0,t) = U_0 \cos \omega t$
 $\delta = 6.5\sqrt{\nu / \omega}$ $u(\infty,t) = 0$

Layer confined constant thickness

Stagnation point flow: $\delta = 2.4\sqrt{\nu / B}$ layer not a function of x since convection balances diffusion

Flat plate boundary layer: $u_x + v_y = 0$
 $uu_x + \nu u_{yy} = \nu u_{yy}$
 $\delta = 4.9\sqrt{\nu x / U}$ $u(x,0) = 0$
 $u(x,\infty) = U$

Layer grows with x due convection

Impulsive wind-driven free surface flow

Water at rest subject to impulsive wind stress τ_0 . Assume unsteady parallel flow $u(z, t)$ and neglect Coriolis force

$$u_t = \nu u_{zz}$$

$$\tau_0 = (\mu u_z)_{\text{air}} = (\mu u_z)_{\text{water}} \quad \mu_{\text{air}} \ll \mu_{\text{water}}$$
$$u_z(0, t) = K = \tau_0 / \mu_{\text{water}} \quad z=0$$

$$u(\infty, t) = 0$$

$$u(z, 0) = 0$$

Similar Stokes 1st problem except the surface condition is for u_z vs u

$$u_z = K \left[1 + \operatorname{erf} \left(\frac{z}{2\sqrt{\nu t}} \right) \right]$$

$$u/K = z \left[1 + \operatorname{erf} \left(\frac{z}{2\sqrt{\nu t}} \right) \right] + 2 \sqrt{\frac{\nu t}{\pi}} e^{-z^2/4\nu t}$$

$$u(0, t) = u_0 = 2K \sqrt{\frac{\nu t}{\pi}}$$

$$\tau_0 = .002 \rho_{\text{air}} (V_{\text{wind}} - u_0)^2$$

For $V_{\text{wind}} = 6 \text{ m/s}$ (12 knots) at $T = 20^\circ\text{C} \Rightarrow u_0 = .6 \text{ m/s}$ for $t = 1 \text{ min}$ at 2.3 m/s after 1 hour, which is greatly overestimated. Actual flow after 1 hour $u_0 = .2 \text{ m/s}$ or 3% V_{wind} due to turbulence and need to use $\mu_t \gg \mu$ in the solution

For Ekman layer viscous effects due to wind shear $\tau(x)$. Assume horizontal uniformity (i.e, $p_x = p_y = 0$), which is justified for $L \sim 100$ km and $H \sim 50$ m. The wind stress acts in the x direction.

$$-fv = \nu u_{zz} \qquad fu = \nu v_{zz}$$

$$\mu u_z = \tau \qquad \text{at } z = 0$$

$$v_z = 0 \qquad \text{at } z = 0 \qquad \underline{\tau} = \tau \hat{i}$$

$$\tau = .002 \rho_{air} (v_{wind} - u(0))$$

$$(u, v) = 0 \qquad \text{at } z = -\infty$$

Multiply v-equation by $i = \sqrt{-1}$ and add to u-equation:

$$\frac{d^2 V}{dz^2} = \frac{if}{\nu} V \qquad V = u + iv$$

$$\qquad \qquad \qquad = \text{complex velocity}$$

$$V = Ae^{(1+i)z/\delta} + Be^{-(1+i)z/\delta}$$

$$\delta = \sqrt{\frac{2\nu}{f}} = \text{Ekman layer thickness}$$

$$B = 0 \text{ for } u(-\infty), v(-\infty) = 0$$

$$\mu \frac{dV}{dz} = \tau \text{ at } z = 0$$

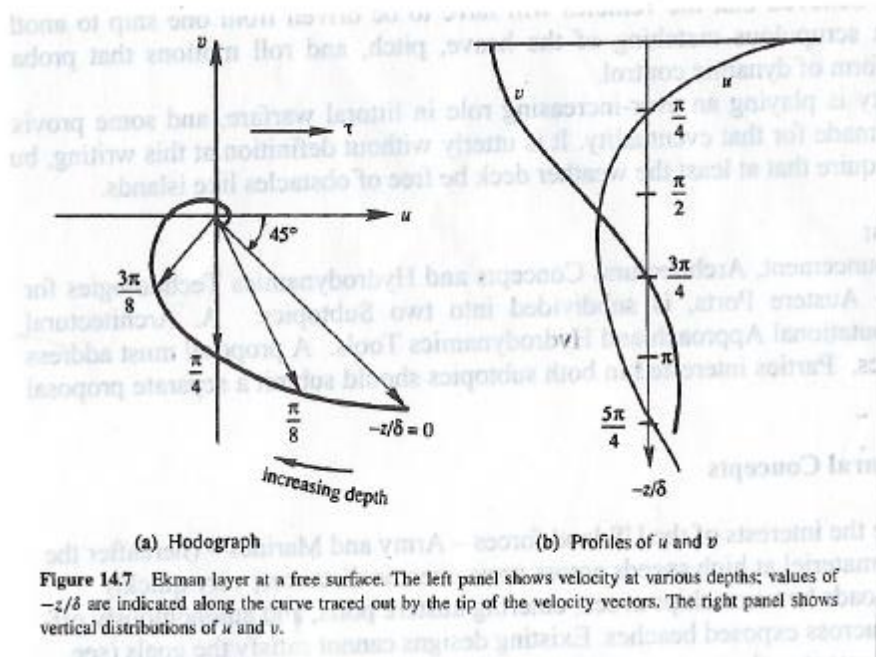
$$\rightarrow A = \frac{\tau\delta(1-i)}{2\rho\nu}$$

$$\text{i.e. } u = \frac{\tau/\rho}{\sqrt{f\nu}} e^{z/\delta} \cos\left(-\frac{z}{\delta} + \frac{\pi}{4}\right) \quad \text{and}$$

$$v = \frac{\tau/\rho}{\sqrt{f\nu}} e^{z/\delta} \sin\left(-\frac{z}{\delta} + \frac{\pi}{4}\right)$$

$V_0 = \frac{\tau/\rho}{\sqrt{f\nu}}$ is the surface-water speed and $\delta = \sqrt{\frac{2\nu}{f}}$ is the penetration depth.

F. Nansen (1902) observed drifting arctic ice drifted 20-40° to the right of the wind, which he attributed to Coriolis acceleration. His student Ekman (1905) derived the solution. Recall $f < 0$ in southern hemisphere, so the drift is to the left of τ .



The surface water velocities are

$$u = V_0 \cos\left(\frac{\pi}{4}\right)$$

$$v = V_0 \sin\left(\frac{\pi}{4}\right)$$

Showing that the surface velocity vector is 45 deg. angle to the right of the wind in the northern hemisphere. As we move below the surface the resultant current vector moves uniformly to the right and decreases exponentially in magnitude forming a logarithmic spiral. At $z=-3D/4$ the current is exactly opposite to the wind and has magnitude $V_0 e^{-3\pi/4} \approx .095V_0$.

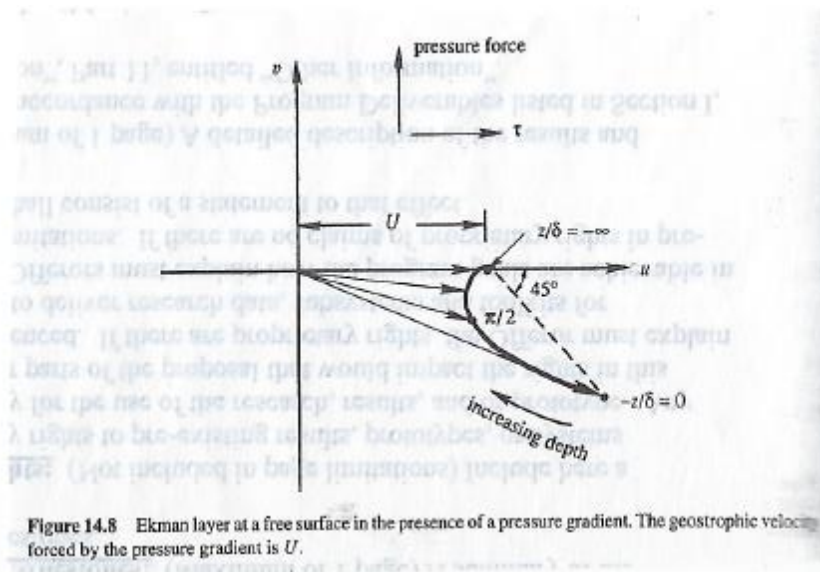
The components of the volume transport are

$$\int_{-\infty}^0 u dz = 0$$

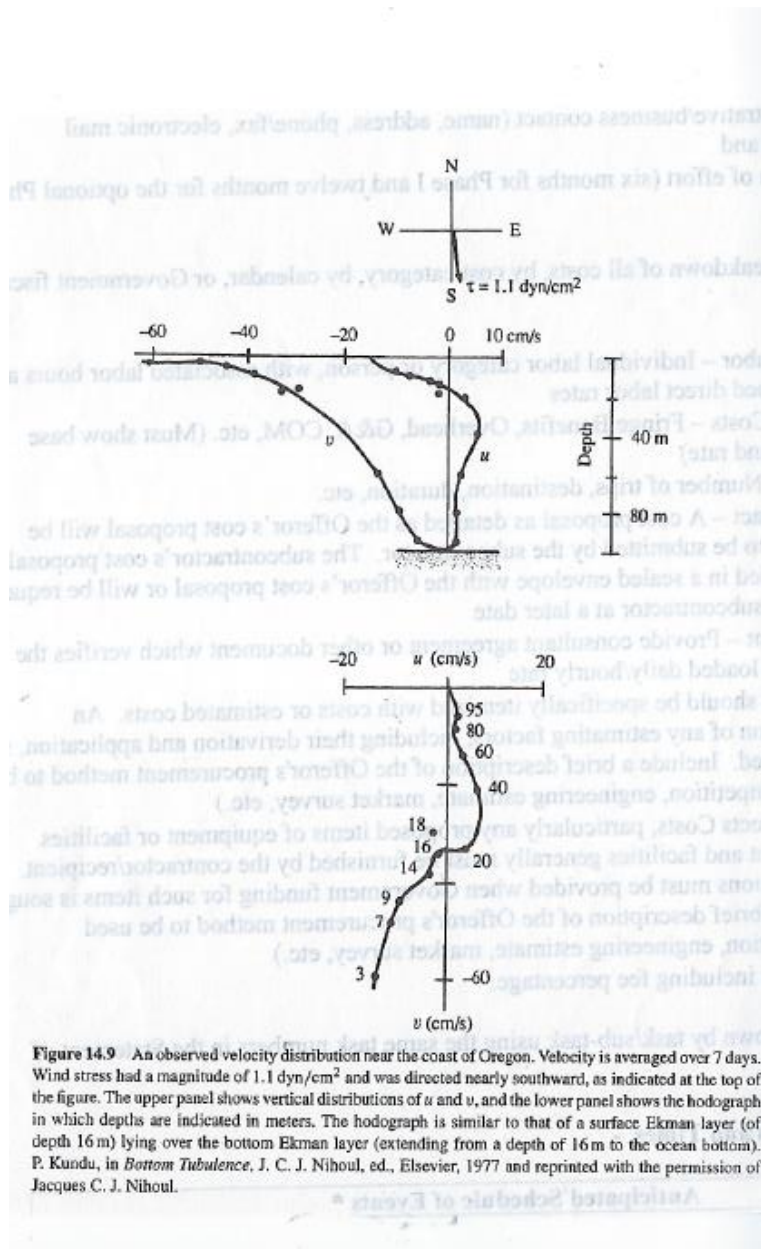
$$\int_{-\infty}^0 v dz = -\tau / \rho f$$

Which shows that the net transport is to the right of the applied stress and independent of v due fact that the depth integrated Coriolis forces is directed to the right of the depth-integrated volume transport balances the wind stress.

The horizontal uniformity can be removed easily if assume p is not function $p(z)$ such that geostrophic solution is additive and combined solution recovers former for large depths $z/\delta \rightarrow -\infty$.



For Ekman layer $V_{\text{wind}} = 6 \text{ m/s}$ and $T=20^\circ \text{ C}$ and latitude $\theta = 41^\circ \text{ N}$ (Rhode Island): Laminar solution $u_0 = 2.7 \text{ m/s}$ at $D = 45 \text{ cm}$, which are too high/low; however, using turbulent ν_t , $u_0 = 2 \text{ cm/s}$ and $D = 100 \text{ m}$, which is more realistic. However, constant eddy viscosity and steady flow are unrealistic such that Ekman layers are not often observed.



The Ekman layer thickness is constant in time and space, which can be explained as follows

$$\omega_x = \omega_y - v_z = -v_z \quad \text{Since } \omega = 0$$

$$\omega_y = v_z - \omega_x = v_z$$

at

$$\text{recall } -fv = v \frac{d^2 v}{dz^2} \quad -fv_z = v \frac{d^2 (v_z)}{dz^2} = v \frac{d^2 \omega_y}{dz^2}$$

$$fv_y = v \frac{d^2 v_y}{dz^2}$$

$$-fv_z = v \frac{d^2 (v_z)}{dz^2} = v \frac{d^2 \omega_x}{dz^2}$$

which shows the vortex diffusion balances the Coriolis force. The vertical fluid lines coincide with the vortex lines. The tilting of the vertical fluid lines as per LHS causes a rate of change of ω_x and ω_y that cancels the viscous diffusion term.