Turbulence Modeling in Computational Fluid Dynamics (CFD)

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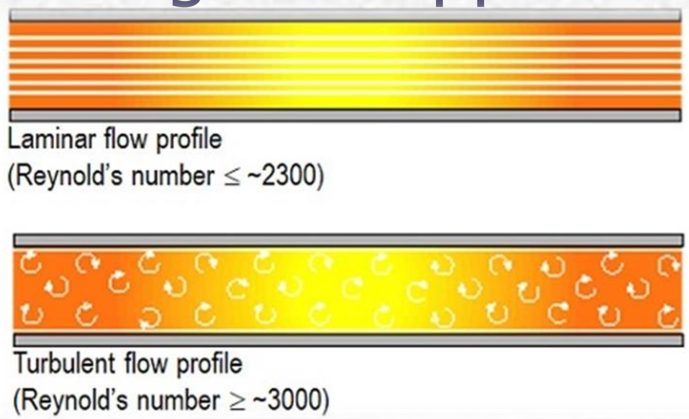


Characteristics of turbulence

- Randomness and fluctuation: $u = U + u'$

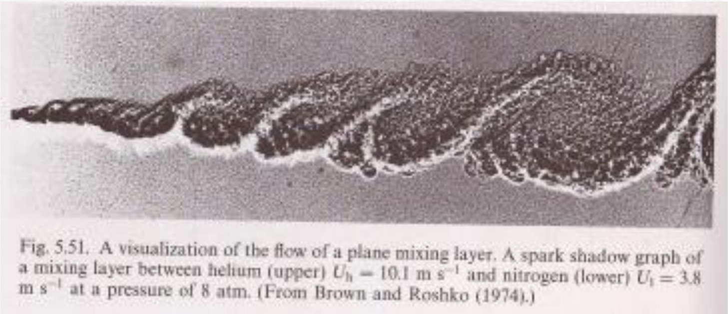
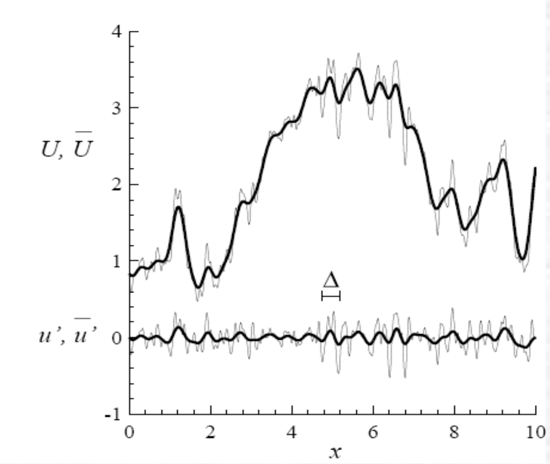


Fig. 5.51. A visualization of the flow of a plane mixing layer. A spark shadow graph of a mixing layer between helium (upper) $U_h = 10.1$ m s^{-1} and nitrogen (lower) $U_l = 3.8$ m s^{-1} at a pressure of 8 atm. (From Brown and Roshko (1974).)



- Nonlinearity: Reynolds stresses from the nonlinear convective terms
- Diffusion: enhanced diffusion of momentum, energy etc.
- Vorticity/eddies/energy cascade: vortex stretching

Characteristics of turbulence

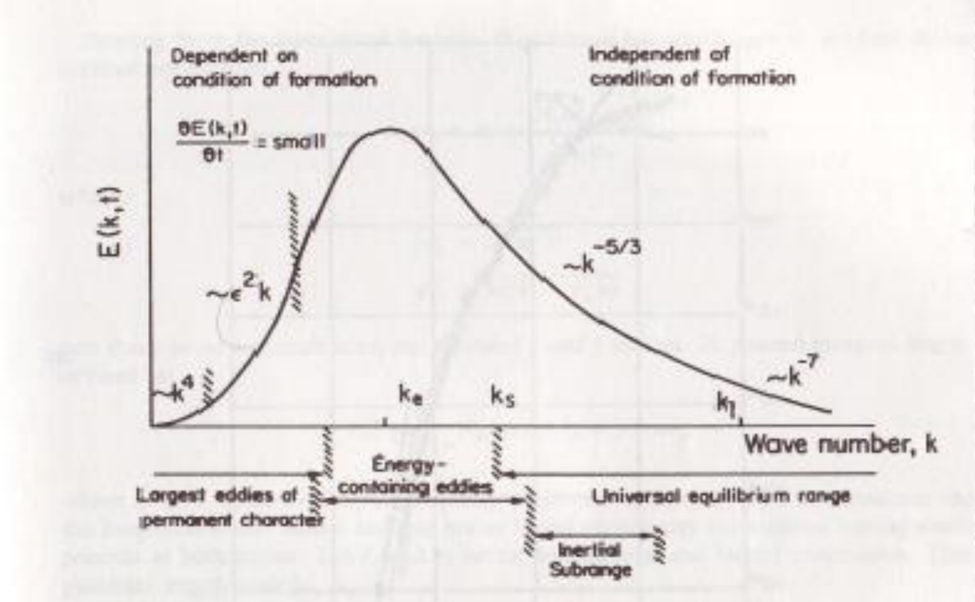


Fig. 6.9(b) Distribution of $E(k, t)$ for various range in isotropic turbulence. (Taken from Hinze, J. O., *Turbulence, and Introduction to its Mechanism and Theory*, McGraw-Hill, New York, 1959. With permission.)

- Dissipation: occurs at smallest scales
- Three-dimensional: fluctuations are always 3D
- Coherent structures: responsible for a large part of the mixing
- A broad range of length and time scales: making DNS very difficult

Approaches to predicting turbulent flows

- AFD, EFD and CFD:
  - AFD: No analytical solutions exist
  - EFD: Expensive, time-consuming, and sometimes impossible (e.g. fluctuating pressure within a flow)
  - CFD: Promising, the need for turbulence modeling
- Another classification scheme for the approaches
  - The use of correlations: $C_D = f\left(\mathrm{Re}\right)$
  - Integral equations: reduce PDE to ODE for simple cases
  - One-point closure: RANS equations + turbulent models
  - Two-point closure: rarely used, FFT of Two-point equations
  - LES: solve for large eddies while model small eddies
  - DNS: solve NS equations directly without any model

Deeper insights on RANS/URANS/LES

Fig. 1.1. Decomposition of the energy spectrum of the solution associated with the Reynolds Averaged Numerical Simulation (symbolic representation).

RANS

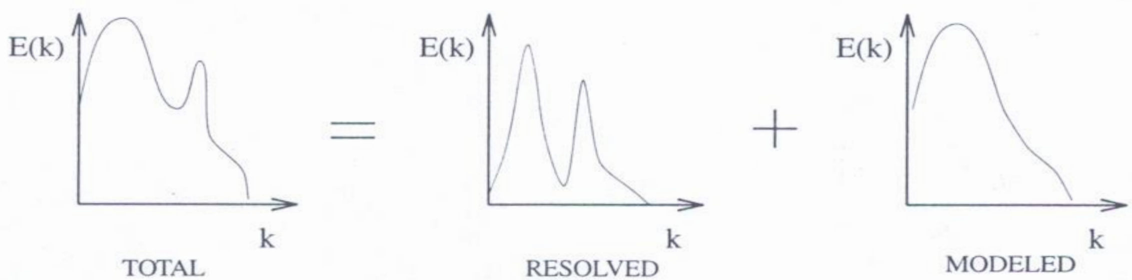


Fig. 1.2. Decomposition of the energy spectrum of the solution associated with the Unsteady Reynolds Averaged Numerical Simulation approach, when a predominant frequency exists (symbolic representation).

URANS

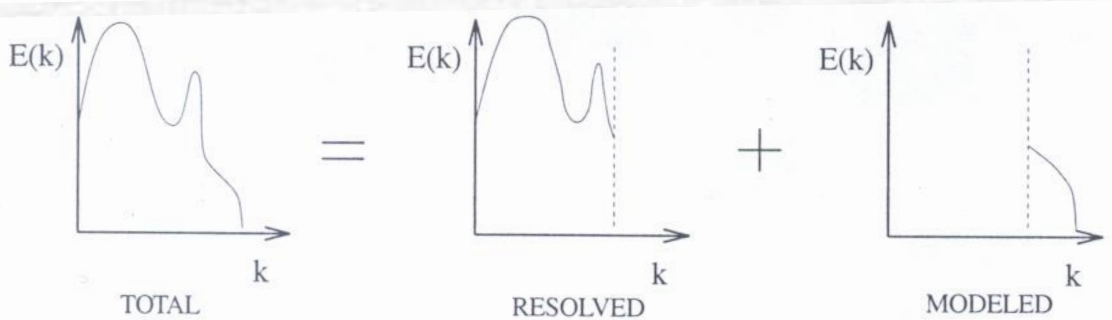


Fig. 1.3. Decomposition of the energy spectrum in the solution associated with large-eddy simulation (symbolic representation).

LES

Reynolds Averaging

- Time averaging: for stationary turbulence
- Spatial averaging: for homogenous turbulence
- Ensemble averaging: for any turbulence
- Phase averaging: for turbulence with periodic motion

RANS equations and unknowns

- RANS equation

$$\rho \frac{\partial U_i}{\partial t} + \rho U_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\mu S_{ji} - \rho \overline{u'_j u'_i}\right)$$

- RANS equation in conservative form

$$\rho \frac{\partial U_i}{\partial t} + \rho \frac{\partial}{\partial x_j}\left(U_j U_i + \overline{u'_j u'_i}\right) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\mu S_{ji}\right)$$

- Numbers of unknowns and equations
  - Unknowns: 10 = P (1) + U (3) + $\left(-\overline{u'_i u'_j}\right)$ (6)
  - Equations: 4 = Continuity (1) + Momentum (3)

The Reynolds-Stress Equation

- Derivation: Taking moments of the NS equation.

Multiply the NS equation by a fluctuating property and time average the product. Using this procedure, one can derive a differential equation for the Reynolds-stress tensor.

$$\frac{\partial \tau_{ij}}{\partial t}+U_k\frac{\partial \tau_{ij}}{\partial x_k}=-\tau_{ik}\frac{\partial U_j}{\partial x_k}-\tau_{jk}\frac{\partial U_i}{\partial x_k}+2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}$$

$$+\left(\overline{\frac{u_i'}{\rho}\frac{\partial p'}{\partial x_j}}+\overline{\frac{u_j'}{\rho}\frac{\partial p'}{\partial x_i}}\right)+\frac{\partial}{\partial x_k}\left(\nu\frac{\partial \tau_{ij}}{\partial x_k}+\overline{u_i'u_j'u_k'}\right)$$

- NEW Equations: 6 = 6 equation for the Reynolds stress tensor
- NEW Unknowns: 22 = 6 + 6 + 10

$$\left(\overline{\frac{u_i'}{\rho}\frac{\partial p'}{\partial x_j}}+\overline{\frac{u_j'}{\rho}\frac{\partial p'}{\partial x_i}}\right)\rightarrow 6 \quad unkowns$$

$$2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}\rightarrow 6 \quad unkowns$$

$$\overline{u_i'u_j'u_k'}\rightarrow 10 \quad unkowns$$

The closure problem of turbulence

- Because of the non-linearity of the Navier-Stokes equation, as we take higher and higher moments, we generate additional unknowns at each level.
- In essence, Reynolds averaging is a brutal simplification that loses much of the information contained in the Navier-Stokes equation.
- The function of turbulence modeling is to devise approximations for the unknown correlations in terms of flow properties that are known so that a sufficient number of equations exist.
- In making such approximations, we close the system.

Characteristics of Wall-Bound Turbulent Flows

- The turbulent boundary layer (zero-pressure gradient) has universal velocity distribution near the wall (inner-layer) (Clauser 1951)

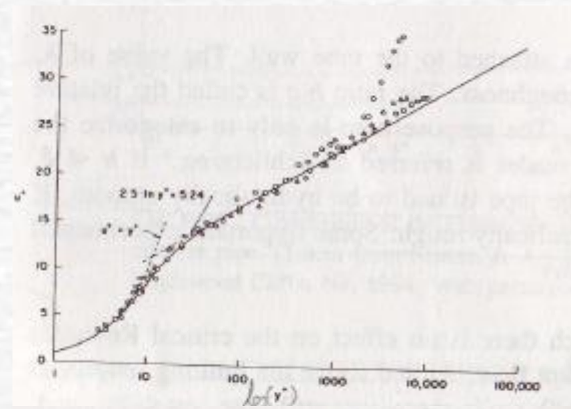


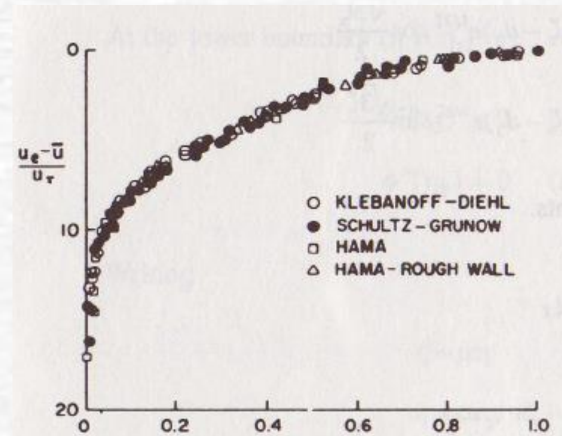
Fig. 6.11 Mean velocity distribution near a smooth wall. (Taken from Cebeci, T. and Smith, A. M. O., *Analysis of Turbulent Boundary Layers*, Academic Press, New York, 1974. With permission.)

$$\frac{u}{u_\tau} = u^+ = \frac{y u_\tau}{\nu} = y^+ \quad : y^+ < 5$$

$$u^+ = \frac{1}{0.4}\ln(y^+) + 5.1 \quad : 10^4 > y^+ \geq 30$$

$$\frac{k}{u^2_\tau} = 0.3 \quad : 10^4 > y^+ \geq 30$$

- The velocity defect when plotted vs. y/δ collapse on a single curve (outer-layer)



$$\frac{u_\infty - u}{u_\tau} = 9.6\left(1 - \frac{y}{\delta}\right)^2$$

Turbulent models (RANS)

- Boussinesq eddy-viscosity approximation
- Algebraic (zero-equation) models
  - Mixing length
  - Cebeci-Smith Model
  - Baldwin-Lomax Model
- One-equation models
  - Baldwin-Barth model
  - Spalart-Allmaras model
- Two-equation models
  - k-ε model
  - k-ω model
- Four-equation (v2f) models
- Reynolds-stress (seven-equation) models

Turbulent models (RANS)

- Boussinesq eddy-viscosity approximation

$$-\overline{u_i' u_j'} = \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3}\delta_{ij} k$$

- Dimensional analysis shows: $\nu_T = C_\mu q L$, where q is a turbulence velocity scale and L is a turbulence length scale. Usually $q = \sqrt{2k}$ where $k = \frac{1}{2}\overline{u_i' u_i'}$ is the turbulent kinetic energy. Models that do not provide a length scale are called incomplete.

Turbulent models (0-eqn RANS)

- Mixing length model: $\nu_T = l_{mix}^2 \left| \frac{dU}{dy} \right|$
- Assume $l_{mix} = \alpha\delta(x)$ for free shear flow, then
  - α=0.180 for far wake
  - α=0.071 for mixing layer
  - α=0.098 for plane jet
  - α=0.080 for round jet
- Comments:
  - Reliable only for free shear flows with different α values
  - Not applicable to wall-bounded flows

Turbulent models (0-eqn RANS)

Cebeci-Smith Model (Two-layer model)

$$\nu_T = \begin{cases} \nu_{T_i}, & y \le y_m \\ \nu_{T_o} & y > y_m \end{cases}$$

Where y_m is the smallest value of y for $\nu_{T_i} = \nu_{T_o}$

Inner layer: $\nu_{T_i} = l_{mix}^2 \left[\left(\frac{\partial U}{\partial y} \right)^2 + \left(\frac{\partial V}{\partial x} \right)^2 \right]^{1/2}$ $\quad l_{mix} = \kappa y \left[1 - e^{-y^+/A_0^+} \right]$

Outer layer: $\nu_{T_0} = \alpha U_e \delta_v^* F_{Kleb}(y;\delta)$

Closure coefficients: $\kappa = 0.40$ $\quad \alpha = 0.0168$ $\quad A^+ = 26\left[1 + y \frac{dP/dx}{\rho U_\tau^2} \right]^{-1/2}$

$$F_{Kleb}(y;\delta) = \left[1 + 5.5 \left(\frac{y}{\delta} \right)^6 \right]^{-1} \qquad \delta_v^* = \int_0^\delta (1 - U/U_e)\, dy$$

Comments:

Three key modifications to mixing length model;

Applicable to wall-bounded flows;

Applicable to 2D flows only;

Not reliable for separated flows;

δ, δ_v^*, U_e difficult to determine in some cases;

Turbulent models (0-eqn RANS)

Baldwin-Lomax Model (Two-layer model)

$$\nu_T = \begin{cases} \nu_{T_i}, & y \le y_m \\ \nu_{T_o} & y > y_m \end{cases}$$

Where y_m is the smallest value of y for $\nu_{T_i} = \nu_{T_o}$

Inner layer: $\nu_{T_i} = l_{mix}^2 |\omega| \qquad l_{mix} = \kappa y \left[1 - e^{-y^+/A_0^+} \right]$

Outer layer: $\nu_{T_0} = \alpha C_{cp} F_{wake} F_{Kleb} \left(y; y_{\max} / C_{Kleb} \right)$

Closure coefficients: $\kappa = 0.40 \quad \alpha = 0.0168 \quad A^+ = 26 \quad C_{cp} = 1.6 \quad C_{Kleb} = 0.3 \quad C_{wk} = 1$

$$F_{Kleb}\left(y; y_{\max}/C_{Kleb} \right) = \left[1 + 5.5 \left(\frac{y}{y_{\max}/C_{Kleb}} \right)^6 \right]^{-1} \qquad \omega = \left[\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \right)^2 + \left(\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z} \right)^2 + \left(\frac{\partial U}{\partial z} - \frac{\partial W}{\partial x} \right)^2 \right]^{1/2}$$

Comments:

Applicable to 3D flows;

Not reliable for separated flows;

No need to determine δ, δ_v^*, U_e

Turbulent models (1-eqn RANS)

Baldwin-Barth model

Kinematic eddy viscosity : $\nu_T = C_\mu \nu \tilde{R}_T D_1 D_2$

Turbulence Reynolds number :

$$\frac{\partial\left(\nu\tilde{R}_T\right)}{\partial t} + U_j \frac{\partial\left(\nu\tilde{R}_T\right)}{\partial x_j} = \left(C_{\varepsilon 2} f_2 - C_{\varepsilon 1}\right)\sqrt{\nu\tilde{R}_T P} + \left(\nu + \nu_T/\sigma_\varepsilon\right)\frac{\partial^2\left(\nu\tilde{R}_T\right)}{\partial x_k \partial x_k} - \frac{1}{\sigma_\varepsilon}\frac{\partial \nu_T}{\partial x_k}\frac{\partial\left(\nu\tilde{R}_T\right)}{\partial x_k}$$

Closure coefficients and auxiliary relations:

$$C_{\varepsilon 1} = 1.2 \quad C_{\varepsilon 2} = 2.0 \quad C_\mu = 0.09 \quad A_0^+ = 26 \quad A_2^+ = 10 \quad \frac{1}{\sigma_\varepsilon} = \left(C_{\varepsilon 1} - C_{\varepsilon 2}\right)\frac{\sqrt{C_\mu}}{\kappa^2} \quad \kappa = 0.41$$

$$P = \nu_T\left[\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - \frac{2}{3}\frac{\partial U_k}{\partial x_k}\frac{\partial U_k}{\partial x_k}\right] \quad D_1 = 1 - e^{-y^+/A_0^+} \quad D_2 = 1 - e^{-y^+/A_2^+}$$

$$f_2 = \frac{C_{\varepsilon 1}}{C_{\varepsilon 2}} + \left(1 - \frac{C_{\varepsilon 1}}{C_{\varepsilon 2}}\right)\left(\frac{1}{\kappa y^+} + D_1 D_2\right)\cdot\left[\sqrt{D_1 D_2} + \frac{1}{\sqrt{D_1 D_2}}\left(\frac{D_2}{A_0^+}e^{-y^+/A_0^+} + \frac{D_1}{A_2^+}e^{-y^+/A_2^+}\right)\right]$$

Turbulent models (1-eqn RANS)

Spalart-Allmaras model

Kinematic eddy viscosity : $\nu_T = \tilde{\nu} f_{v1}$

Eddy viscosity equation:

$$\frac{\partial \tilde{\nu}}{\partial t} + U_j \frac{\partial \tilde{\nu}}{\partial x_j} = c_{b1} \tilde{S} \tilde{\nu} - c_{w1} f_w \left(\frac{\tilde{\nu}}{d} \right)^2 + \frac{1}{\sigma} \frac{\partial}{\partial x_k} \left[\left(\nu + \tilde{\nu} \right) \frac{\partial \tilde{\nu}}{\partial x_k} \right] + \frac{c_{b1}}{\sigma} \frac{\partial \tilde{\nu}}{\partial x_k} \frac{\partial \tilde{\nu}}{\partial x_k}$$

Closure coefficients and auxiliary relations:

$$c_{b1} = 0.1355 \quad c_{b2} = 0.622 \quad c_{v1} = 7.1 \quad \sigma = 2/3 \quad c_{w1} = \frac{c_{b1}}{\kappa^2} + \frac{\left(1 + c_{b2}\right)}{\sigma}$$

$$c_{w2} = 0.3 \quad c_{w2} = 2.0 \quad \kappa = 0.41 \quad \chi = \frac{\tilde{\nu}}{\nu} \quad g = r + c_{w2}\left(r^6 - r\right) \quad r = \frac{\tilde{\nu}}{\tilde{S} \kappa^2 d^2}$$

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3} \quad f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}} \quad f_w = g \left[\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6} \right]^{1/6} \quad \tilde{S} = S + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{v2} \quad S = \sqrt{2 \Omega_{ij} \Omega_{ij}}$$

Turbulent models (1-eqn RANS)

Comments on one-equation models:

1.One-equation models based on turbulence kinetic energy are incomplete as they relate the turbulence length scales to some typical flow dimension. They are rarely used.

2.One-equation models based on an equation for the eddy viscosity are complete such as Baldwin-Barth model and Spalart-Allmaras model.

3.They circumvent the need to specify a dissipation length by expressing the decay, or dissipation, of the eddy viscosity in terms of spatial gradients.

4 Spalart-Allmaras model can predicts better results than Baldwin-Barth model, and much better results for separated flow than Baldwin-Barth model and algebraic models.

5 Also most of DES simulations are based on the Spalart-Allmaras model.

Turbulent models (2-eqn RANS)

k-ε model: $\nu_T = C_\mu k^2 / \varepsilon$

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = \tau_{ij} \frac{\partial U_i}{\partial x_j} - \varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \nu_T/\sigma_k\right) \frac{\partial k}{\partial x_j} \right]$$

$$\frac{\partial \varepsilon}{\partial t} + U_j \frac{\partial \varepsilon}{\partial x_j} = C_{\varepsilon 1} \frac{\varepsilon}{k} \tau_{ij} \frac{\partial U_i}{\partial x_j} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \nu_T/\sigma_\varepsilon\right) \frac{\partial \varepsilon}{\partial x_j} \right]$$

$$C_{\varepsilon 1} = 1.44 \quad C_{\varepsilon 2} = 1.92 \quad C_\mu = 0.09 \quad \sigma_k = 1.0 \quad \sigma_\varepsilon = 1.3 \quad \omega = \varepsilon / \left(C_\mu k\right) \quad l = C_\mu k^{3/2} / \varepsilon$$

k-ω model: $\nu_T = k/\omega$

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = \tau_{ij} \frac{\partial U_i}{\partial x_j} - \beta^* k\omega + \frac{\partial}{\partial x_j}\left[\left(\nu + \sigma^* \nu_T\right) \frac{\partial k}{\partial x_j} \right]$$

$$\frac{\partial \omega}{\partial t} + U_j \frac{\partial \omega}{\partial x_j} = \alpha \frac{\omega}{k} \tau_{ij} \frac{\partial U_i}{\partial x_j} - \beta \omega^2 + \frac{\partial}{\partial x_j}\left[\left(\nu + \sigma \nu_T\right) \frac{\partial \omega}{\partial x_j} \right]$$

$$\alpha = \frac{13}{25} \quad \beta = \beta_0 f_\beta \quad \beta^* = \beta_0^* f_{\beta^*} \quad \sigma = \frac{1}{2} \quad \sigma^* = \frac{1}{2} \quad \beta_0 = \frac{9}{125} \quad f_\beta = \frac{1 + 70\chi_\omega}{1 + 80\chi_\omega}$$

$$\chi_\omega = \left| \frac{\Omega_{ij}\Omega_{jk}S_{ki}}{\left(\beta_0^* \omega\right)^3} \right| \quad \beta_0^* = \frac{9}{100} \quad f_{\beta^*} = \begin{cases} 1, & \chi_k \le 0 \\ \dfrac{1 + 680\chi_k^2}{1 + 400\chi_k^2} & \chi_k > 0 \end{cases} \quad \chi_k = \frac{1}{\omega^3} \frac{\partial k}{\partial x_j} \frac{\partial \omega}{\partial x_j}$$

$$\varepsilon = \beta^* \omega k \qquad l = k^{1/2} / \omega$$

Turbulent models (2-eqn RANS)

Comments on two-equation models:

1. Two-equation models are complete;

2. k-ε and k-ω models are the most widely used two-equation models and a lot of versions exist. For example, a popular variant of k-ω model introduced by Menter has been used in our research code CFDSHIP-IOWA. There are also a lot of low-Reynolds-number versions with different damping functions.

3. k-ω model shows better results than k-ε model for flows with adverse pressure gradient and separated flows as well as better numerical stability.

Turbulent models (4-eqn RANS)

v2f-kε model: $\nu_T = C_\mu \overline{v^2} T$

$$\frac{D\overline{v^2}}{Dt} = kf - \varepsilon \frac{\overline{v^2}}{k} + \frac{\partial}{\partial x_j}\left[(\nu + \nu_t) \frac{\partial \overline{v^2}}{\partial x_j} \right] \qquad L^2 \nabla^2 f - f = \frac{C_1}{T}\left(\frac{\overline{v^2}}{k} - \frac{2}{3} \right) - C_2 \frac{P_k}{k}$$

$$\frac{Dk}{Dt} = P - \varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \qquad \frac{D\varepsilon}{Dt} = C_{\varepsilon 1} \frac{\varepsilon}{k} P - C_{\varepsilon 2} \frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_j} \right]$$

$$C_1 = 0.4 \quad C_2 = 0.3 \quad C_L = 0.3 \quad C_\eta = 70 \quad C_{\varepsilon 2} = 1.9 \quad \sigma_\varepsilon = 1.0$$

$$C_{\varepsilon 1} = 1.3 + 0.25 \Big/ \left[1 + \left(C_L d / 2L \right)^2 \right]^4 \qquad T = \max\left\{ \frac{k}{\varepsilon}, 6\sqrt{\frac{\nu}{\varepsilon}} \right\} \qquad L = \max\left\{ C_L \frac{k^{3/2}}{\varepsilon}, C_\eta \left(\frac{\nu^3}{\varepsilon} \right)^{1/4} \right\}$$

v2f-kω model: $\nu_T = C_\mu k^n \overline{v^2}^{1-n} \big/ \omega$

$$\frac{D\overline{v^2}}{Dt} = kf - 6\overline{v^2} \frac{\varepsilon}{k} + \frac{\partial}{\partial x_j}\left[(\nu + \nu_t) \frac{\partial \overline{v^2}}{\partial x_j} \right] \qquad L^2 \nabla^2 - f = \frac{1}{T}(C_1 - 1)\left(\frac{\overline{v^2}}{k} - \frac{2}{3} \right) - 5\frac{1}{T}\frac{\overline{v'^2}}{k} - C_2 \frac{P_k}{k}$$

$$\frac{Dk}{Dt} = \tau_{ij} \frac{\partial U_i}{\partial x_j} - \beta^* \omega k + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \qquad \frac{D\omega}{Dt} = \alpha \frac{\omega}{k} \tau_{ij} \frac{\partial U_i}{\partial x_j} - \beta \omega^2 \left(\frac{\overline{v^2}}{k} \right)^{1-n} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\omega} \right) \frac{\partial \omega}{\partial x_j} \right]$$

$$\alpha = 0.45977 \quad \beta^* = 0.09 \quad \beta = 3/40 \quad \sigma_k = 1.0 \quad \sigma_\omega = 1.5$$

$$T = \max\left\{ \frac{k}{\varepsilon}, 6\sqrt{\frac{\nu}{\varepsilon}} \right\} \qquad L = \max\left\{ C_L \frac{k^{3/2}}{\varepsilon}, C_\eta \left(\frac{\nu^3}{\varepsilon} \right)^{1/4} \right\}$$

Turbulent models (4-eqn RANS)

Comments on four-equation models:

1. For two-equation models a major problem is that it is hard to specify the proper conditions to be applied near walls.

2. Durbin suggested that the problem is that the Reynolds number is low near a wall and that the impermeability condition (zero normal velocity) is far more important. That is the motivation for the equation for the normal velocity fluctuation.

3. It was found that the model also required a damping function f, hence the name v2f model.

4. They appear to give improved results at essentially the same cost as the k-ε and k-ω models especially for separated flows. Hopefully the v2f-kω model can have better numerical stability than v2f-kε model as their counterparts behave in two-equation models.

Turbulent models (7-eqn RANS)

Some of the most noteworthy types of applications for which models based on the Boussinesq approximation fail are:

1. Flows with sudden changes in mean strain rate
2. Flow over curved surfaces
3. Flow in ducts with secondary motions
4. Flow in rotating fluids
5. Three-dimensional flows
6. Flows with boundary-layer separation

In Reynolds-stress models, the equations for the Reynolds stress tensor are modeled and solved along with the ε-equation:

$$\frac{\partial \tau_{ij}}{\partial t} + U_k \frac{\partial \tau_{ij}}{\partial x_k} = -\tau_{ik}\frac{\partial U_j}{\partial x_k} - \tau_{jk}\frac{\partial U_i}{\partial x_k} + 2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} + \left(\overline{\frac{u_i'}{\rho}\frac{\partial p'}{\partial x_j}} + \overline{\frac{u_j'}{\rho}\frac{\partial p'}{\partial x_i}} \right) + \frac{\partial}{\partial x_k}\left(\nu \frac{\partial \tau_{ij}}{\partial x_k} + \overline{u_i' u_j' u_k'} \right)$$

Turbulent models (7-eqn RANS)

These are some versions of Reynolds stress models:

1. LRR rapid pressure-strain model
2. Lumley pressure-strain model
3. SSG pressure-strain model
4. Wilcox stress- model

Comments on Reynolds stress models:

1.Reynolds stress models require the solution of seven additional PDEs and those equation are even harder to solve than the two-equation models.

2. Although Reynolds stress models have greater potential to represent turbulent flow more correctly, their success so far has been moderate.

3. There is a lot of current research in this field, and new models are often proposed. Which model is best for which kind of flow is not clear due to the fact that in many attempts to answer this question numerical errors were too large to allow clear conclusions to be reached.

Turbulent models (LES)

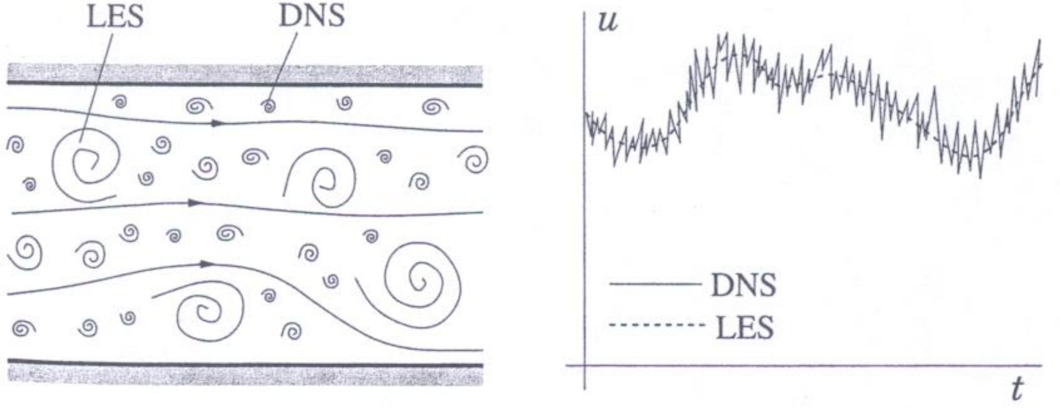


Fig. 9.3. Schematic representation of turbulent motion (left) and the time dependence of a velocity component at a point (right)

- Large scale motions are generally much more energetic than the small scale ones.
- The size and strength of large scale motions make them to be the most effective transporters of the conserved properties.
- LES treats the large eddies more exactly than the small ones may make sense
- LES is 3D, time dependent and expensive but much less costly than DNS.

Turbulent models (LES, filtering)

- LES needs a velocity field that contains only the large scale components of the total field, which is achieved by filtering the velocity field (Leonard, 1974)

$$\bar{u}_i(x) = \int G(x-\xi)u_i(\xi)d\xi$$

- $G(x-\xi)$ is the filter kernel, is a localized function, which includes a Gaussian, a box filter (a simple local average) and a cutoff (a filter which eliminates all Fourier coefficients belonging to wavenumbers above a cutoff)
- Each filter has a length scale associated with it, Δ.
- Eddies of size large than Δ are large eddies while those smaller than Δ are small eddies and need to be modeled.

$$G(x-\xi) = \begin{cases} \frac{1}{\bar{\Delta}} & \text{if } |x-\xi| \le \frac{\bar{\Delta}}{2} \\ 0 & \text{otherwise} \end{cases}$$

Box or top-hat filter

$$G(x-\xi) = \left(\frac{\gamma}{\pi\bar{\Delta}^2}\right)^{1/2} \exp\left(\frac{-\gamma|x-\xi|^2}{\bar{\Delta}^2}\right)$$

Gaussian filter

$$G(x-\xi) = \frac{\sin(k_c(x-\xi))}{k_c(x-\xi)}$$

$$k_c = \frac{\pi}{\Delta}$$

Cutoff filter

Turbulent models (LES, Governing Equations)

- Filtered Navier-Stokes equations (constant density, incompressible):

$$\frac{\partial(\rho \bar{u}_i)}{\partial x_i} = 0$$

$$\frac{\partial(\rho \bar{u}_i)}{\partial t} + \frac{\partial\left(\rho \overline{u_i u_j}\right)}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\right]$$

- Note that: $\overline{u_i u_j} \neq \bar{u}_i \bar{u}_j$

Introducing Subgrid-scale Reynolds Stress $\tau_{ij}^{S} = -\rho\left(\overline{u_i u_j} - \bar{u}_i \bar{u}_j\right)$

- The filter width Δ> grid size h
- The models used to approximate the SGS Reynolds stress are called subgrid-scale (SGS) or subfilter-scale models.

Turbulent models (LES, Smagorinsky model)

- The earliest and most commonly used subgrid scale model is one proposed by Smagorinsky (1963), which is an eddy viscosity model.
- As the increased transport and dissipation are due to the viscosity in laminar flow, it seems reasonable to assume that

$$\tau_{i,j}^{S} - \frac{1}{3}\tau_{kk}^{S}\delta_{i,j} = \mu_t\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) = 2\mu_t \bar{S}_{ij}$$

$\bar{S}_{ij}$ Strain rate of the large scale or resolved field

μ_t Eddy viscosity $\mu_t = C_S^2 \rho \Delta^2 \left|\bar{S}\right|$

$C_S \approx 0.2$ Model constant

- Drawbacks: 1. Cs is not constant
  2. Changes of Cs are required in all shear flows
  3. Need to be reduced near the wall.
  4. Not accurate for complex and/or higher Reynolds number flows.

Turbulent models (LES, Scale-similarity model and Dynamic model)

- Dynamic model:

  1. filtered LES solutions can be filtered again using a filter broader than the previous filter to obtain a very large scale field.
  2. An effective subgrid-scale field can be obtained by subtraction of the two fields.
  3. model parameter can then be computed.

  Advantages: 1. model parameter computed at every spatial grid point and every time step from the results of LES

  2. Self-consistent subgrid-scale model

  3. Automatically change the parameter near the wall and in shear flows

  Disadvantages: backscatter (eddy viscosity<0) may cause instability.

Turbulent models (DES)

- Massively separated flows at high *Re* usually involve both large and small scale vortical structures and very thin turbulent boundary layer near the wall
- RANS approaches are efficient inside the boundary layer but predict very excessive diffusion in the separated regions
- LES is accurate in the separated regions but is unaffordable for resolving thin near-wall turbulent boundary layers at industrial Reynolds numbers
- Motivation for DES: combination of LES and RANS. RANS inside attached boundary layer and LES in the separated regions

DES Formulation

- Modification to RANS models was straightforward by substituting the length scale d_w , which is the distance to the closest wall, with the new DES length scale, $\tilde{l}$ defined as:

$$\tilde{l} = \min(d_w, C_{DES}\Delta) \qquad \Delta = \max(\delta_x, \delta_y, \delta_z)$$

- where C_{DES} is the DES constant, Δ is the grid spacing and is based on the largest dimensions of the local grid cell, and $\delta_x, \delta_y, \delta_z$ are the grid spacing in x, y and z coordinates respectively
- Applying the above modification will result in S-A Based, standard k-ε (or k-ω) based and Menter's SST based DES models, etc.

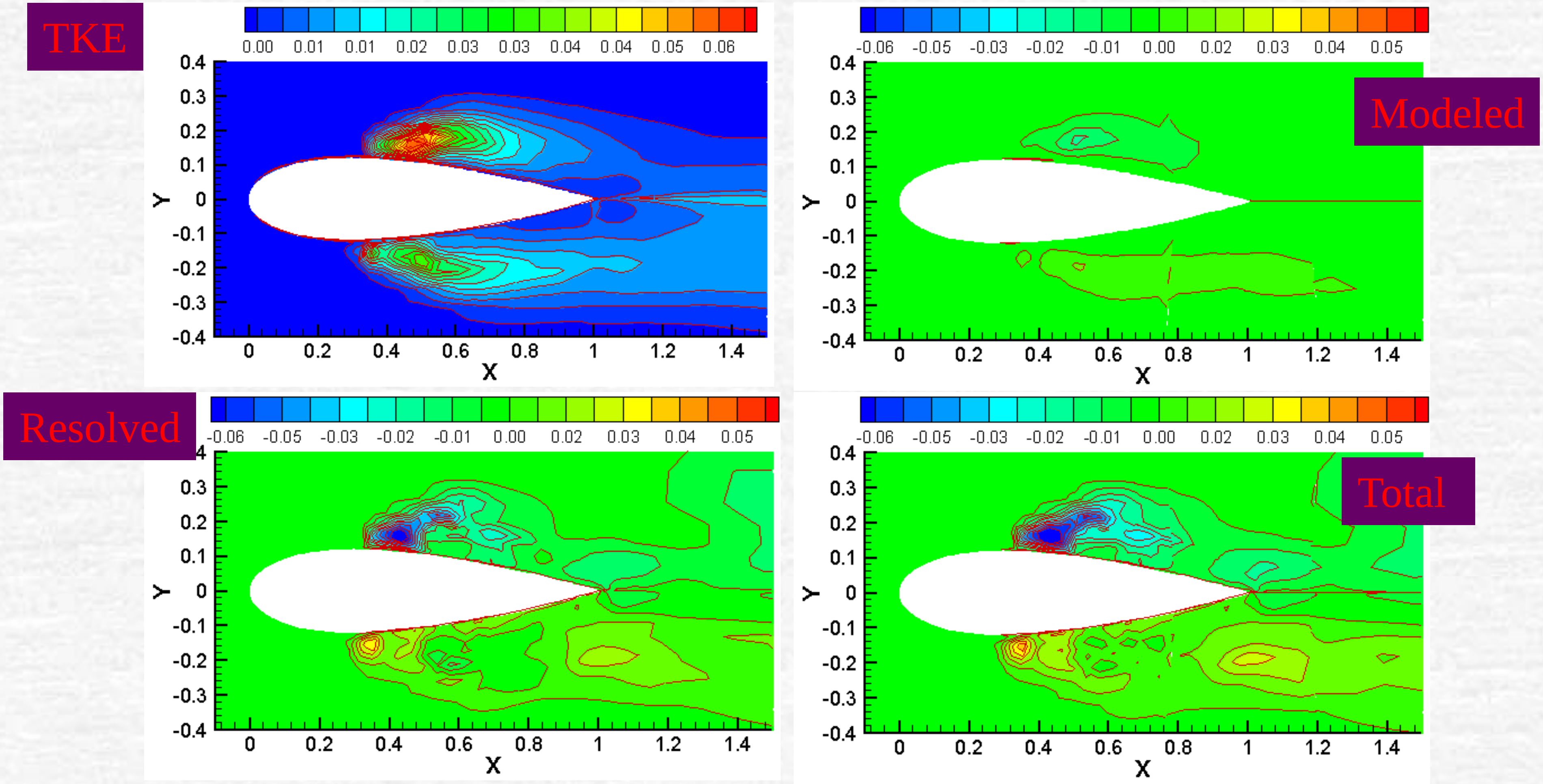
$$D^k_{RANS} = \rho\beta^* k\omega = \rho k^{3/2} / l_{k-\omega}$$

$$D^k_{DES} = \rho k^{3/2} \big/ \tilde{l}$$

$$l_{k-\omega} = k^{1/2} \big/ (\beta^* \omega)$$

$$\tilde{l} = \min(l_{k-\omega}, C_{DES}\Delta)$$

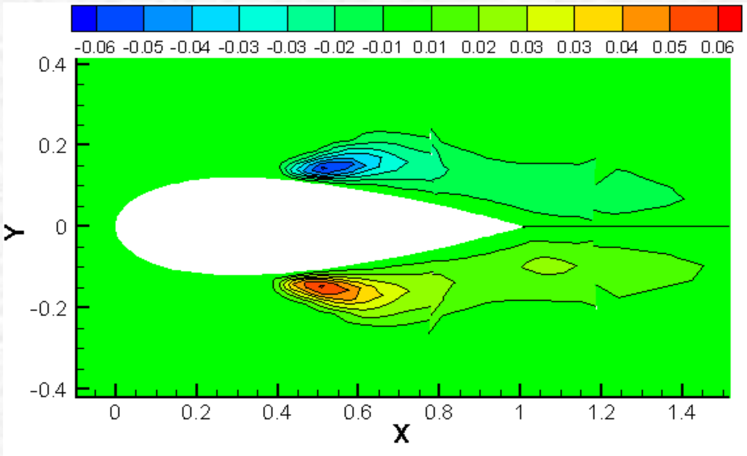
Resolved/Modeled/Total Reynolds stress (DES)



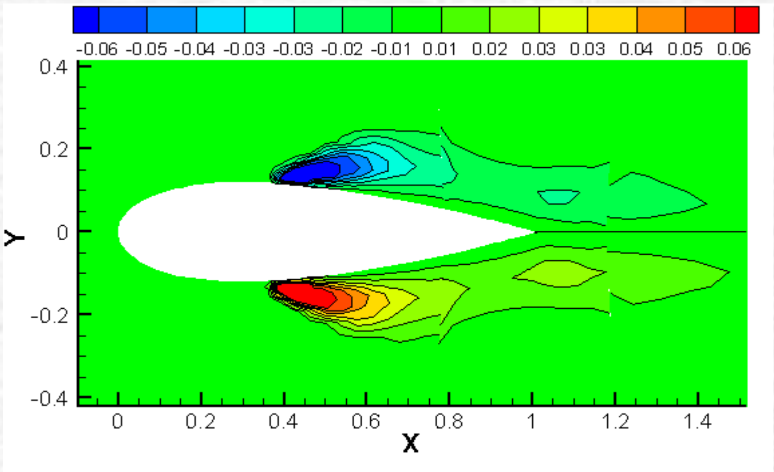
Modeled Reynolds stress: $\overline{u'v'} = -\nu_t \overline{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)}$

Resolved/Modeled/Total Reynolds stress (URANS)

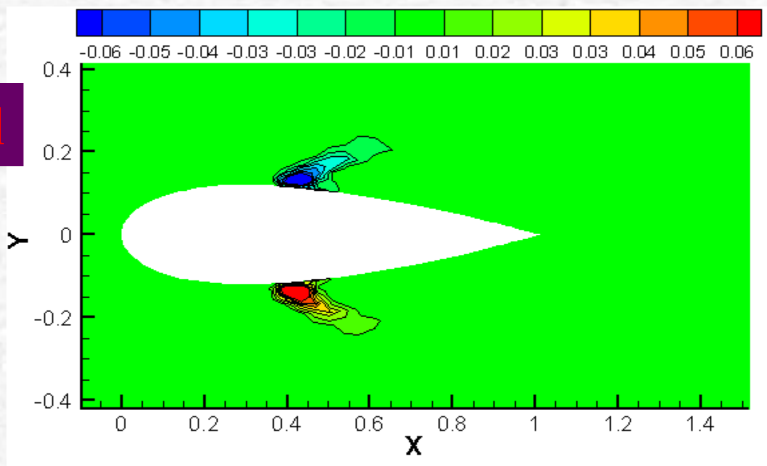
Modeled



Total



Resolved

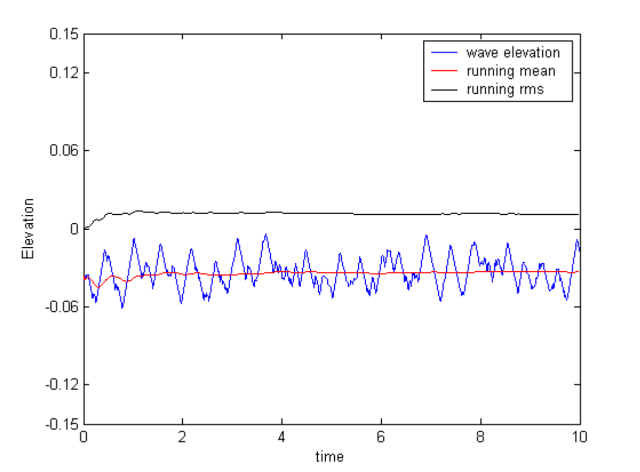


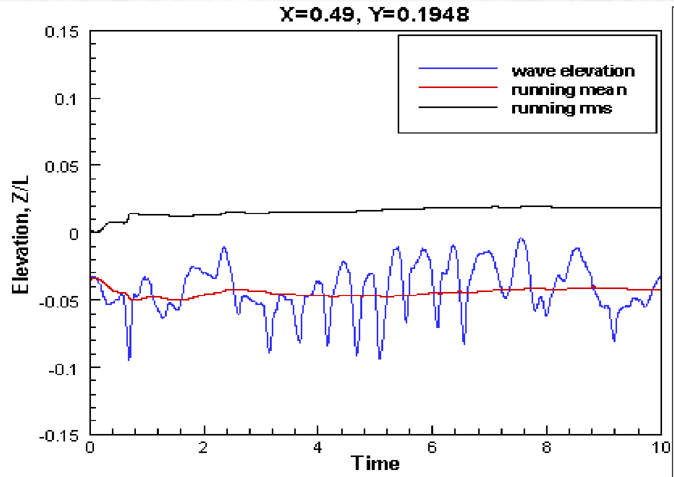
Deep insight into RANS/DES (Point oscillation on free surface with power spectral analysis)

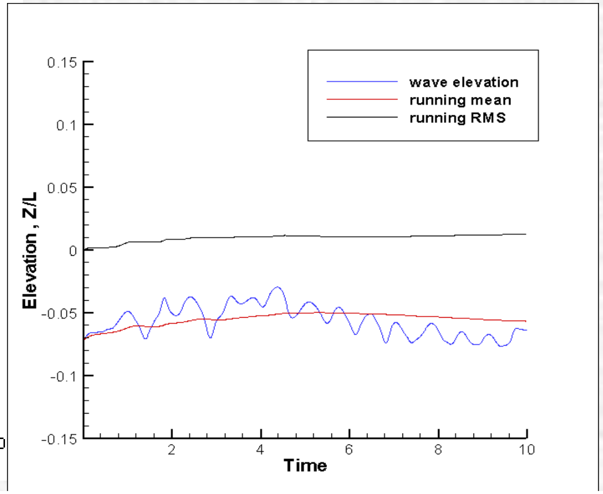
EFD

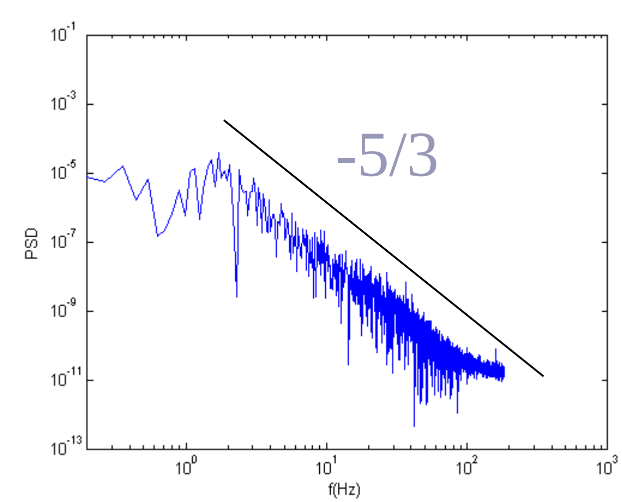
DES

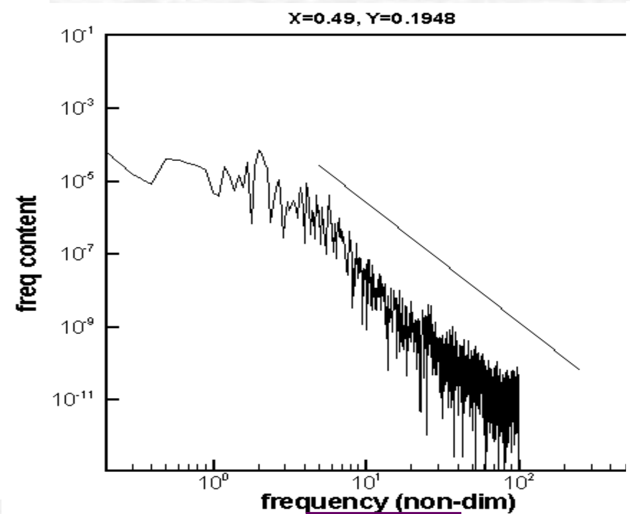
URANS

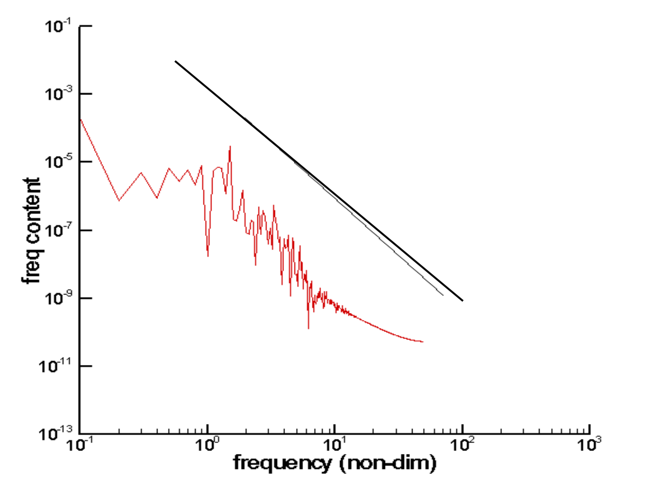












2HZ

Direct Numerical Simulation (DNS)

- DNS is to solve the Navier-Stokes equation directly without averaging or approximation other than numerical discretizations whose errors can be estimated (V&V) and controlled.
- The domain of DNS must be at least as large as the physical domain or the largest turbulent eddy (scale L)
- The size of the grid must be no larger than a viscously determined scale, Kolmogoroff scale, η
- The number of grid points in each direction must be at least L/ η
- The computational cost is proportional to

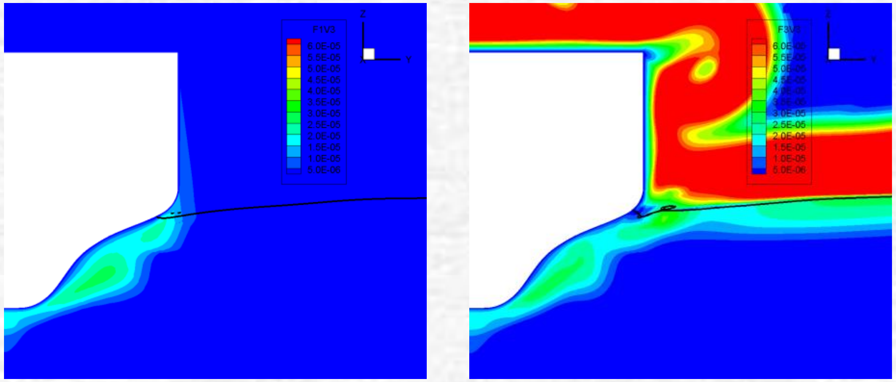
$$\mathrm{Re}_L^{3/4} \approx \left(0.01\mathrm{Re}\right)^{3/4}$$

- Provide detailed information on flow field
- Due to the computational cost, DNS is more likely to be a research tool, not a design tool.

Two-Phase Turbulence

- Most of the turbulence models (RANS and LES) are developed based on single-phase flows. The effect of turbulence on the interface and the interface induced turbulence are not considered.
- The eddy viscosity is found to be over-predicted in the gas area near the interface (Wang et. al, 2010).
- A buoyancy term G_b is added to the TKE

Equation (Devolder et al., 2017/2018):



Eddy viscosity for KCS wave breaking. Left: V4.5; right: V5.5.

- This additional term suppresses the turbulence level at the air-water interface

Examples (Diffuser)

- Asymmetric diffuser with separation is a good test case for turbulence models.
- A inlet channel was added at the diffuser inlet to generate fully developed velocity profile
- Boundary layer in the lower diffuser wall will separate due to the adverse pressure gradient.
- Results shown next include comparisons between V2f and k-ε
- LES simulation of this geometry can be found in:

M. Fatica, H. J. Kaltenbach, and R. Mittal, "Validation of LES in a Plain Asymmetric Diffuser", *center for turbulence research, annual research briefs, 1997*

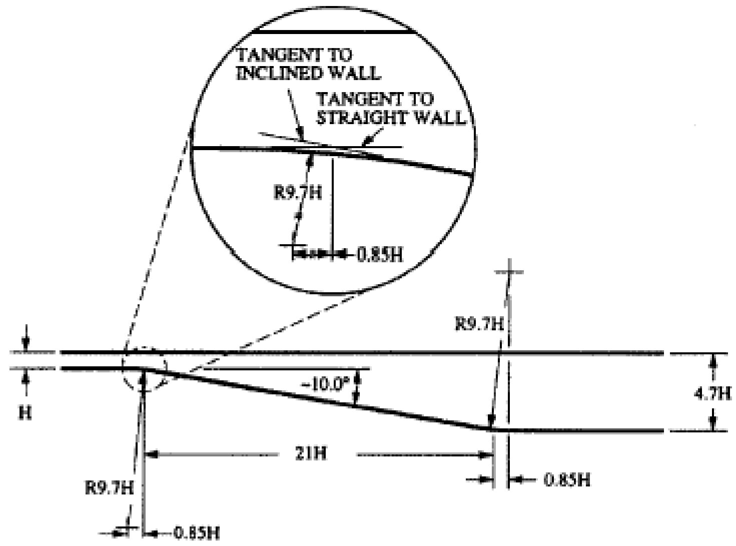


Fig. 1 Asymmetric diffuser geometry

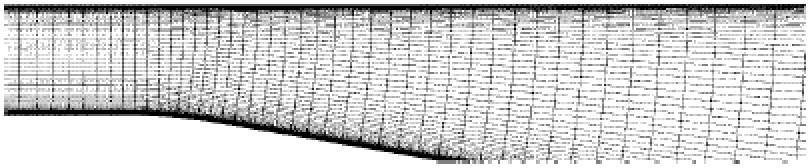
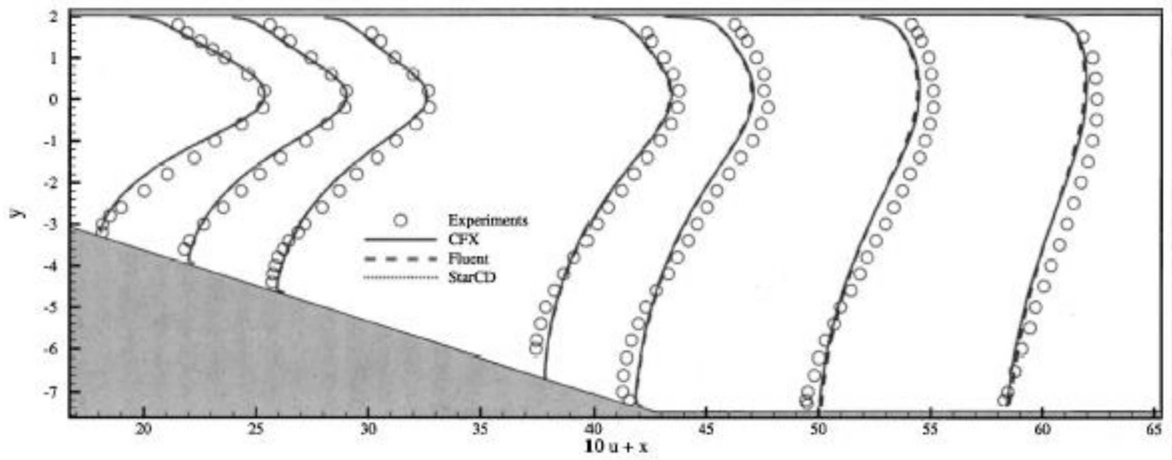


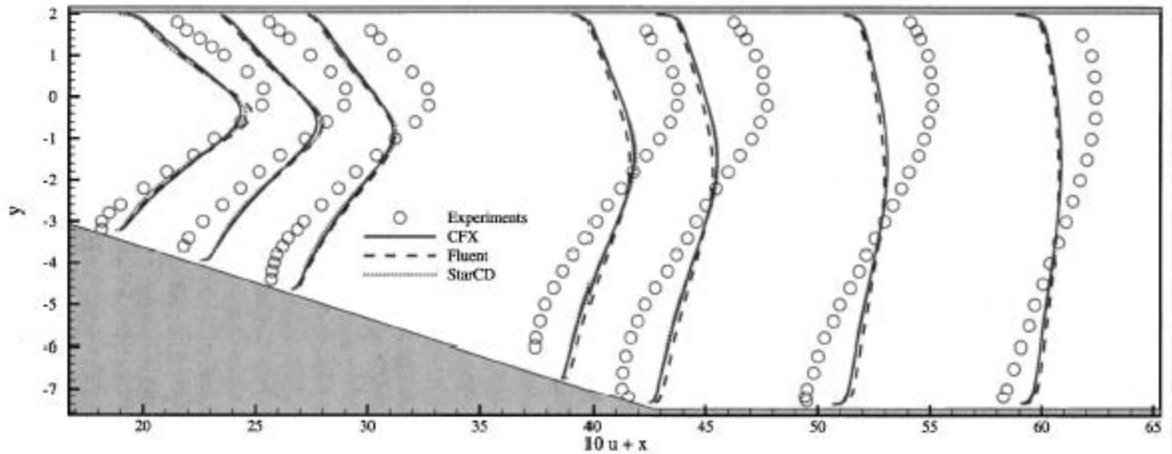
Fig. 2 Computational grid—detail of the channel-diffuser connection

Examples (Diffuser)

- Mean velocity predicted by V2f agreed very well with EFD data, particular the separation region is captured.
- K-ε model fails to predict the separation caused by adverse pressure gradient.

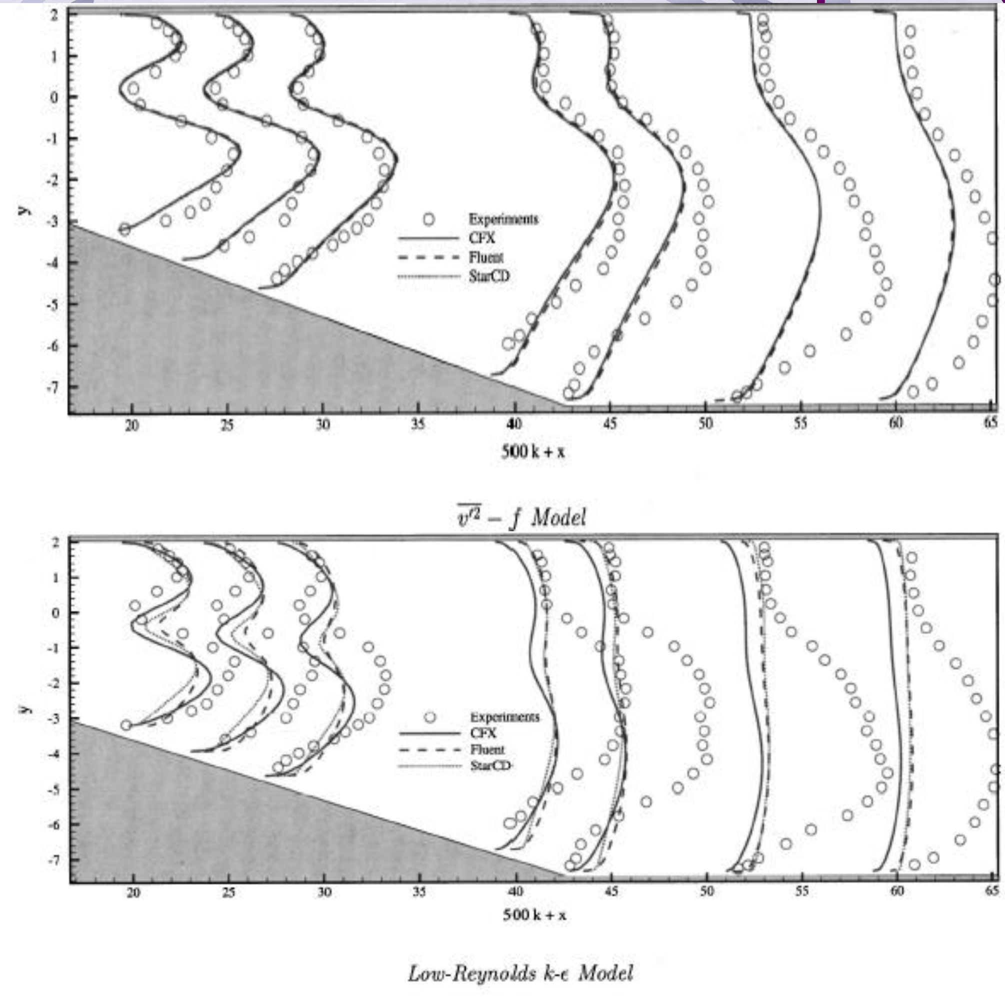


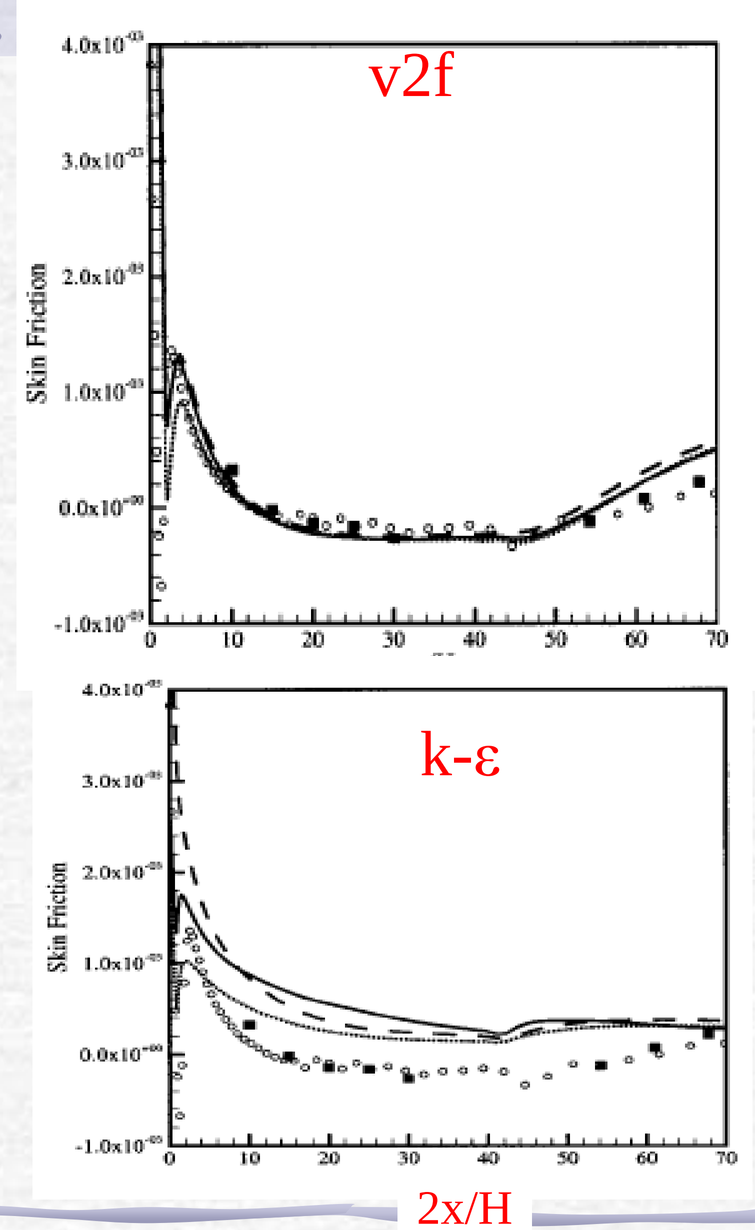
$\overline{v'^2} - f$ *Model*



Low-Reynolds k-ε Model

Examples (Diffuser)





- TKE predicted by V2f agreed better with EFD data than k- ε model, particular the asymmetric distribution.
- Right column is for the skin friction coefficient on the lower wall, from which the separation and reattachment point can be found.

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