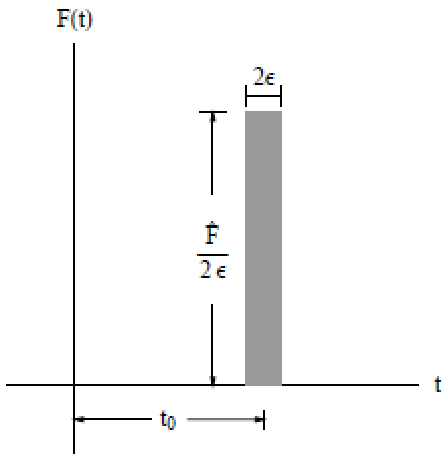
Lecture 14: General Forcing Functions (Optional)

Reading materials: Sections 5.1 and 5.2

1. Impulse function

- An impulse or shock loading is a force that is applied for a very short time.



$$F(t) = \begin{cases} 0; & t < t_0 - \epsilon \\ \frac{\hat{F}}{2\epsilon}; & t_0 - \epsilon < t < t_0 + \epsilon \\ 0; & t > t_0 + \epsilon \end{cases}$$

- Using dirac-delta function

$$F(t) = F_{\text{imp}}\,\delta(t - t_0)$$

where

$$\delta(t - t_0) = 0 \quad t \neq t_0$$

$$\int_{-\infty}^{\infty} \delta(t - t_0)\,\mathrm{dt} = 1 \qquad \int_{-\infty}^{\infty} \delta(t - t_0)\,g(t)\,\mathrm{dt} = g(t_0)$$

Equations of motion

$$m\,\ddot{y}(t) + c\,\dot{y}(t) + k\,y(t) = F_{\mathrm{imp}}\,\delta(t - t_0); \quad y(0) = 0 \quad \dot{y}(0) = 0$$

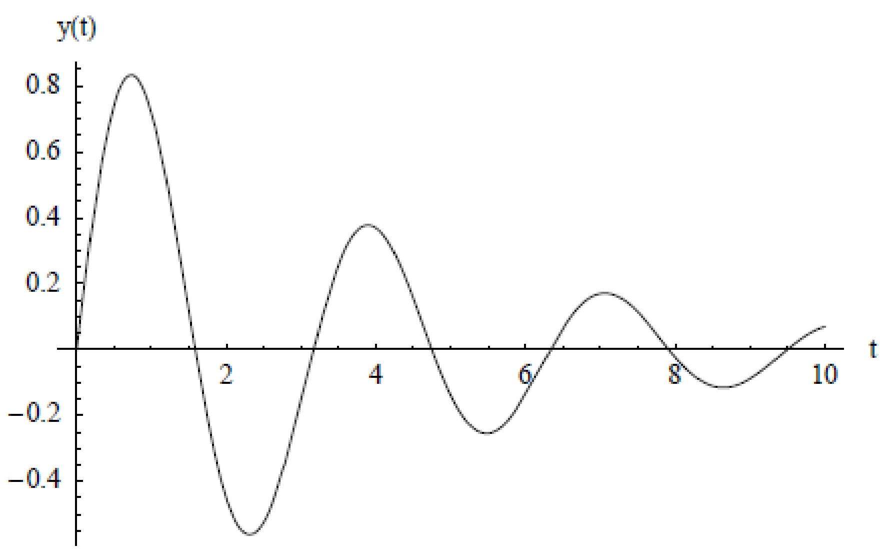
when
$t_0 = 0$

$$m\,\ddot{y}(t) + c\,\dot{y}(t) + k\,y(t) = 0; \quad y(0) = 0 \quad \dot{y}(0) = \frac{F_{\mathrm{imp}}}{m}$$

Solution

$$y(t) = e^{-\xi\,\omega\,t}\left(\frac{F_{\mathrm{imp}}/m}{\omega_d}\sin\omega_d\,t\right) = \frac{F_{\mathrm{imp}}}{m\,\omega_d}\,e^{-\xi\,\omega\,t}\sin\omega_d\,t$$

Example



2. Arbitrary function

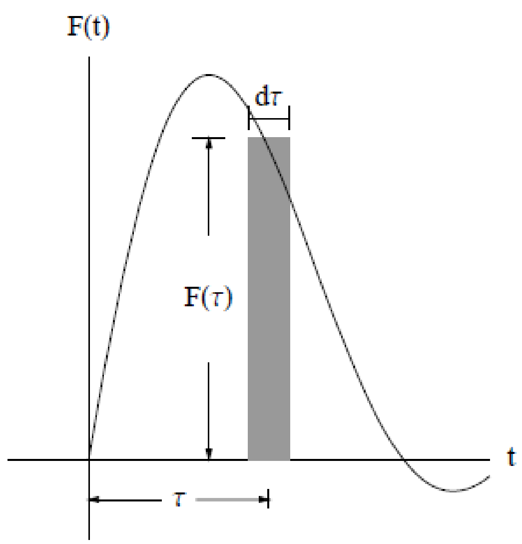
- For an arbitrary dynamic loading, the force can be thought of as infinite series of short duration pulses.

- The motion of a single degree of freedom system as a result of the impulse whose magnitude is $F(\tau)d\tau$ is

$$\mathrm{d}y = \frac{F(\tau)\,\mathrm{d}\tau}{m\,\omega_d}\, e^{-\xi\,\omega\,(t-\tau)} \sin \omega_d (t-\tau); \quad t \geq \tau$$

- The total response via superposition

$$y(t) = \int_0^t \frac{F(\tau)\,\mathrm{d}\tau}{m\,\omega_d}\, e^{-\xi\,\omega\,(t-\tau)} \sin \omega_d (t-\tau) = \frac{1}{m\,\omega_d} \int_0^t F(\tau)\, e^{-\xi\,\omega\,(t-\tau)} \sin \omega_d (t-\tau)\,\mathrm{d}\tau$$



- Ideal step function: The force is suddenly applied and then stays constant afterwards.

$$y(t) = \frac{1}{m\,\omega_d} \int_0^t F(\tau)\, e^{-\xi\,\omega\,(t-\tau)} \sin \omega_d (t-\tau)\,\mathrm{d}\tau = \frac{F}{m\,\omega_d} \int_0^t e^{-\xi\,\omega\,(t-\tau)} \sin \omega_d (t-\tau)\,\mathrm{d}\tau$$

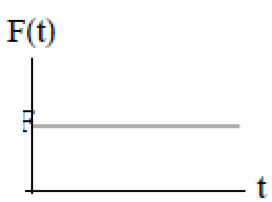
$$y(t) = \frac{F}{m\,\omega_d}\left(\frac{e^{-t\xi\omega}\left(\left(e^{t\xi\omega} - \cos(t\,\omega_d)\right)\omega_d - \xi\,\omega\sin(t\,\omega_d)\right)}{\xi^2\,\omega^2 + \omega_d^2}\right)$$

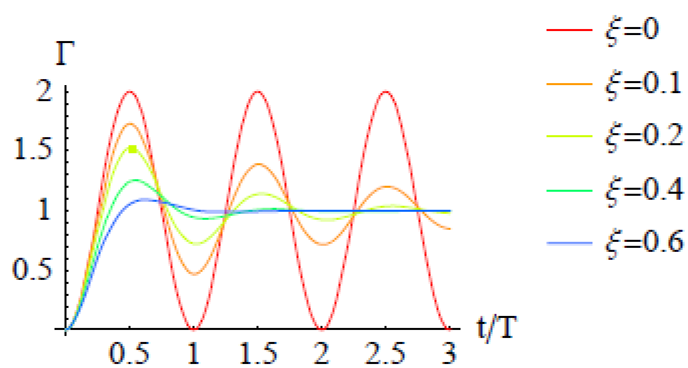
$$y(t) = \frac{F}{k}\left(-e^{-t\xi\omega}\cos\left(t\sqrt{1-\xi^2}\;\omega\right) - \frac{e^{-t\xi\omega}\,\xi\sin\left(t\sqrt{1-\xi^2}\;\omega\right)}{\sqrt{1-\xi^2}} + 1\right)$$

Dynamic load magnification factor

$$\Gamma(t) = -e^{-t\xi\omega}\cos\left(t\sqrt{1-\xi^2}\;\omega\right) - \frac{e^{-t\xi\omega}\,\xi\sin\left(t\sqrt{1-\xi^2}\;\omega\right)}{\sqrt{1-\xi^2}} + 1$$

$$\Gamma(t) = -e^{-\frac{2\pi t\xi}{T}}\cos\left(\frac{2\pi t\sqrt{1-\xi^2}}{T}\right) - \frac{e^{-\frac{2\pi t\xi}{T}}\,\xi\sin\left(\frac{2\pi t\sqrt{1-\xi^2}}{T}\right)}{\sqrt{1-\xi^2}} + 1$$





Rectangular pulse force: The force is suddenly applied and stays constant until t_d. Then the force is suddenly removed.

Forced vibration phase:

$$y(t) = \frac{1}{m\,\omega_d} \int_0^t F\, e^{-\xi\,\omega\,(t-\tau)} \sin \omega_d (t-\tau)\, \mathrm{d}\tau; \quad t \le t_d$$

Free vibration phase:

$$y(t) = e^{-\xi\,\omega\,(t-t_d)} \left(y_{t_d} \cos \omega_d (t-t_d) + \frac{\xi\,\omega\,u_{t_d} + v_{t_d}}{\omega_d} \sin \omega_d (t-t_d) \right); \quad t > t_d$$

(notice: y_{td} should be u_{td})

U_{td} and v_{td} are displacement and velocity at time t_d obtained from the forced vibration phase.

$$u_{t_d} = \frac{F}{k}\left(-e^{-\xi\,\omega\,t_d} \cos(t_d\,\omega_d) - \frac{e^{-\xi\,\omega\,t_d}\,\xi \sin(t_d\,\omega_d)}{\sqrt{1-\xi^2}} + 1\right);$$

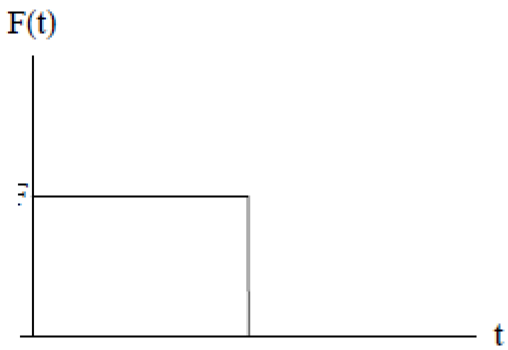
$$v_{t_d} = \frac{F}{k}\left(\frac{e^{-\xi\,\omega\,t_d}\left(\xi\,\omega\left(\sqrt{1-\xi^2}\,\cos(t_d\,\omega_d) + \xi \sin(t_d\,\omega_d)\right) + \left(\sqrt{1-\xi^2}\,\sin(t_d\,\omega_d) - \xi \cos(t_d\,\omega_d)\right)\omega_d\right)}{\sqrt{1-\xi^2}}\right)$$

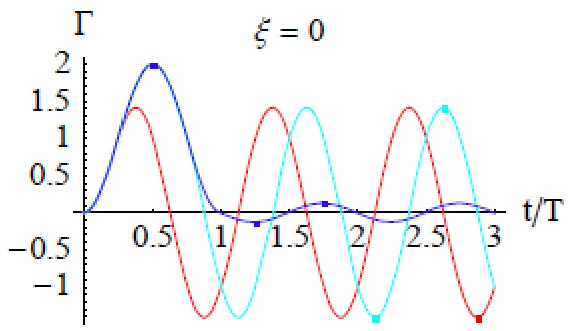
Therefore, the solution becomes

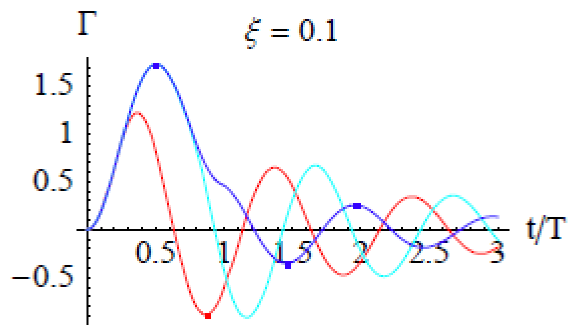
$$y(t) = \frac{F}{k}\left(-e^{-t\,\xi\,\omega} \cos(t\,\omega_d) - \frac{e^{-t\,\xi\,\omega}\,\xi \sin(t\,\omega_d)}{\sqrt{1-\xi^2}} + 1\right); \qquad 0 \le t \le t_d$$

$$y(t) = \frac{F}{k}\Bigg(\frac{1}{(\xi^2-1)\,\omega} e^{-t\xi\omega}\Bigg(\omega\Big(\cos((t-t_d)\,\omega_d)\Big(e^{\xi\omega t_d}(\xi^2-1) - \cos(t_d\,\omega_d)(\xi^2-1) + \xi\sqrt{1-\xi^2}\,\sin(t_d\,\omega_d)\Big) - e^{\xi\omega t_d}\,\xi\sqrt{1-\xi^2}\,\sin((t-t_d)\,\omega_d)\Big) + \sin\Big(\sqrt{1-\xi^2}\,\omega\,(t-t_d)\Big)\Big(\xi\cos\Big(\sqrt{1-\xi^2}\,\omega\,t_d\Big) - \sqrt{1-\xi^2}\,\sin\Big(\sqrt{1-\xi^2}\,\omega\,t_d\Big)\Big)\omega_d\Bigg)\Bigg);$$

$t > t_d$







Example

Triangular pulse force

Forced vibration phase:

$$y(t) = \frac{1}{m\omega}\int_0^t F(1-\tau/t_d)\,\sin\omega(t-\tau)\,d\tau; \quad t \le t_d$$

Free vibration phase:

$$y(t) = u_{t_d}\cos\omega(t-t_d) + \frac{v_{t_d}}{\omega}\sin\omega(t-t_d); \quad t > t_d$$

$$y(t) = \frac{F}{k}\left(-\cos(t\,\omega) + \frac{\sin(t\,\omega) - t\,\omega}{\omega\,t_d} + 1\right); \qquad 0 \le t \le t_d$$