Lecture 19: Continuous Systems

Reading materials: Sections 7.2, 7.3, and 7.1

1. Introduction

- Discrete systems vs Continuous systems

- We have so far dealt with discrete systems where mass, damping, and elasticity were assumed to be present only at certain discrete points in the system.

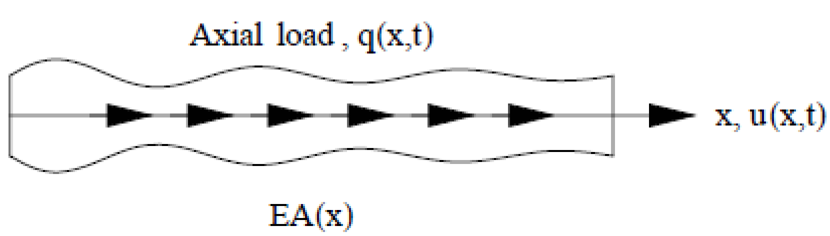
- In continuous systems it is not possible to identify discrete masses, damping, or springs. We must consider the continuous distribution of the mass, damping, and elasticity and assume that each of the infinite number of points of system can vibrate.

- If a system is modeled as a discrete one, the governing equations are ODEs.

- If a system is modeled as a continuous one, the governing equations are PDEs.

2. Axial Vibration of Bars

- Governing Equation



Initial conditions and boundary conditions

Free vibration solution for uniform bars

Free vibration of a fixed-free bar

- Natural frequencies of a bar carrying a mass

Vibrations of a bar subjected to initial force

3. Torsional vibration of a shaft or rod

Governing Equation

$$\frac{\partial}{\partial x}\left(G\,I_p\,\frac{\partial\phi}{\partial x}\right) = \rho\,I_p\,\ddot{\phi}$$

$$\left(\frac{G}{\rho}\right)\frac{\partial^2\phi(x,t)}{\partial x^2} = \frac{\partial^2\phi(x,t)}{\partial t^2}$$

Initial conditions and boundary conditions

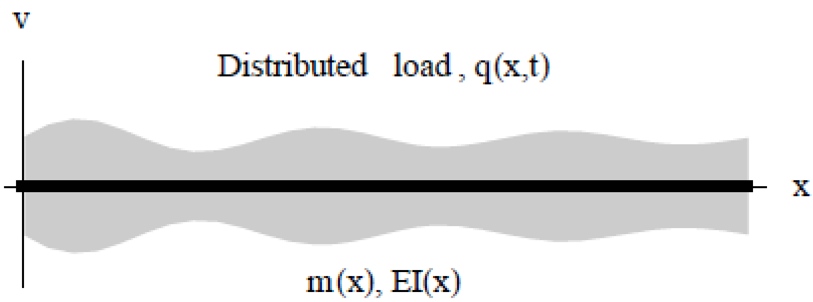
$$\phi(x,\,0) = \phi_0; \quad \dot{\phi}(x,\,0) = \dot{\phi}_0$$

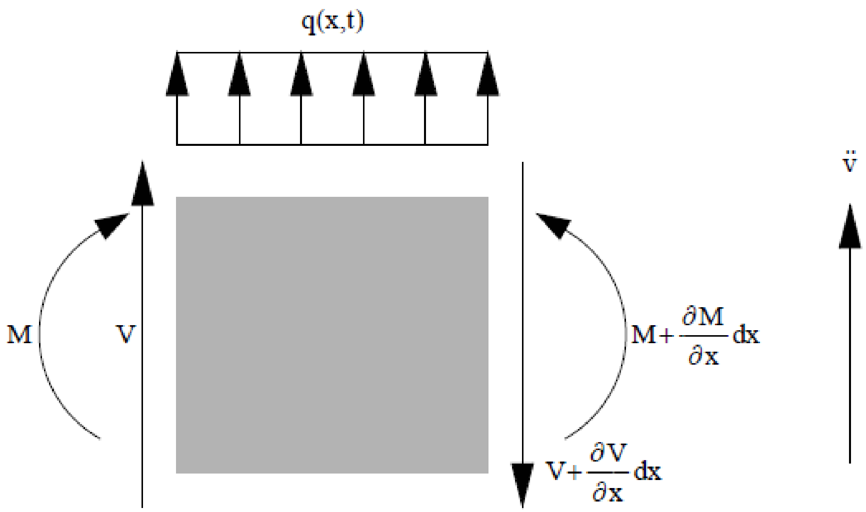
$$\phi(0,\,t) = 0; \quad G\,I_p\,\frac{\partial\phi(L,t)}{\partial x} = F(t)$$

Solution

4. Transverse vibration of beams

Governing Equation





$$m\,dx\,\ddot{v}(x,t) = q(x,t)\,dx + V - \left(V + \frac{\partial V}{\partial x}dx\right) \quad \Longrightarrow \quad m\,\ddot{v}(x,t) + \frac{\partial V}{\partial x} = q(x,t)$$

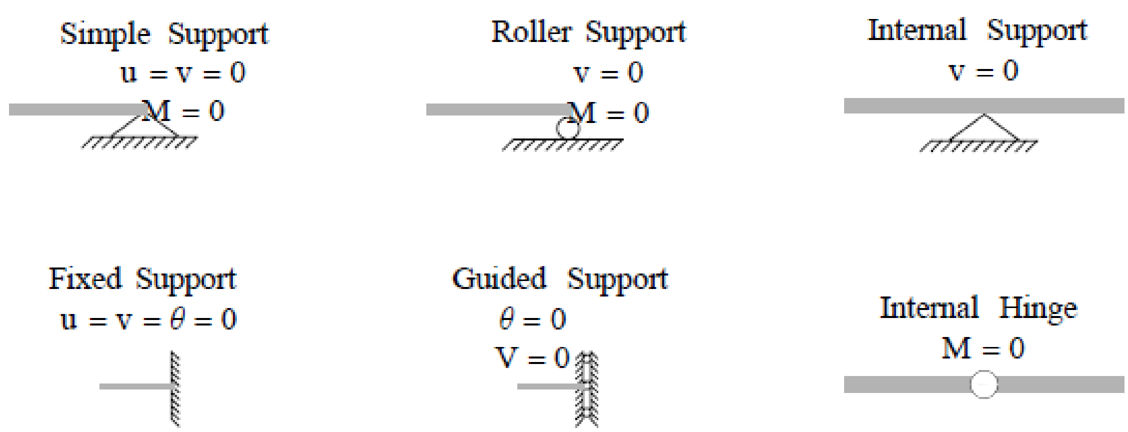
$$V = \frac{\partial M}{\partial x} = \frac{\partial}{\partial x}\left(EI\frac{\partial^2 v}{\partial x^2}\right)$$

$$m\,\ddot{v}(x,t) + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v}{\partial x^2}\right) = q(x,t)$$

Initial conditions and boundary conditions

$$v(x,\,0) = v_0; \quad \dot{v}(x,\,0) = \dot{v}_0$$

Specified quantity	*Left side of section at x_0*	*Right side of section at x_0*
Displacement	$v(x_0) = v_{x0}$	$v(x_0) = v_{x0}$
Rotation	$\theta(x_0) \equiv \dfrac{\partial v(x_0)}{\partial x} = \theta_{x0}$	$\theta(x_0) \equiv \dfrac{\partial v(x_0)}{\partial x} = \theta_{x0}$
Moment	$-E\,I\,\dfrac{\partial^2 v(x_0)}{\partial x^2} = M_{x0}$	$E\,I\,\dfrac{\partial^2 v(x_0)}{\partial x^2} = M_{x0}$
Shear	$E\,I\,\dfrac{\partial^3 v(x_0)}{\partial x^3} = F_{x0}$	$-E\,I\,\dfrac{\partial^3 v(x_0)}{\partial x^3} = F_{x0}$



Free vibration solution for uniform beams

Equation of motion

$$m\,\frac{\partial^2 v(x,t)}{\partial t^2} + E\,I\,\frac{\partial^4 v(x,t)}{\partial x^4} = 0$$

$$\frac{\partial^2 v(x,t)}{\partial t^2} + c^2\,\frac{\partial^4 v(x,t)}{\partial x^4} = 0 \qquad c = \sqrt{\mathrm{EI}/m}$$

Separation of variable method

$$v(x,\,t) = Y(x)\,f(t)$$

$$Y(x)\left(\frac{d^2 f(t)}{\mathrm{dt}^2}\right) + c^2\,f(t)\left(\frac{d^4\,Y(x)}{\mathrm{dx}^4}\right) = 0$$

$$c^2\,\frac{1}{Y(x)}\left(\frac{d^4\,Y(x)}{\mathrm{dx}^4}\right) = \frac{-1}{f(t)}\left(\frac{d^2 f(t)}{\mathrm{dt}^2}\right)$$

The original PDE becomes two ODEs

$$\frac{-1}{f(t)}\left(\frac{d^2 f(t)}{\mathrm{dt}^2}\right) = \omega^2 \quad \Longrightarrow \quad \frac{d^2 f(t)}{\mathrm{dt}^2} + \omega^2\,f(t) = 0$$

$$c^2\,\frac{1}{Y(x)}\left(\frac{d^4\,Y(x)}{\mathrm{dx}^4}\right) = \omega^2 \quad \Longrightarrow \quad \frac{d^4\,Y(x)}{\mathrm{dx}^4} - \lambda^4\,Y(x) = 0$$

$$\lambda^4 = \frac{\omega^2}{c^2} = \frac{m\,\omega^2}{E\,I} \quad \Longrightarrow \quad \omega = \lambda^2\sqrt{\frac{E\,I}{m}}$$

Solution for the first ODE

$$f(t) = A \sin \omega t + B \cos \omega t$$

Solution for the second ODE

$$Y(x) = C\, e^{\psi x}$$

$$\psi^4 C\, e^{\psi x} - \lambda^4 C\, e^{\psi x} = 0 \quad \Longrightarrow \quad \psi^4 = \lambda^4$$

$$\psi_1 = \lambda; \quad \psi_2 = -\lambda; \quad \psi_3 = i\,\lambda; \quad \psi_4 = -i\,\lambda$$

$$Y(x) = C_1\, e^{\lambda x} + C_2\, e^{-\lambda x} + C_3\, e^{i \lambda x} + C_3\, e^{-i \lambda x}$$

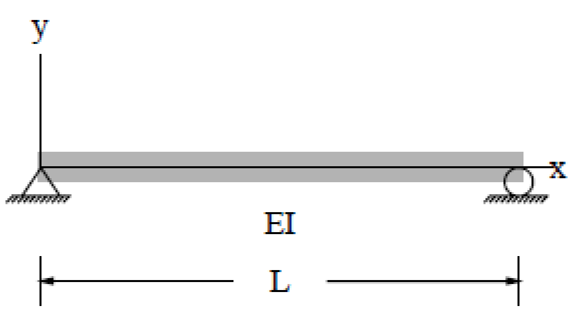
or

$$Y(x) = A_1 \cosh \lambda x + A_2 \sinh \lambda x + A_3 \cos \lambda x + A_4 \sin \lambda x$$

General solution of the original PDE

$$v(x, t) = Y(x)\, f(t) = (A_1 \cosh \lambda x + A_2 \sinh \lambda x + A_3 \cos \lambda x + A_4 \sin \lambda x)\,(A \sin \omega t + B \cos \omega t)$$

Free vibration solution for a simply supported beams



Boundary conditions

$$v(0,\,t) = 0; \quad E\,I\,\frac{\partial^2 v(0,,t)}{\partial x^2} = 0; \quad v(L,\,t) = 0; \quad E\,I\,\frac{\partial^2 v(L,t)}{\partial x^2} = 0$$

so

$$Y(0) = 0; \quad \frac{d^2\,Y(0)}{dx^2} = 0; \quad Y(L) = 0; \quad \frac{d^2\,Y(L)}{dx^2} = 0$$

Apply BCs

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ \lambda^2 & 0 & -\lambda^2 & 0 \\ \cosh(L\,\lambda) & \sinh(L\,\lambda) & \cos(L\,\lambda) & \sin(L\,\lambda) \\ \lambda^2\cosh(L\,\lambda) & \lambda^2\sinh(L\,\lambda) & -\lambda^2\cos(L\,\lambda) & -\lambda^2\sin(L\,\lambda) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

To obtain non-trivial solution, we have

$$-4\,\lambda^4\sin(L\,\lambda)\sinh(L\,\lambda) = 0$$

Therefore,

$$\lambda_n\,L = n\,\pi; \quad \Longrightarrow \quad \lambda_n = \frac{n\,\pi}{L}; \quad n = 1,\,2,\,3,\,\ldots$$

$$\omega_n = \lambda_n^2\sqrt{\frac{E\,I}{m}} = \left(\frac{n\,\pi}{L}\right)^2\sqrt{\frac{E\,I}{m}}$$

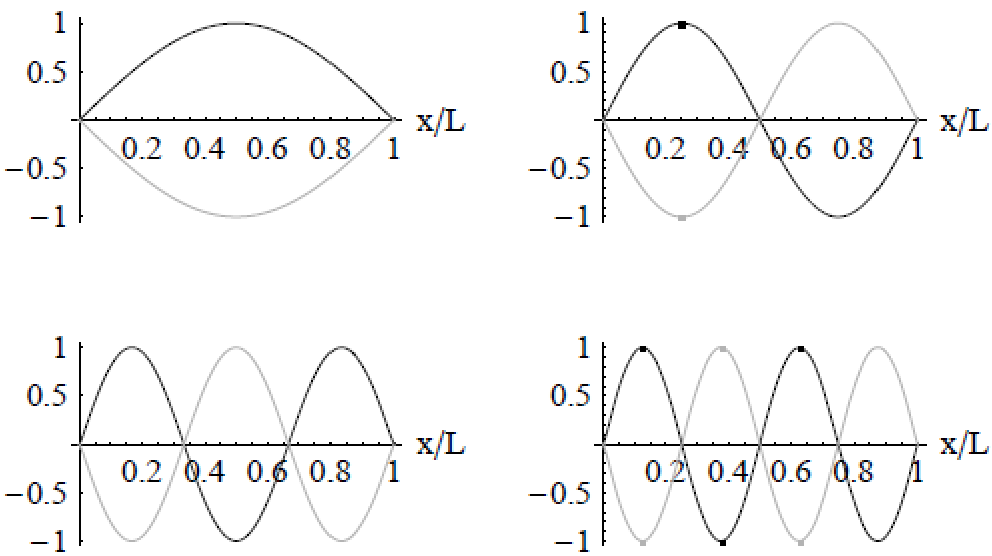
The coefficients are

$$\{A_2 \to -\operatorname{csch}(L\,\lambda)\sin(L\,\lambda),\, A_4 \to 1,\, A_1 \to 0,\, A_3 \to 0\}$$

and the mode shapes are

$$Y_n(\mathrm{x}) = \sin\left(\frac{n\,\pi\,x}{L}\right)$$

The first four mode shapes



Complete free vibration solution

$$v(x,\ t) = Y(x)\,f(t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\,\pi\,x}{L}\right)\left(A_n \sin\omega_n\,t + B_n \cos\omega_n\,t\right)$$

If the following initial conditions are given

$$v(x,\ 0) = \sin\left(\frac{\pi\,x}{L}\right); \quad \dot{v}(x,\ 0) = 0$$

We have

$$v(x,\ t) = \sin\left(\frac{\pi\,x}{L}\right)\cos\omega_1\,t$$

$$\omega_1 = \left(\frac{\pi}{L}\right)^2 \sqrt{\frac{E\,I}{m}}$$

5. Modal superposition approach (optional)

Orthogonality property of free vibration mode shapes

The free vibration mode shapes of continuous systems satisfy orthogonality condition.

$$\int_0^L \left(m\, Y_i\, Y_j \right) \mathrm{dx} = 0$$

$$\int_0^L \left(\mathrm{EI}\, \frac{d^2 Y_i}{\mathrm{dx}^2}\, \frac{d^2 Y_j}{\mathrm{dx}^2} \right) \mathrm{dx} = 0$$

Modal superposition

$$v(x,\, t) = \sum_{i=0}^{\infty} Y_i(x)\, z_i(t)$$

$$\sum_{i=0}^{\infty} m\, Y_i\, \frac{\partial^2 z_i}{\partial t^2} + E\, I\, \frac{\partial^4 Y_i}{\partial x^4}\, z_i = q(x,\, t)$$

$$\sum_{i=0}^{\infty} \int_0^L \left(m\, Y_i\, Y_j\, \frac{\partial^2 z_i}{\partial t^2} + E\, I\, \frac{\partial^4 Y_i}{\partial x^4}\, Y_j\, z_i \right) \mathrm{dx} = \int_0^L Y_j\, q(x,\, t)\, \mathrm{dx}$$

Modal equations

$$\int_0^L \left(m\, Y_n^2\, \frac{\partial^2 z_n}{\partial t^2} + E\, I\, \frac{\partial^4 Y_n}{\partial x^4}\, Y_n\, z_n \right) \mathrm{dx} = \int_0^L Y_n\, q(x,\, t)\, \mathrm{dx}; \quad n = 1,\, 2,\, \ldots$$

Modal mass: $M_n = \int_0^L m\, Y_n^2\, \mathrm{dx}$

Modal stiffness: $K_n = \int_0^L E\, I\, \frac{\partial^4 Y_n}{\partial x^4}\, Y_n\, \mathrm{dx} \equiv \int_0^L E\, I \left(\frac{\partial^2 Y_n}{\partial x^2} \right)^2 \mathrm{dx}$

Modal load: $F_n(t) = \int_0^L Y_n\, q(x,\, t)\, \mathrm{dx}$

$$M_n \ddot{z}_n + K_n z_n = F_n; \quad n = 1, 2, \ldots$$

since

$$\omega_n^2 = K_n / M_n$$

$$\ddot{z}_n + \omega_n^2 z_n = \frac{1}{M_n} F_n; \quad n = 1, 2, \ldots$$

$$v(x, t) = \sum_{n=0}^{\infty} Y_n(x) z_n(t)$$