Lecture 21: Energy Method (Con't)

Reading materials: Sections 8.4 and 8.5

1. Rayleigh's energy method for MDOF systems

- Discrete MDOF systems can be reduced to equivalent SDOF systems by using the Rayleigh's energy method. The assumption needed is regarding the deformation shape of the system.

- Equations of motion

$$\boldsymbol{m}\,\ddot{\boldsymbol{y}}(t) + \boldsymbol{k}\,\boldsymbol{y}(t) = \boldsymbol{0}$$

- Assume the system vibrates as the follows,

$$\boldsymbol{y}(t) = \boldsymbol{\psi}\,u(t)$$

then

$$\mathbf{m}\boldsymbol{\psi}\ddot{u}(t) + \mathbf{k}\boldsymbol{\psi}u(t) = 0$$

multiplying by $\boldsymbol{\psi}^T$

$$\overline{m}\,\ddot{u}(t) + \overline{k}\,u(t) = 0$$

where

$$\overline{m} = \boldsymbol{\psi}^T \boldsymbol{m}\,\boldsymbol{\psi} \quad \text{and} \quad \overline{k} = \boldsymbol{\psi}^T \boldsymbol{k}\,\boldsymbol{\psi}$$

- Natural frequency,

$$\omega = \sqrt{\frac{\bar{k}}{\bar{m}}} = \sqrt{\frac{\psi^T \mathbf{k} \psi}{\psi^T \mathbf{m} \psi}}$$

- Example,

$$\mathbf{m} = \begin{pmatrix} 0.05823 & 0 & 0 \\ 0 & 0.04658 & 0 \\ 0 & 0 & 0.03494 \end{pmatrix} \text{lb}-s^2/\text{in}; \qquad \mathbf{k} = \begin{pmatrix} 137.78 & -111.88 & 0 \\ -111.88 & 286.7 & -174.82 \\ 0 & -174.82 & 174.82 \end{pmatrix} \text{lb/in}$$

Assume the deformation shape is obtained from static deflection due to the weights.

$m_1 g = 22.5001$ lb; $m_2 g = 17.9985$ lb; $m_3 g = 13.5008$ lb

Load vector, $\mathbf{r} = \{22.5001, 17.9985, 13.5008\}$

Static deflection

$$\psi = \mathbf{k}^{-1}\mathbf{r} = \{2.08492, 2.36646, 2.44369\}$$

$$\bar{k} = \psi^T \mathbf{k} \psi = 122.495 \qquad \bar{m} = \psi^T \mathbf{m} \psi = 0.722623$$

$$\omega = \sqrt{\frac{\bar{k}}{\bar{m}}} = 13.0198 \text{ rad/s}$$

2. Rayleigh's energy method for continuous systems

- Take Beams as an example

- The kinetic energy of the beam

$$T = \frac{1}{2}\int_0^l \dot{y}^2 dm = \frac{1}{2}\int_0^l \dot{y}^2 \rho A(x) dx$$

- The maximum kinetic energy can be found by assuming a harmonic variation $y(x,t) = Y(x)\cos\omega t$

$$T_{\max} = \frac{\omega^2}{2}\int_0^l \rho A(x) Y^2(x) dx$$

- The potential energy of the beam is the same as the work done in deforming the beam. By neglecting the work done by the shear forces, we have

$$U = \frac{1}{2}\int_0^l M d\theta = \frac{1}{2}\int_0^l (EI\frac{\partial^2 y}{\partial x^2})\frac{\partial^2 y}{\partial x^2} dx$$

and

$$U_{\max} = \frac{1}{2}\int_0^l EI(x)(\frac{d^2 Y(x)}{\partial x^2})^2 dx$$

- By equating $T_{\max}$ to $U_{\max}$, we have

$$w^2 = \frac{\int_0^l EI(x)(\frac{d^2 Y(x)}{\partial x^2})^2 dx}{\int_0^l \rho A(x) Y^2(x) dx}$$

- Example: Fundamental frequency of a tapered beam

3. Rayleigh-Ritz method

- For continuous systems.

- The basis of the FEM.

- Rayleigh's method approximates a continuous system by an equivalent SDOF system via assuming a single deformation shape.

- In Rayleigh-Ritz method, a continuous system is reduced to a discrete MDOF system. The number of DOF is equal to the number of Ritz modes chosen.

- Considering the case of beam bending. The beam deflection is approximated by

$$y(x,\, t) = \sum_{i=1}^{n} \psi_i(x)\, u_i(t) \equiv \boldsymbol{\psi}^T \boldsymbol{u}$$

$$\frac{\mathrm{d}y(x,t)}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\mathrm{d}u_i}{\mathrm{d}t}\, \psi_i(x) \equiv \boldsymbol{\psi}^T \dot{\boldsymbol{u}}$$

$$\frac{d^2\, y(x,t)}{\mathrm{d}x^2} = \sum_{i=1}^{n} \frac{d^2\, \psi_i}{\mathrm{d}x^2}\, u_i(t) \equiv \boldsymbol{B}^T \boldsymbol{u}$$

where

$$\dot{\boldsymbol{u}} = \left(\frac{du_1}{dt} \quad \frac{du_2}{dt} \quad \ldots \quad \frac{du_n}{dt} \right)^T$$

$$\boldsymbol{B}^T = \left(\frac{d^2 \psi_1}{dx^2} \quad \frac{d^2 \psi_2}{dx^2} \quad \ldots \quad \frac{d^2 \psi_n}{dx^2} \right)$$

Other expressions

$$\left(\frac{dy(x,t)}{dt}\right)^2 = \left(\boldsymbol{\psi}^T \dot{\boldsymbol{u}}\right)^2 = \left(\boldsymbol{\psi}^T \dot{\boldsymbol{u}}\right)^T \boldsymbol{\psi}^T \dot{\boldsymbol{u}} \equiv \dot{\boldsymbol{u}}^T \boldsymbol{\psi}\, \boldsymbol{\psi}^T \dot{\boldsymbol{u}}$$

$$\left(\frac{d^2 y(x,t)}{dx^2}\right)^2 = \left(\boldsymbol{B}^T \boldsymbol{u}\right)^2 = \left(\boldsymbol{B}^T \boldsymbol{u}\right)^T \boldsymbol{B}^T \boldsymbol{u} \equiv \boldsymbol{u}^T \boldsymbol{B}\, \boldsymbol{B}^T \boldsymbol{u}$$

Potential due to strain energy energy

$$U_s = \frac{1}{2} \int_0^L \mathrm{EI} \left[\frac{d^2 y(x,t)}{dx^2} \right]^2 dx = \frac{1}{2} \boldsymbol{u}^T \int_0^L \mathrm{EI}\, \boldsymbol{B}\, \boldsymbol{B}^T dx\, \boldsymbol{u} \equiv \frac{1}{2} \boldsymbol{u}^T \boldsymbol{k}\, \boldsymbol{u}$$

Stiffness

$$\boldsymbol{k} = \int_0^L \mathrm{EI}\, \boldsymbol{B}\, \boldsymbol{B}^T dx$$

Work done by the external force

$$W = \int_0^L q(x,\, t)\, y(x,\, t)\, dx = \int_0^L q\, \boldsymbol{\psi}^T dx\, \boldsymbol{u} \equiv \boldsymbol{r}^T \boldsymbol{u}$$

where

$$\boldsymbol{r} = \int_0^L q\, \boldsymbol{\psi}\, dx$$

Kinetic energy

$$T = \frac{1}{2}\int_0^L m\left[\frac{\mathrm{dy}(x,t)}{\mathrm{dt}}\right]^2 \mathrm{dx} = \frac{1}{2}\,\dot{\boldsymbol{u}}^T \int_0^L m\,\psi\,\psi^T\,\mathrm{dx}\,\dot{\boldsymbol{u}} \equiv \frac{1}{2}\,\dot{\boldsymbol{u}}^T\,\boldsymbol{m}\,\dot{\boldsymbol{u}}$$

where

$$\boldsymbol{m} = \int_0^L m\,\psi\,\psi^T\,\mathrm{dx}$$

Equations of motion

$$U = U_s - W = \frac{1}{2}\,\boldsymbol{u}^T\,\boldsymbol{k}\,\boldsymbol{u} - \boldsymbol{r}^T\,\boldsymbol{u}$$

$$\frac{d}{\mathrm{dt}}(T+U) = 0 \quad \Longrightarrow \quad \frac{d}{\mathrm{dt}}\left[\frac{1}{2}\,\dot{\boldsymbol{u}}^T\,\boldsymbol{m}\,\dot{\boldsymbol{u}} + \frac{1}{2}\,\boldsymbol{u}^T\,\boldsymbol{k}\,\boldsymbol{u} - \boldsymbol{r}^T\,\boldsymbol{u}\right] = 0$$

$$\frac{d}{\mathrm{dt}}\left[\frac{1}{2}\,\dot{\boldsymbol{u}}^T\,\boldsymbol{m}\,\dot{\boldsymbol{u}}\right] = \dot{\boldsymbol{u}}^T\,\boldsymbol{m}\,\frac{d}{\mathrm{dt}}[\dot{\boldsymbol{u}}] = \dot{\boldsymbol{u}}^T\,\boldsymbol{m}\,\ddot{\boldsymbol{u}}$$

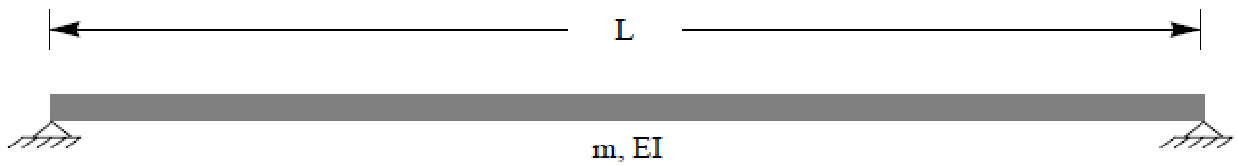
$$\frac{d}{\mathrm{dt}}\left[\frac{1}{2}\,\boldsymbol{u}^T\,\boldsymbol{k}\,\boldsymbol{u}\right] = \frac{d}{\mathrm{dt}}\left[\boldsymbol{u}^T\right]\boldsymbol{k}\,\boldsymbol{u} = \dot{\boldsymbol{u}}^T\,\boldsymbol{k}\,\boldsymbol{u}$$

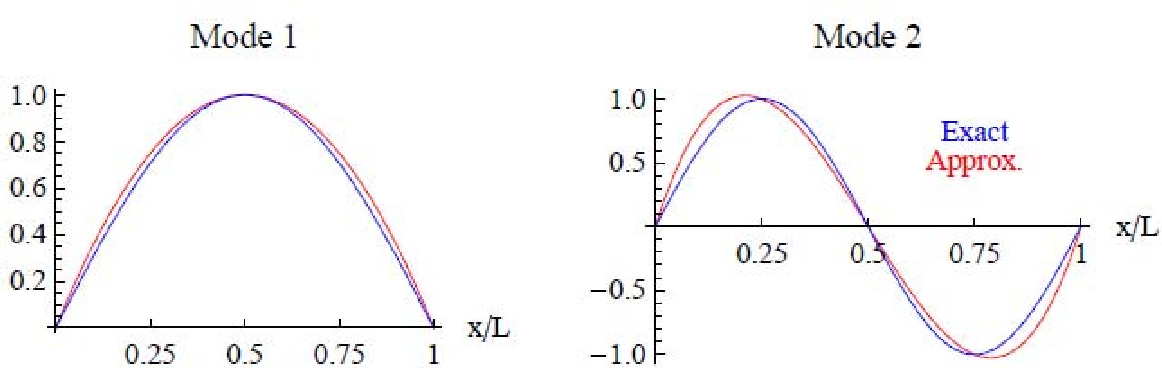
$$\frac{d}{\mathrm{dt}}\left[\boldsymbol{r}^T\,\boldsymbol{u}\right] = \boldsymbol{r}^T\,\frac{d}{\mathrm{dt}}[\boldsymbol{u}] = \boldsymbol{r}^T\,\dot{\boldsymbol{u}} \equiv \dot{\boldsymbol{u}}^T\,\boldsymbol{r}$$

$$\frac{d}{\mathrm{dt}}(T+U) = 0 \quad \Longrightarrow \quad \dot{\boldsymbol{u}}^T[\boldsymbol{m}\,\ddot{\boldsymbol{u}} + \boldsymbol{k}\,\boldsymbol{u} - \boldsymbol{r}] = 0$$

$$\boldsymbol{m}\,\ddot{\boldsymbol{u}}(t) + \boldsymbol{k}\,\boldsymbol{u}(t) = \boldsymbol{r}(t)$$

Example


L
m, EI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0.8
0.6
0.4
0.2
0.25
0.5
0.75
1
x/L
Mode 2
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0.5
−0.5
−1.0
0.25
0.5
0.75
1
x/L
Exact
Approx.