Lecture 5: damped Free Vibration (Optional)

Reading materials: Section 2.2

1. Introduction

Free vibration equations of motion of a damped multi-degree of freedom system are

2. Complex frequencies and mode shapes

For underdamped systems the roots of characteristic equation are complex conjugate pairs. Each pair gives one mode shape.

Example: Compute natural frequencies and mode shapes for a damped two DOF system.

$$\boldsymbol{m} = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}; \qquad \boldsymbol{c} = \begin{pmatrix} 2.7 & -0.3 \\ -0.3 & 1.3 \end{pmatrix}; \qquad \boldsymbol{k} = \begin{pmatrix} 27 & -3 \\ -3 & 3 \end{pmatrix}$$

$$\mathrm{Det}\left[\psi^2\,\boldsymbol{m} + \psi\,\boldsymbol{c} + \boldsymbol{k}\right] = 0 \Longrightarrow 9\,\psi^4 + 14.4\,\psi^3 + 57.42\,\psi^2 + 41.4\,\psi + 72 = 0$$

$$\psi_1 = -0.40408 - 1.48991\,i; \qquad \psi_1^* = -0.40408 + 1.48991\,i;$$
$$\psi_2 = -0.39592 - 1.78891\,i; \qquad \psi_2^* = -0.39592 + 1.78891\,i$$

$$\left[\psi_1^2\,\boldsymbol{m} + \psi_1\,\boldsymbol{c} + \boldsymbol{k}\right]\boldsymbol{\phi}^{(1)} = \boldsymbol{0} \qquad \boldsymbol{\phi}^{(1)} = \begin{pmatrix} 1 \\ 2.15117 + 2.70099\,i \end{pmatrix}$$

$$\boldsymbol{\phi}^{(1*)} = \begin{pmatrix} 1 \\ 2.15117 - 2.70099\,i \end{pmatrix}$$

$$\boldsymbol{\phi}^{(2)} = \begin{pmatrix} 1 \\ -0.984508 + 2.56501\,i \end{pmatrix} \qquad \boldsymbol{\phi}^{(2*)} = \begin{pmatrix} 1 \\ -0.984508 - 2.56501\,i \end{pmatrix}$$

3. Damped free vibration solution

Generally, the frequencies and mode shapes are complex conjugate pairs

$$\psi_m = \alpha_m + i\,\beta_m; \quad \psi_m^* = \alpha_m - i\,\beta_m; \quad m = 1, \ldots, n$$

$$\boldsymbol{\phi}^{(m)} = \boldsymbol{p}^{(m)} + i\,\boldsymbol{q}^{(m)}; \quad \boldsymbol{\phi}^{(m*)} = \boldsymbol{p}^{(m)} - i\,\boldsymbol{q}^{(m)}; \quad m = 1, \ldots, n$$

Then, the solution can be written as linear combination in terms of the roots of the complex frequency

$$\boldsymbol{y}(t) = \sum_{m=1}^{n} (a_m + i\,b_m)\, e^{(\alpha_m + i\,\beta_m)\,t}(\boldsymbol{p}^{(m)} + i\,\boldsymbol{q}^{(m)}) + (a_m - i\,b_m)\, e^{(\alpha_m - i\,\beta_m)\,t}(\boldsymbol{p}^{(m)} - i\,\boldsymbol{q}^{(m)})$$

There are 2n undertermined coefficients which can be dertermined via initial conditions. The solution can be expressed as the following using Euler's equations

$$\boldsymbol{y}(t) = \sum_{m=1}^{n} e^{\alpha_m\,t}[\boldsymbol{v}_m \sin(\beta_m\,t) + \boldsymbol{w}_m \cos(\beta_m\,t)]$$

$$\boldsymbol{v}_m = -2\,(a_m\,\boldsymbol{q}^{(m)} + b_m\,\boldsymbol{p}^{(m)})$$

$$\boldsymbol{w}_m = 2\,(a_m\,\boldsymbol{p}^{(m)} - b_m\,\boldsymbol{q}^{(m)})$$

Example: In the previous example, initial conditions are

$$y_1(0) = 1; \quad y_2(0) = 0; \quad \dot{y}_1(0) = 0; \quad \dot{y}_2(0) = 0$$

$$\psi_1 = -0.40408 - 1.48991\,i; \qquad \psi_1^* = -0.40408 + 1.48991\,i;$$
$$\psi_2 = -0.39592 - 1.78891\,i; \qquad \psi_2^* = -0.39592 + 1.78891\,i;$$

$$\alpha_1 = -0.40408; \qquad \beta_1 = -1.48991$$
$$\alpha_2 = -0.39592; \qquad \beta_2 = -1.78891$$

$$\boldsymbol{\phi}^{(1)}=\begin{pmatrix} 1 \\ 2.15117 + 2.70099\, i \end{pmatrix} \Longrightarrow \boldsymbol{p}^{(1)}=\begin{pmatrix} 1 \\ 2.15117 \end{pmatrix} \text{ and } \boldsymbol{q}^{(1)}=\begin{pmatrix} 0 \\ 2.70099 \end{pmatrix}$$

$$\boldsymbol{\phi}^{(2)}=\begin{pmatrix} 1 \\ -0.984508 + 2.56501\, i \end{pmatrix} \Longrightarrow \boldsymbol{p}^{(2)}=\begin{pmatrix} 1 \\ -0.984508 \end{pmatrix} \text{ and } \boldsymbol{q}^{(2)}=\begin{pmatrix} 0 \\ 2.56501 \end{pmatrix}$$

$$\begin{aligned} \boldsymbol{y}(t) = {} & 2\, e^{\alpha_1 t}\left[-\left(a_1\, \boldsymbol{q}^{(1)} + b_1\, \boldsymbol{p}^{(1)}\right) \sin(\beta_1 t) + \left(a_1\, \boldsymbol{p}^{(1)} - b_1\, \boldsymbol{q}^{(1)}\right) \cos(\beta_1 t)\right] + \\ & 2\, e^{\alpha_2 t}\left[-\left(a_2\, \boldsymbol{q}^{(2)} + b_2\, \boldsymbol{p}^{(2)}\right) \sin(\beta_2 t) + \left(a_2\, \boldsymbol{p}^{(2)} - b_2\, \boldsymbol{q}^{(2)}\right) \cos(\beta_2 t)\right] \end{aligned}$$

Based on initial conditions

$$\{a_1 \to 0.17148,\ b(1) \to -0.425705,\ a_2 \to 0.32852,\ b(2) \to 0.465994\}$$