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METHOD AND CODE FOR THE VISUALIZATION OF MULTIVARIATE SOLIDS

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ABSTRACT

This paper is devoted to a method and computer code for the automatic visualization of multivariate solids. Example of a multivariate solids arise in computer aided geometric design when a geometric entity is swept in space, where the totality of points touched by the entity is called the swept volume and is characterized by an equation of many parameters. The method and code are presented in an integrated manner and are aimed at providing the reader with a replicable computer algorithm. The formulation for is based on the implicit function theorem; is applicable to the visualization of solids of any number of parameters; and produces the exact boundary representation. Considering the solid as a manifold (possibly with boundaries), it is shown that further stratification of the various submanifolds yields varieties that can be depicted in $\mathbf{R}^3$. A measure of the computational complexity is presented to give the reader a sense of robustness of the method. The code is developed using a symbolic manipulator and is presented with a number of examples.

INTRODUCTION

A broadly applicable analytic formulation for the visualization of multivariate solids is presented that is based on a rigorous mathematical approach. The basic problem which this work addresses is the following: It is required to visualize a solid that is characterized by a single equation of three or more parameters parameters (e.g., a surface has two parameters, the sweep of a surface along a curve yields a solid of three parameters, the extrusion of the solid along an axis yield an equation of a solid parameterized in four parameters, etc.). Sweeping in this paper is defined as the motion of a geometric entity along a given path while specifying its

orientation. The objective is to find the new solid volume which represents all of the points in space which the object has occupied at some time during the motion. In this paper, we present a method that is comprehensive and that yields a formulation for visualizing the exact solid. The main challenge for this formulation has been the lack (and proclaimed difficulty) of computer implementation. Therefore, the motivation for this work is to demonstrate a simple step-by-step computer program that implements our proposed formulation. While the code represents thousands of lines of computer programs, it is presented in concise and replicable form using a symbolic manipulator.

The topic of multivariate solid visualization has not received considerable attention in the past because of the delicate mathematics associated with its underlying formulation. The need for implementing swept volumes in many disciplines such as manufacturing automation, is what has recently enticed various researchers to address this issue (see Kieffer and Litvin 1991, Abdel-Malek and Yeh 199b, Blackmore, et al. 1997).

This paper presents a mathematical method that employs symbolic manipulation to implement methods adapted from differential geometry and differential topology towards the visualization of multivariate solids. The underlying problem for all application areas addressed herein is characterized by the definition of a configuration space (a manifold, possibly with singularities) for a point in $\mathbf{R}^n$, the movement of which is constrained by a series of parameters. The term “manifold” will be used in a non-rigorous fashion for brevity of exposition (although it may have boundaries and singularities).

Simple sweep technology in commercial systems also exists but is limited by the type of swept entity and the number

of consecutive sweeps. Moreover, approximations are used to represent the resulting swept volume. Indeed, when consecutive sweeps are implemented, the error significantly increases because the result of each individual sweep operation is approximated. The approximation is necessary because of the limitation imposed by current formulations that allow only three parameters (two for the swept entity and one for the sweep path). The main goal of this research is to investigate means and methods for implementing this technology into commercial code in a fast and efficient manner.

We will first review prior work in the field. The background formulation will then be presented in Section 3 in an integrated and systematic manner taking into account both implicitly and parametrically defined geometric entities. The determination of all parts comprising the manifold (manifold stratification) will be presented in the context of determining all varieties of the submanifold (Sections 3-5). Computer implementation and computational complexity of the method will be presented in terms of the processing time required. Illustrative examples will be shown and future directions of this research will be discussed. Future directions of this research will be addressed in Section 12.

PRIOR RESEARCH

Trivariate B-Spline and Bezier solids have been treated by a number of researchers. Stanton, *et al.* (1977), Casale and Stanton (1985), and Farouki and Hinds (1985) all discuss the trivariate form, but avoid the equations of construction of the general boundary surfaces. Lasser (1985) discusses the general trivariate B-spline form, the generation of points in the volume and the generation of derivatives of these solids. Sederberg and Parry (1986) utilized the freeform trivariate B-spline solid for deformations. They embed an object in a deformable region of space defined by the trivariate solid such that each point of the object has a unique parameterization that defines its position in the region. The trivariate region is then altered by moving its control points. Using the trivariate form gives great flexibility to the definition of the deformable regions, and gives few parameters, which can be used to control the deformation. Rappaport, *et al.* (1996) have incorporated physical properties into freeform trivariate solids. Reus, *et al.* (1992) treated the trivariate tensor-product solid in a physical manner.

The most rigorous work pertaining to the determination of the boundary of swept volumes is the use of sweep differential techniques (Blackmore and Leu 1992, Blackmore, *et al.* 1992, Weld and Leu 1991, and Leu, *et al.* 1986). In this work, the sweep differential equation (called SDE) and the boundary flow methods are shown to be related to the Lie group theory structure of Euclidean motions. The extension of the sweep differential equation method is also used to determine deformed sweeps (Blackmore, *et al.* 1994). This work presents an elegant method for determining the swept volume with applications to robotics and to NC verification

(Blackmore, *et al.* 1997a and Leu, *et al.* 1997). A summary of this work presented by the authors can be found in Blackmore, *et al.* (1997b).

Other formulations for addressing the sweep generated by deformable objects was reported by Boussac and Crosnier (1996). Because of the importance of the NC verification problem, there have been many works addressing simulation technologies such as those reported by Menon and Voelker (1992), Liang, *et al.* (1997), Liu, *et al.* (1996), Jerard and Drysdale (1988), Voelker and Hunt (1985). Envelope theory has met with limited success at evaluating the swept volume as first presented by Wang and Wang (1988) and Sambandan and Wang (1989), and followed by many reports extending the theory. For example, Martin and Stephenson (1990) evaluated the swept volume of a three-dimensional object moving along an arbitrary path and have highlighted the implementation of the swept volume technique as an additional tool in Constructive Solid Geometry (CSG). To date, the envelope theory has not been shown to solve consecutive sweeping with more than three parameters, nor has it been extended to determine voids in the volume. Other reports that have addressed the sweeping of objects in space include Klok (1986), Ahn, *et al.* (1993), Bronsvort, *et al.* (1989), and Coquillart (1987), Ilies and Shapiro (1999), Juttler and Wagner (1996 and 1999).

Expanding the "domain of solid modeling using swept volumes" has also been asserted by researchers to be of great importance (Hu and Ling 1996 and Ganter *et al.* 1993). Learning and creating new methods to represent complex solid models is a goal that is of importance to many industries that implement CAD technology.

Kieffer and Litvin (1991) developed an algorithm for the determination of all surfaces and edges that may bound the swept volume. Computations of volumes swept by polyhedral objects in order to compute viewpoints for monitoring objects and features in an active robot work-cell is reported by Abrams and Allen (1995). Another type of mathematical approach is defined by Schroeder and Lorensen (1994) as the minimization of the sum of distances from the moving object (a polygon). The instantaneous screw theory, adopted from kinematics, is also used to generate a swept volume (Parida and Mudur 1994). More recently, sweep techniques were used to define boundaries of trivariate solids for visualization purposes (Joy and Duchaineau 1999 and Madrigal and Joy 1999).

In prior work, we have demonstrated (Abdel-Malek and Yeh 1997a, Abdel-Malek, *et al.* 1997) a general formulation for identifying singular surfaces that may exist on the boundary of a swept volume. The method was implemented to robotics analysis (Abdel-Malek and Yeh 1997b, Abdel-Malek, *et al.* 1999a) and to NC verification (Abdel-Malek, *et al.* 1999b). Recently, it was shown that a quadratic form on a surface characterizing the acceleration of a point in the volume identifies the boundary (Abdel-Malek, *et al.* 1998) of the volume in a systematic manner. Furthermore, a methodology

was developed using a formulation adapted from kinematics (called the Denavit-Hartenberg representation method) to accurately and systematically represent consecutive sweeps (Abdel-Malek and Othman 1999, Blackmore, *et al.* 1999, Abdel-Malek, *et al.* 1999c).

FORMULATION FOR IMPLICIT AND PARAMETRIC SOLIDS

The purpose of this section is to present the underlying formulation in an integrated manner characterizing the sweep of both implicitly and parametrically defined surfaces to generate solids, by means of defining the manifold with singularities, performing a manifold stratification, and determining all varieties and visualizing manifold strata. We present the complete formulation in a referenced manner pertinent to definitions and theorems adopted from Differential Geometry and Differential Topology (Spivak 1976, Guillemin and Pollack 1974, Lu 1976).

Consider a geometric entity parametrized in terms of one or more variables as a (3×1) vector given by $\Gamma(\mathbf{u})$, where $\mathbf{u} = [u_1, \dots, u_n]^T$ in n -dimensional space $\mathbf{R}^n$. This could be a curve, a surface or volume such as the sphere $\Gamma(\mathbf{u}) = [u_1 \cos u_2 \cos u_3 \quad u_1 \cos u_2 \sin u_3 \quad u_1 \sin u_2]^T$. In order to generalize the formulation, we consider boundaries imposed on $\Gamma(\mathbf{u})$ in the form of constraints on the parameters u_i characterized by inequalities in the form of $u_i^L \leq u_i \leq u_i^U$.

We will also consider surfaces implicitly defined on K where $\Gamma(\mathbf{u}) = [u_1 \quad u_2 \quad u_3]^T$ and where this surface will be represented by a number of inequalities, with no restriction on the number of inequalities nor on the type of variables. Let K be represented by

$$K := \left\{ \mathbf{u}: \ell_1^L \leq f_1(\mathbf{u}) \leq \ell_1^U, \dots, \ell_N^L \leq f_N(\mathbf{u}) \leq \ell_N^U, \right. \\ \left. u_1^L \leq u_1 \leq u_1^U, \dots, u_M^L \leq u_M \leq u_M^U \right\} \quad (1)$$

where $\mathbf{f} = [f_1, \dots, f_N]^T$ and $f_i(\mathbf{u})$ denotes an expression representing the surface; ℓ_i is a limit; the upper subscript L indicates the lower limit; U denotes the upper limit; N is the number of expressions; and M is the number of variables.

The sweep of a geometric entity is governed by an orientation matrix $\mathbf{R}$ and a path. Each path will be considered as a second geometric entity parametrized in terms of one or more variables as a (3×1) vector $\Psi_1(v_1)$. This entity also has a boundary defined by $v_1^L \leq v_1 \leq v_1^U$. The manifold (possibly with corners, generated by the sweep of $\Gamma(u_1, \dots, u_n)$ on $\Psi_1(v_1)$) is defined by the vector

$$\mathbf{N}_1(\mathbf{q}) = \mathbf{R}_1(v_1)\Gamma(\mathbf{u}) + \Psi_1(v_1) \quad (2)$$

where $\mathbf{q}$ is a subset of the Euclidean defined by $\mathbf{q} = [q_1 \quad \dots \quad q_n]^T = [u_1 \quad \dots \quad u_n \quad v_1]^T$ and $\mathbf{R}_1(v_1)$ is the (3×3) rotation matrix defining the orientation of the swept entity. In fact, $\mathbf{N}_1(\mathbf{q})$ characterizes the set of all points that belong to the manifold. Another sweep motion yields an expanded swept volume in the form of

$$\mathbf{N}_2(\mathbf{q}) = \mathbf{R}_2(v_2)\mathbf{N}_1(\mathbf{q}) + \Psi_2(v_2) = \mathbf{R}_2\mathbf{R}_1\Gamma + \mathbf{R}_2\Psi_1 + \Psi_2 \quad (3)$$

where now $\mathbf{q} = [u_1 \quad \dots \quad u_n \quad v_1 \quad v_2]^T$. The generalized case yields a space characterized by the vector function ξ , such that

$$\xi(\mathbf{q}) = \prod_{i=1}^{n+m} \mathbf{R}_i \Gamma + \sum_{j=1}^{n+m} \left(\prod_{i=j+1}^{n+m-1} [\mathbf{R}_i] \Psi_j \right) + \Psi_{n+m} \quad (4)$$

where $\mathbf{q} = [\mathbf{u}^T \quad \mathbf{v}^T]^T$, $\mathbf{u} = [u_1 \quad \dots \quad u_n]^T$, and $\mathbf{v} = [v_1 \quad \dots \quad v_m]^T$. Note that $\xi(\mathbf{q})$ is defined on the Euclidean space $\mathbf{R}^{n+m}$ where $\xi: \mathbf{R}^3 \rightarrow \mathbf{R}^{n+m}$, which is a Manifold possibly with corners. To impose inequality constraints, it was shown (Abdel-Malek and Yeh 1997a) that a constraint of the form $q_i^{\min} \leq q_i \leq q_i^{\max}$ can be transformed into an equation by introducing a new set of generalized coordinates λ_i such that

$$q_i = a_i + b_i \sin \lambda_i; \quad i = 1, \dots, n+m \quad (5)$$

where $a_i = (q_i^{\max} + q_i^{\min})/2$ and $b_i = (q_i^{\max} - q_i^{\min})/2$ are the mid-point and half-range, respectively. At a point in the manifold, a vector constraint function $\Phi(\mathbf{q}^*)$ with parametrized inequality constraints is defined as

$$\Phi(\mathbf{q}^*) = \begin{bmatrix} \xi(\mathbf{q}) \\ u_i - 0.5(u_i^U + u_i^L) - 0.5(u_i^U - u_i^L) \sin \lambda_i \\ v_j - 0.5(v_j^U + v_j^L) - 0.5(v_j^U - v_j^L) \sin \lambda_{n+j} \end{bmatrix} \quad (6)$$

where $i = 1, \dots, n$ and $j = 1, \dots, m$ and where $\mathbf{q}^* = [\mathbf{u}^T \quad \mathbf{v}^T \quad \lambda^T]^T = [q_1 \quad \dots \quad q_{2n+2m}]^T$ is the vector of all generalized coordinates and $\lambda = [\lambda_1, \dots, \lambda_{n+m}]^T$ is the vector of new generalized coordinates. The vector function $\Phi(\mathbf{q}^*)$ represents every point in the swept volume.

For the implicit formulation, the constraint is defined by

$$\Phi(\mathbf{q}^*) = \begin{bmatrix} \xi(\mathbf{q}) \\ f_i(\mathbf{u}) - 0.5(\ell_i^U + \ell_i^L) - 0.5(\ell_i^U - \ell_i^L) \sin \lambda_i \\ u_j - 0.5(u_j^U + u_j^L) - 0.5(u_j^U - u_j^L) \sin \lambda_{n+j} \\ v_k - 0.5(v_k^U + v_k^L) - 0.5(v_k^U - v_k^L) \sin \lambda_{n+n+k} \end{bmatrix} \quad (7)$$

where $i = 1, \dots, N$; $j = 1, \dots, n$; and $k = 1, \dots, m$ where the vector of generalized coordinates has been augmented to

$$\mathbf{q}^* = [x_1, \dots, x_N, v_1, \dots, v_m, \lambda_1, \dots, \lambda_{N+n+m}]^T \quad \text{or}$$

$$\mathbf{q}^* = [\mathbf{u}^T \quad \mathbf{v}^T \quad \lambda_1^T \quad \lambda_2^T]^T, \text{ where}$$

$$\lambda_1 = [\lambda_1 \quad \dots \quad \lambda_N]^T, \text{ and } \lambda_2 = [\lambda_{N+1} \quad \dots \quad \lambda_{N+n+m}]^T.$$

IMPLICIT FUNCTION THEOREM

In order to have a well-posed formulation, constraints that are used to model the geometry of this problem should be independent, except at certain critical surfaces in the manifold where the Jacobian becomes singular. However, the sweep Jacobian is not square. It is important, therefore, that there not be open sets in the space of the generalized parameters in which the constraints are redundant. At any interior point of the constraint space $\mathbf{q}^*$, the sweep Jacobian for parametric sweeps is defined as

$$\Phi_{q^*} = \begin{bmatrix} \xi_q & \mathbf{0} \\ \mathbf{I} & \mathbf{q}_\lambda \end{bmatrix} \quad (8)$$

where $\mathbf{q}_\lambda = \partial \mathbf{q} / \partial \lambda$ and $\xi_q = \partial \xi / \partial \mathbf{q}$. For implicit sweeps, the Jacobian is derived as

$$\Phi_{q^*} = \begin{bmatrix} \xi_q & \mathbf{0} & \mathbf{0} \\ \mathbf{f}_q & \mathbf{f}_{\lambda_1} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{q}_{\lambda_2} \end{bmatrix} \quad (9)$$

where $\mathbf{f}_q = \partial \mathbf{f} / \partial \mathbf{q}$; $\mathbf{f}_{\lambda_1} = \partial \mathbf{f} / \partial \lambda_1$; and $\mathbf{q}_{\lambda_2} = \partial \mathbf{q} / \partial \lambda_2$.

Since the swept entities are smooth and continuously differentiable, then the sweep Jacobian exists for both parametrically and implicitly defined surfaces. In both cases, if the rank of the mapping ξ_q is maximal, i.e. 3, the image

under Φ of a small neighborhood of $\mathbf{q}^*$ will be a three-manifold in three-space. At a point $\mathbf{q}^*$ where rank of Φ_{q^*} is

less than 3, the sweep Jacobian becomes singular, or rank-deficient. One might describe this deficiency as “redundancy” in that some number of parameters in the constraint space become dependent, possibly lowering the dimension of the image set. Redundancy occurs when the Jacobian Φ_{q^*} is rank-

deficient. Candidate varieties that are boundary to the manifold are identified by computing values that create a locus of points where the Jacobian has lower rank than maximal. Varieties are computed when the locus of points has a Jacobian with a lower rank. Let ∂W characterize a manifold with corners as

$$\partial W \subset \{ \text{Rank } \Phi_{q^*}(\mathbf{q}^*) \leq \alpha, \text{ for some } \mathbf{q}^* \} \quad (10)$$

where the rank is at most $(n+m+3)$ for parametric and $(n+m+N+3)$ for implicit sweeps.

VARIETIES

In order to enforce the implicit function theorem, square subJacobians are determined. For example, a 3×4 Jacobian and a rank deficiency of one, there are four 3×3 subJacobians. Equating the determinants of the subJacobians to zero forms analytic functions. The set of common zeros defined by this finite number of analytic functions are called varieties. In the case of implicit surfaces, these varieties are readily computed and visualized. In the case of parametric entities, the rank deficiency conditions yield analytic functions equated to zero that must be simultaneously solved. The solutions to the set of functions are singular sets defined by the vector $\mathbf{p}_i$, where $\mathbf{q}^T = [\mathbf{p}_i^T \quad \mathbf{d}^T]$, i.e., $\mathbf{d}$ contains the remaining non-constant parameters.

Substituting $\mathbf{p}_i^T$ into $\xi(\mathbf{q})$ yields strata denoted by V_i as

$$V_i: \xi(\mathbf{d}) = \xi(\mathbf{q}, \mathbf{p}_i) \quad i = 1, \dots, \beta \quad (11)$$

Where β is the number of varieties. In the case of implicitly defined entities, the vector $\xi(\mathbf{d})$ is accompanied by the associated set of functions $\mathbf{f} = [f_1, \dots, f_N]^T$. Note that the stratification process is to decompose a manifold (Eq. 10) into a finite disjoint (or union) of closed submanifolds such that

$$\partial W = V_1 \cap V_2 \cap \dots \cap V_\beta \quad (12)$$

While some strata can be further decomposed, it is only necessary to stratify those that cannot readily be visualized.

Some of these varieties are $\in \mathbf{R}^3$ and are rank deficient. This indicates that the resulting submanifold can be further stratified and is composed of many parts (called strata). The stratification process is to decompose a variety V into a finite, disjoint union or closed manifolds.

MANIFOLD STRATIFICATION

For the case when the stratum $\xi(\mathbf{d})$ cannot be visualized, where $\xi_i(\mathbf{d}_i): \mathbf{d}_i(\lambda) \in \mathbf{R}^3 \rightarrow \mathbf{R}^3$, and where λ is the corresponding vector of slack variables $\lambda = [\lambda_k \quad \lambda_\ell \quad \lambda_m]^T$, the Jacobian of the variety is

$$\xi_\lambda = \xi_d \mathbf{d}_\lambda \quad (13)$$

where $\xi_d = \partial \xi / \partial \mathbf{d}$ and $\mathbf{d}_\lambda = \partial \mathbf{d} / \partial \lambda$. For a parameter at its limit (q_i^{limit}), the second block matrix of Eq. (13) is rank deficient. An elementary matrix of row operations $\mathbf{E}_i$ applied to $\mathbf{d}_\lambda$ yields a row echelon form such that

$$\mathbf{E}_i[\mathbf{d}_\lambda] = \mathbf{E}_{RE} \quad (14)$$

$$\text{where } \mathbf{E}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and } \mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

where the subscript denotes the parameter number, and $\mathbf{E}_{RE}$ is a row echelon form. This same matrix applied to ξ_d^T yields

$$\mathbf{E}_i[\xi_d]^T = \begin{bmatrix} \Lambda \\ \mathbf{0} \end{bmatrix} \quad (15)$$

$$\text{where } \Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \end{bmatrix} \quad (16)$$

where $\dim(\Lambda) = (2 \times 3)$ and $\dim(\Lambda_j) = (2 \times 1)$ for $j = 1, 2, 3$.

For Λ , the application of the implicit function theorem yields three analytic functions that are equated to zero as

$$|\Lambda_1 \Lambda_2| = 0; \quad |\Lambda_2 \Lambda_3| = 0; \quad \text{and} \quad |\Lambda_1 \Lambda_3| = 0 \quad (17)$$

Simultaneous solutions to the three equations are the varieties of this submanifold, i.e., a second stratification process (see Section A.18). This process can be repeated for each resulting stratum that cannot be visualized.

COMPUTER IMPLEMENTATION

In this section, we illustrate the implementation of the above theoretical framework into a computer program that automatically performs the analysis and visualization of multivariate solids. Mathematica® was used to develop this code and to visualize the volume. For commercial applications, it is recommended that this be converted into a more efficient basic language or embedded in hardware. In order to keep the code listing to a minimum, the presentation will adhere to tasks that are vital to the completion of the algorithm.

Mathematica® code

I. Definitions

$qL1=0;$ $qU1=5\pi/4;$ $qL2=-\pi/4;$ $qU2=3\pi/4;$ $qL3=0;$ $qU3=5\pi/4;$ $qL4=0;$ $qU4=2\pi;$

$xE[q1_q2_q3_q4_]:=T04[q1,q2,q3,q4].\{5,0,0,1\}$
 $x4[q1_q2_q3_q4_]:=\{xE[q1,q2,q3,q4][[1]],xE[q1,q2,q3,q4][[2]],xE[q1,q2,q3,q4][[3]]\};$ $p=25;$

II. Sweep Jacobian computation

$\text{MatrixPartials}[\text{funs_List}, \text{vars_List}]:=\text{Outer}[\text{D}, \text{funs}, \text{vars}];$
 $z[i_j_]:=0 \text{ if } i=0 \text{ or } j=0; \text{ fns}=\text{x4}[q1,q2,q3,q4]; \text{ vrs}=\{q1,q2,q3,q4\};$

III. Computing the square sub-jacobians

$\text{Mm2}=\text{Table}[\text{Simplify}[\text{Minors}[\text{MatrixPartials}[\text{fns}, \text{vrs}], 3]]];$

IV. Determination of Varieties

$a2=\text{Solve}[\{\text{Minors}[\text{MatrixPartials}[\text{fns}, \text{vrs}], 3]==\text{Array}[z, \{1, 4\}]\}, \{q2\}];$

$m2=\text{Dimensions}[a2];$

$\text{If}[m2[[2]]!=0, \{\text{Array}[t2, m2[[1]]];$

$\text{Do}[t2[i]=q2/.a2[[i, 1]], \{i, 1, m2[[1]], 1\}]; j=0;$

$\text{Do}[\text{If}[\text{Im}[t2[i]]==0, j=j+1, \text{Skip}], \{i=1, m2[[1]], 1\}];$

$\text{Array}[s2, j]; \text{Do}[s2[i]=t2[[i, 1, j, 1]]];$

$\text{Do}[\text{If}[s2[i]==0, s2[j+1]=\text{Pi}, \text{No}], \{i, 1, j, 1\}];$

$qs2=\text{Cases}[\{s2[i], \{i=1, j+1, 1\}\}, s2_; s2<=qU2 \& \& s2>=qL2];$
 $d2=\text{Dimension}[qs2];$

V. Compute the stratum for each variety

$\text{If}[d2[[1]]>=1, \{\text{g}[p]:=\text{ParametricPlot3D}[\text{x4}[q1, qs2[[1]], q3, qL4], \{q1, qL1, qU1\}, \{q3, qL3, qU3\}, \text{Compiled}->\text{False}],$
 $p=p+1, \text{No}];$

$\text{If}[d2[[1]]>=1, \{\text{g}[p]:=\text{ParametricPlot3D}[\text{x4}[q1, qs2[[1]], q3, q$

$U4], \{q1, qL1, qU1\}, \{q3, qL3, qU3\}, \text{Compiled}->\text{False}],$
 $p=p+1, \text{No}];$

$\text{If}[d2[[1]]>=1, \{\text{g}[p]:=\text{ParametricPlot3D}[\text{x4}[q1, qs2[[1]], qL3, q4], \{q1, qL1, qU1\}, \{q3, qL3, qU3\}, \text{Compiled}->\text{False}],$
 $p=p+1, \text{No}];$

$\text{If}[d2[[1]]>=1, \{\text{g}[p]:=\text{ParametricPlot3D}[\text{x4}[q1, qs2[[1]], qU3, q4], \{q1, qL1, qU1\}, \{q3, qL3, qU3\}, \text{Compiled}->\text{False}],$
 $p=p+1, \text{No}]; \dots$

[Repeat above procedures with respect to other variables and save the graphics of each singular set]

VI. Manifold stratification

$\text{xx1}[q2_q3_q4_]:=\text{x4}[qL1, q2, q3, q4];$

$\text{Fns}=\text{xx1}[q2, q3, q4];$

$\text{Vrs}=\{q2, q3, q4\};$

$a11=\text{Solve}[\{\text{Minors}[\text{MatrixPartials}[\text{fns}, \text{vrs}], 3]==\text{Array}[z, \{1, 1\}]\}, \{q2\}]; m11=\text{Dimensions}[a11];$

If additional singularities exist, repeat step II to find out the singularities which are in the constraint limits save its resulting graphics in the array $\text{g}[p]$.

Repeat above procedures to plot the reduced rank deficiency corresponding to $qU1, qL2, qU2, \dots, qL4, qU4$

VII. Evaluating the strata of rank deficiency equal to 2

$\text{g}[1]:=\text{ParametricPlot3D}[\text{x4}[qL1, qL2, q3, q4], \{q3, qL3, qU3\}, \{q4, qL4, qU4\}, \text{Compiled}->\text{False}];$

$\text{g}[2]:=\text{ParametricPlot3D}[\text{x4}[qL1, qU2, q3, q4], \{q3, qL3, qU3\}, \{q4, qL4, qU4\}, \text{Compiled}->\text{False}]; \dots \dots$

$\text{g}[24]:=\text{ParametricPlot3D}[\text{x4}[q1, q2, qU3, qL4], \{q1, qL1, qU1\}, \{q2, qL2, qU2\}, \text{Compiled}->\text{False}];$

VIII. Visualization of the final swept volume

$\text{Show}[\text{Table}[\text{g}[k], \{k, 1, p-1, 1\}]]$

ILLUSTRATIVE EXAMPLES

1. Consider a solid defined by

$$K:=\left\{\mathbf{u}: \frac{u_3^2}{16} = \frac{(u_1-3)^2}{9} + \frac{u_2^2}{4}, 0 \leq u_1 \leq 6, -2 \leq u_2 \leq 2\right\} \quad (18)$$

Note that this is a surface in $\mathbf{R}^3$ given by

$$\frac{z^2}{16} = \frac{(x-3)^2}{2} + \frac{y^2}{4}. \quad \text{The sweep is defined by the rotation}$$

$$\mathbf{R}(v_1) = \begin{bmatrix} \cos(\pi v_1) & -\sin(\pi v_1) & 0 \\ \sin(\pi v_1) & \cos(\pi v_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \text{for } 0 \leq v_1 \leq 2, \text{ where}$$

every point in the swept volume is given by

$$\xi = \Psi(v_1) + \mathbf{R}(v_1) \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T \quad \text{and where there will be}$$

no translation $\Psi = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$. The parameters are changed to generalized coordinates defined by the vector

$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^T \quad \text{where}$$

$$q_3 = \text{Sqrt}[16(\frac{(q_1-3)^2}{9} + \frac{q_2^2}{4})]$$

$$0 \leq q_1 \leq 6, \text{ and } -2 \leq q_2 \leq 2 \quad (19)$$

$$\mathbf{R}(q_4) = \begin{bmatrix} \cos(\pi q_4) & -\sin(\pi q_4) & 0 \\ \sin(\pi q_4) & \cos(\pi q_4) & 0 \\ 0 & 0 & 1 \end{bmatrix}; 0 \leq q_4 \leq 2 \quad (20)$$

The vector function of Eq. (7) describing the sweep with unilateral constraints after parametrization is written as

$$\Phi(\mathbf{q}^*) = \begin{bmatrix} q_1 \cos(\pi q_4) - q_2 \sin(\pi q_4) \\ q_2 \cos(\pi q_4) + q_1 \sin(\pi q_4) \\ q_3 \\ q_3 - \text{Sqrt}[16 * (\frac{(q_1-3)^2}{9} + \frac{q_2^2}{4})] \\ q_1 - 3 - 3 \sin q_5 \\ q_2 - 2 \sin q_6 \\ q_4 - 1 - \sin q_7 \end{bmatrix} \quad (21)$$

where $\mathbf{q}^* = [q_1 \dots q_7]^T$. Block matrices of the sweep Jacobian are

$$\xi_{\mathbf{q}} = \begin{bmatrix} \cos \pi q_4 & -\sin \pi q_4 & 0 & -\pi q_2 \cos \pi q_4 - \pi q_1 \sin \pi q_4 \\ \sin \pi q_4 & \cos \pi q_4 & 0 & \pi q_1 \cos \pi q_4 - \pi q_2 \sin \pi q_4 \\ 0 & 0 & 1 & 0 \end{bmatrix};$$

$$[\mathbf{f}_{\mathbf{q}} \quad \mathbf{f}_{\mathbf{a}}] = \begin{bmatrix} \frac{-4(q_1-3)}{9\text{Sqrt}[\frac{1}{9}(q_1-3)^2 + \frac{q_2^2}{4}]} & \frac{-q_2}{\text{Sqrt}[\frac{1}{9}(q_1-3)^2 + \frac{q_2^2}{4}]} & 1 & 0 \end{bmatrix}$$

Varieties are computed as $\mathbf{p}_1, \dots, \mathbf{p}_7$, where $\mathbf{p}_1 = \{q_5 = \frac{\pi}{2}\}$;

$$\mathbf{p}_2 = \{q_5 = -\frac{\pi}{2}\}; \quad \mathbf{p}_3 = \{q_6 = \frac{\pi}{2}\}; \quad \mathbf{p}_4 = \{q_6 = -\frac{\pi}{2}\};$$

$$\mathbf{p}_5 = \{q_7 = \frac{\pi}{2}\} \quad ; \quad \mathbf{p}_6 = \{q_7 = -\frac{\pi}{2}\}; \quad \mathbf{p}_7 = \{q_2 = 0\}. \text{ In order}$$

to visualize the strata, we substitute each variety into $\xi(\mathbf{q})$ as shown in Fig. 1.

Fig. 1 (a) Stratum due to $\mathbf{p}_1$ (b due to $\mathbf{p}_2$ (c) due to $\mathbf{p}_5$
The solid is shown in Fig. 2.

Fig. 2 A 4-parameter solid

COMPUTATIONAL TIME

The strength of the presented formulation is evidenced in the ability to obtain exact solutions for any number of sweep parameters and for both implicitly and parametrically defined entities. The code written in Mathematica® can be rendered efficient by converting it in a lower level programming language or embedded in hardware. Nevertheless, and to give the reader a sense of computational complexity, we have attempted to measure the computational (CPU) time required for the various examples. Furthermore, a relatively high computational cost is associated with obtaining manifold strata. This is only available when varieties with more than one-to-one mapping occur. The following examples were computed using a Pentium II PC with 64 MB RAM running at 300MHz.

Examples with $n + m = 4$

Time used = 10.987 seconds: A 4-parameters solid

Time used = 4.021 seconds, Self intersection example. A 4-parameter solid.

Examples with $n + m = 5$

Time used = 12.812 seconds

Workspace of the 5DOF (RRPRR) robot.

CONCLUSIONS

The algorithm and experimental computer code presented in this paper are a culmination of efforts to study formulations for analyzing and visualizing multivariate solids. It was shown that the method encompasses geometric entities defined in both parametric and implicit forms. It was also shown that the rigorous mathematical formulation using manifold stratification is capable of addressing multivariate solid representations. Unilateral constraints imposed on the solid were also taken into consideration. Exact representation of the solid is achieved and it was shown that the methodology is suitable for computer implementation. While the computer code developed using Mathematica® is not optimized for commercial use, it is an illustration of the potential of this method for efficient implementation. The ultimate goal is to implement the formulation as an integral part of commercial CAD systems or as a stand-alone module. Moreover, to address the visualization of the multivariate *freeform* solid.

Computational time measured for various examples using the Mathematica® experimental code provides an indication of the efficiency of the code, in view of implementing such algorithms in a more appropriate computer language or hardware.

Results are shown that demonstrate the robustness of the method in visualizing the multivariate solid. The principal contribution of this method is characterized by its ability to obtain an exact representation for visualization purposes, and in its ability to obtain closed-form equations of its boundary.

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