Technical Note

On Swept Volume Formulations: Implicit Surfaces

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Abstract

Recent developments in formulations for generating swept volumes have made a significant impact on the efficiency of employing such algorithms and on the extent to which formulations can be used in representing complex shapes. In this paper, we outline a method for employing the representation of implicit surfaces using the Jacobian rank deficiency condition presented earlier for the sweep of parametric surfaces. A numerical and broadly applicable analytic formulation is developed that yields the exact swept volume.

Keywords: Swept volume, implicit surfaces, Jacobian rank deficiency, solid modeling

Introduction

Swept volumes have become increasingly important in modern Computer-Aided Design because of the need to represent the trace of objects that have experienced motion. Such applications are numerous and encompass solid modeling, manufacturing automation, robot analysis, collision detection, and computer graphics. In this report we present new results pertaining to the sweep of implicit surfaces using the Jacobian rank deficiency conditions [1-5]. Recent work published in the special edition of *Computer-Aided Design* [6] dedicated to the subject of swept volumes has outlined a method for trimming swept volumes.

For a review of prior work in this field, the reader is referred to Abdel-Malek and Yeh [1] for a comprehensive review. The work presented in [6] is the continuation of a robust method introduced in many reports [7-10] that employs the concept of a differential sweep equation. While the method is aimed at computing trimming curves in order to remove points that are in the interior of the boundary of the swept volume, the technique by which the authors have formulated the problem is of interest. Surfaces and solids *implicitly* represented were considered in the sweep equation. We shall adapt this same representation but we will apply the Jacobian rank deficiency method (never before used for implicitly defined equations) to obtain singular sets. These sets will describe (implicitly) surfaces that exist on the exterior and in the interior of the swept volume. Combining these surfaces (referred to as singular surfaces) yields an *exact representation* of the swept volume.

Although parametric representations are more widely used and are generally much simpler to treat than implicit representations, we focus our attention in this report on the latter. Swept volumes are extensively used in many fields (Solid Modeling in CAD is perhaps the most common) and implicit

representations are commonly utilized to represent surfaces because of their simplicity to use and store. In such cases, a CAD program must have the ability to perform sweep operations on the implicitly represented solid without converting the equation to parametric form. Such an example is given below characterized by the modeling and representation of a mechanical spring element, generated by the sweep of a solid sphere along a helical curve.

The Jacobian rank deficiency method was developed for parametric surfaces [1-4] and was shown to handle self-intersections, multiple parameter sweeping, and solid property computations. The method was never applied to implicitly defined surfaces because of the lack of formulations for proper representation of such surfaces in a consistent numerical algorithm. This report will outline a systematic method for computing the swept volume of implicit surfaces. The uniqueness of the method is in determining the swept volume generated by the sweep of a solid (represented in implicit form) along a curve (characterized as a function of one parameter). Prior to this work, there has never been any published results addressing the computation of implicit swept volumes with more than three parameters. In fact, the only paper that has dealt with a related subject to the sweep of implicit solids is that of Hu and Ling [11] where they have addressed the sweeping of the natural quadric surfaces (i.e., the result is still limited to three parameters and only deals with quadric surfaces).

In order to compare with the method recently presented by Blackmore, *et al.* [6], we shall treat one of the same self-intersecting examples addressed therein. We will first address the general problem using the Jacobian rank deficiency method. It will be shown that self-intersections are inherently considered and do not require additional trimming computations.

Formulation for Implicit Sweeps

In order to remain consistent with the notation used in Blackmore, *et al.*, where a smooth rigid sweep σ may be represented in the form

$$\sigma(\mathbf{w}) = \xi(t) + \mathbf{A}(t)\mathbf{x} \quad (1)$$

where $\mathbf{x} = [x_1, \dots, x_M]^T$ and $\mathbf{w} = [\mathbf{x}^T t]^T$, and where ξ and $\mathbf{A}$ are, respectively, smooth ($=\mathbf{C}^\infty$) vector and matrix valued functions with $\xi(0) = \mathbf{0}$, $\mathbf{A}(0) = \mathbf{I}$ the identity matrix, and $\mathbf{A}(t)$ is a real (3×3) orthonormal matrix (when $M = 3$) for every $t \in [0, 1]$. For the purpose of developing the formulation, it is required only that the object being swept be defined as the set of points satisfying finitely many equalities and inequalities for differentiable functions.

In order to develop a method that is consistent and general for any implicit surface, we will represent the object (manifold) W by a number of inequalities, with no restriction on the number of inequalities nor on the type of variables. Let W be represented by

$$W := \left\{ \mathbf{x}: \ell_1^L \leq f_1(\mathbf{x}) \leq \ell_1^U, \dots, \ell_N^L \leq f_N(\mathbf{x}) \leq \ell_N^U, \right. \\ \left. x_1^L \leq x_1 \leq x_1^U, \dots, x_M^L \leq x_M \leq x_M^U \right\} \quad (2)$$

where $\mathbf{f} = [f_1, \dots, f_N]^T$ and $f_i(\mathbf{x})$ denotes an expression representing the surface, ℓ_i is a limit, the upper subscript L indicates the lower limit, U denotes the upper limit, N is the number of expressions, and k is the number of variables. It should be noted that this is the general case, but an object may be represented by any number of expressions and any number of variables that may or may not have limits.

In order to develop a framework for applying the rank deficiency condition to implicit sweeps, we shall employ a parametrization technique in converting the representation in Eq. (2) to a constraint function. A constraint equation of the form $q_i^L \leq q_i \leq q_i^U$ is transformed into an equation by introducing a new generalized coordinate λ_i such that the inequality constraint can be rewritten as

$$q_i = a_i + b_i \sin \lambda_i \quad (3)$$

where $a_i = (q_i^L + q_i^U)/2$ and $b_i = (q_i^U - q_i^L)/2$ are the mid-point and half-range of the inequality constraint [1]. As a result of this transformation, it is now possible to define the sweep by

$$\Phi(\mathbf{z}) := \begin{bmatrix} \sigma(\mathbf{w}) \\ f_1(\mathbf{x}) - 0.5(\ell_1^U + \ell_1^L) - 0.5(\ell_1^U - \ell_1^L) \sin \lambda_1 \\ \dots \\ f_N(\mathbf{x}) - 0.5(\ell_N^U + \ell_N^L) - 0.5(\ell_N^U - \ell_N^L) \sin \lambda_N \\ x_1 - 0.5(x_1^U + x_1^L) - 0.5(x_1^U - x_1^L) \sin \lambda_{N+1} \\ \dots \\ x_M - 0.5(x_M^U + x_M^L) - 0.5(x_M^U - x_M^L) \sin \lambda_{N+M} \\ t - 0.5 - 0.5 \sin \lambda_{N+M+1} \end{bmatrix} \quad (4)$$

where $\mathbf{z} = [x_1, \dots, x_M, t, \lambda_1, \dots, \lambda_{N+M+1}]^T = [z_1, \dots, z_{N+2M+2}]^T$ or using partitioned coordinates $\mathbf{z} = [\mathbf{w}, \lambda_1, \lambda_2]^T$ where $\mathbf{w} = [x_1, \dots, x_M, t]^T$, $\lambda_1 = [\lambda_1, \dots, \lambda_N]^T$, and $\lambda_2 = [\lambda_{N+1}, \dots, \lambda_{N+M+1}]^T$. Note that the parameter t has also been included in $\Phi(\mathbf{z})$.

Rank deficiency conditions first developed by Abdel-Malek and Yeh [1] can now be applied to determine singular sets of the Jacobian $\Phi_z(\mathbf{z}) = \partial\Phi/\partial\mathbf{z}$ defined by

$$\Phi_z = \begin{bmatrix} \sigma_{\mathbf{w}} & \mathbf{0} & \mathbf{0} \\ \mathbf{f}_{\mathbf{w}} & \mathbf{f}_{\lambda_1} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{w}_{\lambda_2} \end{bmatrix} \quad (5)$$

where $\sigma_{\mathbf{w}} = \partial\sigma/\partial\mathbf{w}$; $\mathbf{f}_{\mathbf{w}} = \partial\mathbf{f}/\partial\mathbf{w}$; $\mathbf{f}_{\lambda_1} = \partial\mathbf{f}/\partial\lambda_1$; and $\mathbf{w}_{\lambda_2} = \partial\mathbf{w}/\partial\lambda_2$. The Implicit Function Theorem mandates that there not be open sets in the space of $\mathbf{z}$ in which the constraints are redundant.

Redundancy occurs when the Jacobian Φ_z , is rank-deficient which will subsequently define *singular surfaces* in the manifold. Singularities as a result of the Jacobian rank deficiency condition are denoted by s_i . An important remark must be made. Compare this Jacobian with that developed for the sweep of parametric surfaces and denoted by

$$\Phi_q^* = \begin{bmatrix} \sigma_q & \mathbf{0} \\ \mathbf{I} & \mathbf{q}_\lambda \end{bmatrix} \quad (6)$$

where $\mathbf{q}$ is the vector of parameters and $\mathbf{q}^* = [\mathbf{q}^T \lambda^T]^T$. In this case, the singularity of the Jacobian was determined in two distinct categories [4]. However, for the Jacobian of implicit surfaces (Eq. 5), there is an additional submatrix given by $[\mathbf{f}_w \mathbf{f}_{\lambda_1} \mathbf{0}]$. For the Jacobian to have a rank deficiency, a row in this matrix must equal zero, which yields a number of singularities that relate both $\mathbf{f}_w$ and $\mathbf{f}_{\lambda_1}$ (typically as a function of several variables). This should become more clear in the illustrative examples below. This submatrix generates functions that relate the change in the surface coordinates with respect to the sweep parameter.

For a non-square matrix, rank deficiency conditions are imposed by setting the determinants of certain square Jacobian submatrices equal to zero. Their simultaneous solution yields s_i . These singularities define singular surfaces. The main difference between this formulation and that developed for the sweep of parametric surfaces is the direct generation of singular parametric surfaces (called varieties) in the latter case by direct substitution of s_i into σ .

Illustrative Example

Consider the solid treated in [6] using the SEDE trimming approach. Let W be the solid cylinder representing a flat-end machine tool in space defined by

$$W = \{\mathbf{x} : -\frac{1}{4} \leq x_1^2 + x_2^2 - \frac{1}{4} \leq 0, -1 \leq x_3 \leq 1\} \quad (7)$$

The goal is to compute the swept volume of W generated by the sweep

$$\xi(\mathbf{x}, t) = \begin{bmatrix} 0 \\ 4t \\ 0 \end{bmatrix} + \mathbf{R}(t) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (8)$$

where $\mathbf{R}$ is a basic rotation matrix about the $\mathbf{x}$ -axis defined by $\mathbf{R}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi t) & -\sin(\pi t) \\ 0 & \sin(\pi t) & \cos(\pi t) \end{bmatrix}$

and where the sweep parameter t is bounded as $0 \leq t \leq 1$. The sweep can be written as

$$\xi(\mathbf{z}) = \begin{bmatrix} x_1 \\ 4t + x_2 \cos(\pi t) - x_3 \sin(\pi t) \\ x_2 \sin(\pi t) + x_3 \cos(\pi t) \\ x_1^2 + x_2^2 - 0.25 - 0.25 \sin(\lambda_1) \\ x_3 - \sin(\lambda_2) \\ t - 0.5 - 0.5 \sin(\lambda_3) \end{bmatrix} = \begin{bmatrix} z_1 \\ 4z_4 + z_2 \cos(\pi z_4) - z_3 \sin(\pi z_4) \\ z_2 \sin(\pi z_4) + z_3 \cos(\pi z_4) \\ z_1^2 + z_2^2 - 0.25 - 0.25 \sin(z_5) \\ z_3 - \sin(z_6) \\ z_4 - 0.5 - 0.5 \sin(z_7) \end{bmatrix} \quad (9)$$

where $\mathbf{z} = [z_1, z_2, z_3, z_4, z_5, z_6, z_7]^T$. The Jacobian matrix is computed as

$$\Phi_z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(\pi z_4) & -\sin(\pi z_4) & 4 - \pi z_3 \cos(\pi z_4) - \pi z_2 \sin(\pi z_4) & 0 & 0 & 0 \\ 0 & \sin(\pi z_4) & \cos(\pi z_4) & \pi z_2 \cos(\pi z_4) - \pi z_3 \sin(\pi z_4) & 0 & 0 & 0 \\ 2z_1 & 2z_2 & 0 & 0 & -0.125 \cos z_5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\cos z_6 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -0.5 \cos z_7 \end{bmatrix} \quad (10)$$

To apply the rank deficiency condition, and because the Jacobian is non-square (dimension 6×7), we compute the seven (6×6) sub-Jacobians (denoted by J_i), compute their determinants, and equate them to zero. The simultaneous zeros of these determinants are singular sets that yield the so-called *Singular Surfaces*. The determinants are evaluated as:

$$\det[J_1] = 0 \quad (11)$$

$$\begin{aligned} \det[J_2] = & 0.25 \cos(\pi z_4) \cos z_5 \cos z_6 \cos z_7 - 0.19635 z_3 \cos^2(\pi z_4) \cos z_5 \cos z_6 \cos z_7 \\ & - 0.19635 z_3 \sin^2(\pi z_4) \cos z_5 \cos z_6 \cos z_7 = 0 \end{aligned}$$

$$\text{which can be further simplified to } \det[J_2] = \cos z_5 \cos z_6 \cos z_7 (-0.19635 z_3 + 0.25 \cos(\pi z_4)) = 0 \quad (12)$$

Note that derivations and algebraic manipulations of the determinants of the Jacobian are carried out using a symbolic manipulator (Mathematica®).

$$\det[J_3] = \cos z_5 \cos z_6 \cos z_7 (-0.196 z_2 + 0.25 \sin(\pi z_4)) = 0 \quad (13)$$

$$\det[J_4] = -0.0625 \cos z_5 \cos z_6 \cos z_7 = 0 \quad (14)$$

$$\det[J_5] = \cos z_6 \cos z_7 (\pi z_2 z_3 - 4 z_2 \cos(\pi z_4)) = 0 \quad (15)$$

$$\det[J_6] = \cos z_5 \cos z_7 (0.196 z_2 - 0.25 \sin(\pi z_4)) = 0 \quad (16)$$

$$\det[J_7] = 0.125 \cos z_5 \cos z_6 = 0 \quad (17)$$

The singularity sets as a result of the simultaneous solution of Eqs. (11-17) are:

$$\begin{aligned} s_1 &= \{z_5 = \pi/2, \text{ and } z_6 = \pi/2\}, & s_2 &= \{z_5 = \pi/2, \text{ and } z_6 = -\pi/2\}, & s_3 &= \{z_5 = -\pi/2, \text{ and } z_6 = \pi/2\}, \\ s_4 &= \{z_5 = -\pi/2, \text{ and } z_6 = -\pi/2\}, & s_5 &= \{z_5 = \pi/2, \text{ and } z_7 = \pi/2\}, & s_6 &= \{z_5 = \pi/2, \text{ and } z_7 = -\pi/2\}, \\ s_7 &= \{z_5 = -\pi/2, \text{ and } z_7 = \pi/2\}, & s_8 &= \{z_5 = -\pi/2, \text{ and } z_7 = -\pi/2\}, & s_9 &= \{z_5 = \pi/2, \text{ and } z_2 = 0\}, \\ s_{10} &= \{z_5 = -\pi/2, \text{ and } z_2 = 0\}, & s_{11} &= \{z_6 = \pi/2, \text{ and } z_7 = \pi/2\}, & s_{12} &= \{z_6 = \pi/2, \text{ and } z_7 = -\pi/2\}, \\ s_{13} &= \{z_6 = -\pi/2, \text{ and } z_7 = \pi/2\}, & s_{14} &= \{z_6 = -\pi/2, \text{ and } z_7 = -\pi/2\}, & s_{15} &= \{z_5 = \pi/2, \text{ and } z_3 = 4/\pi \cos(\pi z_4)\}, \\ s_{16} &= \{z_5 = -\pi/2, \text{ and } z_3 = 4/\pi \cos(\pi z_4)\}, & s_{17} &= \{z_6 = \pi/2, \text{ and } z_2 = 1.27 \sin(\pi z_4)\}, & s_{18} &= \{z_6 = -\pi/2, \text{ and } \\ & & & & z_2 = 0.25/0.196 \sin(\pi z_4)\}; \end{aligned}$$

For each singular set, a geometric entity must be determined. Depending on the rank of the Jacobian deficiency, this entity could be in the form of a curve or a surface. To illustrate, consider $s_1 = \{z_5 = \pi/2, \text{ and } z_6 = \pi/2\}$. For $z_5 = \pi/2$ (recall that $z_5 = \lambda_1$) the parametrization $x_1^2 + x_2^2 = 0.25 + 0.25 \sin(\lambda_1)$, yields $x_1^2 + x_2^2 = 0.25$. Similarly, $z_6 = \pi/2$ (recall that $z_6 = \lambda_2$ indicates that $x_3 = 1$). Hence this singularity is defined by $s_1 = \{x_1^2 + x_2^2 = 0.25, x_3 = 1\}$. For each singularity set, a similar analysis is performed and all singularities are listed below.

$$\begin{aligned} s_2 &= \{x_1^2 + x_2^2 = 0.25, x_3 = -1\}, \\ s_3 &= \{x_1^2 + x_2^2 = 0, x_3 = 1\}, s_4 = \{x_1^2 + x_2^2 = 0, x_3 = -1\}, \\ s_5 &= \{x_1^2 + x_2^2 = 0.25, t = 1\}, s_6 = \{x_1^2 + x_2^2 = 0.25, t = 0\}, \\ s_7 &= \{x_1^2 + x_2^2 = 0, t = 1\}, s_8 = \{x_1^2 + x_2^2 = 0, t = 0\}, \\ s_9 &= \{x_1^2 + x_2^2 = 0.25, x_2 = 0\}, s_{10} = \{x_1^2 + x_2^2 = 0, x_2 = 0\}, \\ s_{11} &= \{x_3 = 1, t = 1\}, s_{12} = \{x_3 = 1, t = 0\}, \\ s_{13} &= \{x_3 = -1, t = 1\}, s_{14} = \{x_3 = -1, t = 0\}; \\ s_{15} &= \{x_1^2 + x_2^2 = 0.25, x_3 = 4/\pi \cos(\pi t)\}, s_{16} = \{x_1^2 + x_2^2 = 0, x_3 = 4/\pi \cos(\pi t)\}, \\ s_{17} &= \{x_3 = 1, x_2 = 0.25/0.196 \sin(\pi t)\}, s_{18} = \{x_3 = -1, x_2 = 0.25/0.196 \sin(\pi t)\}; \end{aligned}$$

Note that singular sets s_{15} - s_{18} have variables that are parametrized as a function of another variable. Also note that sets $s_3, s_4, s_7, s_8, s_{10}, s_{16}, s_{17}, s_{18}$ are curves. Surfaces due to singularity sets $s_1, s_2, s_5, s_6, s_9, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}$, are depicted in Fig. 1.

Fig. 1 Here

Combining all surfaces yields the swept volume as shown in the two views of Fig. 2.

Fig. 2 Here

Parametric Sweep

In order to compare the two formulations for the Jacobian rank deficiency condition (for both implicit and parametric surfaces), we shall examine the same sweep but we use a parametric representation of the solid cylinder given by

$$\mathbf{x}(q_1, q_2, q_4) = [q_4 \cos q_1 \quad q_4 \sin q_1 \quad q_2]^T \quad (18)$$

on the surface parametrized as $\xi(q_3) = [0 \quad 4q_3 \quad 0]^T$ with the same rotation matrix as above $\mathbf{R}(q_3)$,

where $-\pi \leq q_1 \leq \pi$, $-1 \leq q_2 \leq 1$, $0 \leq q_3 \leq 1$, and $0 \leq q_4 \leq 0.5$. The sweep equation is defined by

$$\sigma(\mathbf{q}) = \begin{bmatrix} q_4 \cos q_1 \\ 4q_3 + q_4 \cos(\pi q_3) \sin q_1 - q_2 \sin(\pi q_3) \\ q_2 \cos(\pi q_3) + q_4 \sin q_1 \sin(\pi q_3) \end{bmatrix} \quad (19)$$

The Jacobian is a 3×4 non-square matrix. Applying the rank deficiency conditions yields four sub-Jacobians that are equated to zero as

$$\det[J_1] = q_4(-\pi q_2 + 4 \cos(\pi q_3)) \sin q_1 = 0 \quad (20)$$

$$\det[J_2] = q_4 = 0 \quad (21)$$

$$\det[J_3] = q_4(\pi q_4 \sin q_1 - 4 \sin(\pi q_3)) = 0 \quad (22)$$

$$\det[J_4] = \cos q_1(\pi q_2 - 4 \cos(\pi q_3)) = 0 \quad (23)$$

Simultaneous solutions to the above four equations yields the singular sets listed below. Considering the definition for rank deficiency, reduced rank deficiency, and constraint singularity [1], the total sets are computed as follows:

$$\begin{aligned} p_1 = \{q_1 = -\pi, q_4 = 0.5\}, \quad p_2 = \{q_1 = 0, q_4 = 0.5\}, \quad p_3 = \{q_2 = -1, q_3 = 0\}, \quad p_4 = \{q_2 = -1, q_3 = 1\}, \\ p_5 = \{q_2 = 1, q_3 = 0\}, \quad p_6 = \{q_2 = 1, q_3 = 1\}, \quad p_7 = \{q_2 = -1, q_4 = 0.5\}, \quad p_8 = \{q_2 = 1, q_4 = 0.5\}, \\ p_9 = \{q_3 = 0, q_4 = 0.5\}, \quad p_{10} = \{q_3 = 1, q_4 = 0.5\}, \quad p_{11} = \{q_3 = 0.318 \text{ ArcCos}(0.785q_2), q_4 = 0.5\} \end{aligned}$$

These singular sets are equivalent (in parametric form) to those computed using the implicit sweep formulation. In particular, consider the singularity p_{11} (in the parametric formulation) where $q_3 = 0.318 \text{ ArcCos}(0.785q_2)$. This is equivalent to $x_3 = 4/\pi \cos(\pi t)$ (or $t = (1/\pi) \text{ArcCos}(x_3\pi/4) = .318 \text{ArcCos}(.785x_3)$) in s_{15} and s_{16} (of the implicit formulation).

Mechanical Spring Example

Consider the modeling of a spring element in a Mechanical Computer Aided Design (MCAD) system. Although one may be able to model the spring using parametric methods (i.e., using the B-rep approach), it will not adequately represent the solid properties associated with it. However, using the proposed implicit sweeping method, we are able to readily sweep a solid sphere along a spiral to generate the equation of a solid spring element. Again, the emphasis here is on obtaining an exact representation of the intended design.

Consider the solid sphere represented in implicit form as

$$W = \{\mathbf{x}: 0 \leq x_1^2 + x_2^2 + x_3^2 \leq 4\}$$

The goal is to compute the swept volume of W generated by the sweep

$$\xi(\mathbf{x}, t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 10\cos(t) \\ 10\sin(t) \\ 10t \tan(\pi/9) \end{bmatrix}$$

where the sweep parameter t is bounded as $0 \leq t \leq 4\pi$. The sweep can be written as

$$\xi(\mathbf{z}) = \begin{bmatrix} z_1 + 10\cos(z_4) \\ z_2 + 10\sin(z_4) \\ z_3 + 10z_4 \tan(\pi/9) \\ z_1^2 + z_2^2 + z_3^2 - 2 - 2\sin(z_5) \\ z_4 - 2\pi - 2\pi \sin(z_6) \end{bmatrix}$$

where $\mathbf{z} = [z_1, \dots, z_6]^T$. The Jacobian matrix is computed as

$$\Phi_z = \begin{bmatrix} 1 & 0 & 0 & -10\sin(z_4) & 0 & 0 \\ 0 & 1 & 0 & 10\cos(z_4) & 0 & 0 \\ 0 & 0 & 1 & 10\tan(\pi/9) & 0 & 0 \\ 2z_1 & 2z_2 & 2z_3 & 0 & -2\cos(z_5) & 0 \\ 0 & 0 & 0 & 1 & 0 & -2\pi\cos(z_6) \end{bmatrix}$$

The determinants of the sub-Jacobians are evaluated as:

$$\begin{aligned} \det[J_1] &= 2\cos(z_5) \\ \det[J_2] &= 40\pi\cos(z_6)[z_2\cos(z_4) - z_1\sin(z_4) + z_3\tan(\pi/9)] \\ \det[J_3] &= 4\pi\cos(z_5)\cos(z_6) \\ \det[J_4] &= 40\pi\cos(z_5)\cos(z_6)\tan(\pi/9) \\ \det[J_5] &= -40\pi\cos(z_4)\cos(z_5)\cos(z_6) \\ \det[J_6] &= -40\pi\cos(z_5)\cos(z_6)\sin(z_4) \end{aligned}$$

Simultaneously solving the six equations above for singular sets and substituting into the sweep equation yields the following singular surfaces.

$$\begin{aligned} s_1 &= \{z_1^2 + z_2^2 + z_3^2 = 4, z_4 = 4\pi\}. \\ s_2 &= \{z_1^2 + z_2^2 + z_3^2 = 4, z_4 = 0\}. \\ s_3 &= \{z_1^2 + z_2^2 + z_3^2 = 0, z_4 = 4\pi\}. \\ s_4 &= \{z_1^2 + z_2^2 + z_3^2 = 0, z_4 = 0\}. \\ s_5 &= \{z_1^2 + z_2^2 + z_3^2 = 4, z_3 = 0, z_2 = z_1 \tan(z_4)\}. \\ s_6 &= \{z_1^2 + z_2^2 + z_3^2 = 0, z_3 = 0, z_2 = z_1 \tan(z_4)\}. \end{aligned}$$

Combining all singular entities yields the solid shown in Fig. 3.

Fig. 3 Here

The calculations are much simpler than those required to depict the parametric form of the solid and only three singular surfaces were needed.

Example 3

In order to demonstrate that not only limits are needed to obtain the complete representation of the solid, but also that the rank deficiency is indeed necessary, consider the sweep represented by

$$W = \{\mathbf{x}: 0 \leq x_1 \leq 1, x_2 = 0, 0 \leq x_3 \leq 1\}$$

and the following rotation matrices and path.

$$\xi(\mathbf{x}, t) = \begin{bmatrix} \cos(\pi t) & 0 & \sin(\pi t) \\ 0 & 1 & 0 \\ -\sin(\pi t) & 0 & \cos(\pi t) \end{bmatrix} \begin{bmatrix} \cos(\pi t) & -\sin(\pi t) & 0 \\ \sin(\pi t) & \cos(\pi t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 6t \\ 0 \end{bmatrix}$$

where the sweep parameter t is bounded as $0 \leq t \leq 1$. The swept volume is shown in Fig. 4.

Fig. 4 Here

Conclusions

The method based on a Jacobian rank deficiency condition developed for computing the sweep of parametric surfaces was expanded in this report to a broadly applicable formulation for implicit surfaces. This was accomplished by reformulating the parametrization method in implicit form.

Note that in the foregoing analysis, neither the dimension of $\mathbf{x}$ nor the number of inequality expressions has been limited. As a result, the formulation is valid for n -dimensional manifolds. Also note that the resulting swept volume is exact as no approximations have been made. It is also noted that the formulation is developed as an algorithm suitable for computer implementation using commercial symbolic manipulators with an ability to perform derivation and plotting of surfaces.

References

1. Abdel-Malek, K and Yeh, HJ "Geometric Representation of the Swept Volume Using Jacobian Rank Deficiency Conditions," *Computer-Aided Design*, Vol. 29, No. 6, (1997), pp457-468.
2. Abdel-Malek, K, Yeh, HJ, and Othman, S, 1998, " Swept Volumes: Void and Boundary Identification," *Computer-Aided Design*, Vol. 30, No. 13, pp1009-1018.
3. Abdel-Malek, K, Yeh, HJ, and Rouwady, R, 1997b "A Unified Formulation for Swept Volume and Robot Workspace Analysis," *Proceedings of the ASME Design Automation Conference*, Sacramento, CA, September 14-17.
4. Abdel-Malek, Seaman, W, and Yeh, HJ, "NC Verification of up to 5 Axis Machining Processes using Manifold Stratification," (in press) *ASME Journal of Manufacturing Science and Engineering*.
5. Abdel-Malek, K. and Othman, S., (1999), "Multiple Sweeping using the Denavit-Hartenberg Representation Method," *Computer Aided Design*, Vol. 31, pp. 567-583.
6. Blackmore, D, Samulyak, R, and Leu, MC, "Trimming swept volumes", *Computer-Aided Design*, Vol. 31(3), pp215-224.
7. Blackmore, D, Leu, MC, Wang, LP, Jiang, H, 1997, "Swept Volumes: a retrospective and prospective view," *Neural Parallel and Scientific Computations*, Vol. 5, pp81-102.
8. Blackmore, D and Leu, M C 'Analysis of Swept Volume via Lie Groups and Differential Equations' *The International Journal of Robotics Research* Vol. 11(6) (1992) pp516-537.
9. Leu, M C, Park, S H, and Wang, K K 'Geometric Representation of Translational Swept Volumes and Its Applications' *ASME Journal of Engineering for Industry*, Vol. 108 (1986) pp113-119.
10. Blackmore, D, Leu, M C, and Shih, F 'Analysis and Modeling of Deformed Swept Volumes' *Computer-Aided Design* Vol. 25(4) (1994), pp315-326.
11. Hu, Z.J. and Ling, Z.K., 1996, "Swept Volumes Generated by the Natural Quadric Surfaces", *Computers and Graphics*, Vol. 20, No. 2, pp. 263-274.

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Figures

(a) singularity s_1 (b) singularity s_2

Fig. 1 Singular surfaces due to singularities

(c) singularity s_5 and (d) singularity s_6

(e) singularity s_9 and (f) singularity s_{15}

Fig. 1 (h) singularity s_{11} (i) singularity s_{12}

(j) singularity s_{13} (k) singularity s_{14}

Fig. 2 Two views of the swept volume (same example treated in Blackmore, *et al.* using SEDE trimming methods)

Fig. 3 The mechanical spring element using implicit representation (exact)

(a) The solid (first view)

(b) The solid (second view)

Fig. 4 The resulting solid