

# A Mathematical Method for Ergonomic-Based Design: Placement

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## **Abstract**

A rigorous mathematical formulation for ergonomic design based on obtaining and visualizing the workspace of human limbs is herein presented. The methodology and formulation presented in this paper are aimed at placing the human with respect to specified targets, whereby optimizing a given human performance measure. These measures are developed as mathematical cost functions that can be maximized or minimized. For example, as a result of this analysis, a method to place a human operator in an assembly line while minimizing the person's stress at each joint can be achieved. Other cost functions such maximizing reachability or maximizing dexterity are considered. The method is characterized by two steps: (1) determine the boundary envelope (also called reach envelope) of a human limb in closed form as and (2) Move the workspace envelope towards optimizing the cost function while satisfying all constraints. By defining a new position and orientation of the limb's workspace with respect to the target points, it is possible to establish an ergonomic design that satisfies the given constraints. The strengths of this method are in its ability to visualize the placement-reach design problem and in its broadly applicable mathematical formulation well suited for computer implementation. Furthermore, because the ergonomic design problem is indeed an optimization process, the use of optimization techniques in this work lends itself to addressing other standing problems. The formulation and code are demonstrated using a number of examples.

**Keywords:** Ergonomic design, placement, reachability, reach envelope, optimization.

## Introduction

Ergonomic design of workspaces surrounding human limbs has been a long-standing problem. Consider the placement of human operators in an assembly line or seating a person at a control panel where tools and instruments must be reached. Human modeling and simulation programs have commercially appeared that support the ergonomic design process and allow for digital mannequins to be manipulated. The goal of this work is to increase the capability of such programs in order to automatically calculate the best location and orientation of the human with respect to predefined target points. The result will be a location of the human that will allow for the most comfortable reach!

We propose a general numerical approach for the design of an environment wherein a number of points to be reached (called target points) are specified and where a mathematical method is developed to position and orient the limb's origin (e.g., the torso if the arm is considered). Used iteratively, the method presents an effective mathematically based approach to ergonomic design. Because of its analytic nature, visualization of the limb's reach envelope and placement are readily made available to the designer and are well suited for computer implementation.

Thus far, the idea of introducing a mathematical model for ergonomic modeling of human motion was unforeseeable. For example, a method for the determination of ergonomic parameters that relate people to objects in space was proposed by Costa, et al. (1997). The authors state that mathematical models of human movements are complex to define and hard to solve and suggest the use of Artificial intelligence in Neural Systems (ANS) as an approach to the problem. Indeed, collection of data for simulation of human movement has been done by many researchers (Ciungradi, et al. 1998), but there has never been a fundamental rigorous approach to the problem. Furthermore, the study of human motion has led designers to produce ergonomic-based workstations, by addressing work postures, work height; adjustable chairs; foot/hand use; gravity; momentum within normal work area (Kong 1990), all in an experimental manner, which is costly and is difficult to use in a design scenario (i.e., difficult to implement optimization methods for design). The proposed work will provide a viable venue to address such issues.

Limbs have been modeled as four-link systems consisting of trunk, upper arm, lower arm, and hand, being regarded as a redundant manipulator with a total of eight degrees of freedom (DOF) (Jung, et al. 1992; 1997; Jun and Kee 1996). The authors stated that inverse kinematics were used for solving the system and that the joint range availability was used as a performance function in order to guarantee local optimality (Jung and Park 1994). Inverse kinematics is an expression given to the mathematics used in calculating joint variables given the position and orientation of the end-link (e.g., hand). While inverse kinematics of an eight DOF system is not only difficult to obtain, but is also not reliable because of the many redundant solutions that may arise. Indeed, the concept of inverse kinematics in reachability analysis and ergonomic design should only be considered for non-redundant systems. Before defining the problem, we introduce terms that are necessary for this discussion.

1. **Specified point ( $s_p$ ):** a point embedded into the limb (e.g., the tip of a finger) whose motion will be tracked (Fig. 1a).
2. **Target points ( $p_i$ ):** are points defined in space and are to be touched by the specified point on the human limb as shown in Fig. 1a.
3. **Base Coordinate System (BCS):** is the first coordinate system in a series of consecutively numbered triads (Fig. 1a). The goal is to place the BCS such that the limb will satisfy the given constraints (e.g. reach all points, or maximize a given function).
4. **World Coordinate System (WCS):** is a fixed invariant global coordinate system.
5. **Workspace envelope:** is an envelope that encloses the totality of points that can be touched by the specified point  $s_p$  on the limb [also called the *Reach Envelope* (Molenbroek 1998)].
6. **Drive the workspace:** is an expression used to characterize impelling (causing a motion) of the workspace to minimize/maximize a cost function and will be used throughout this work to denote the repositioning and reorienting of the human.

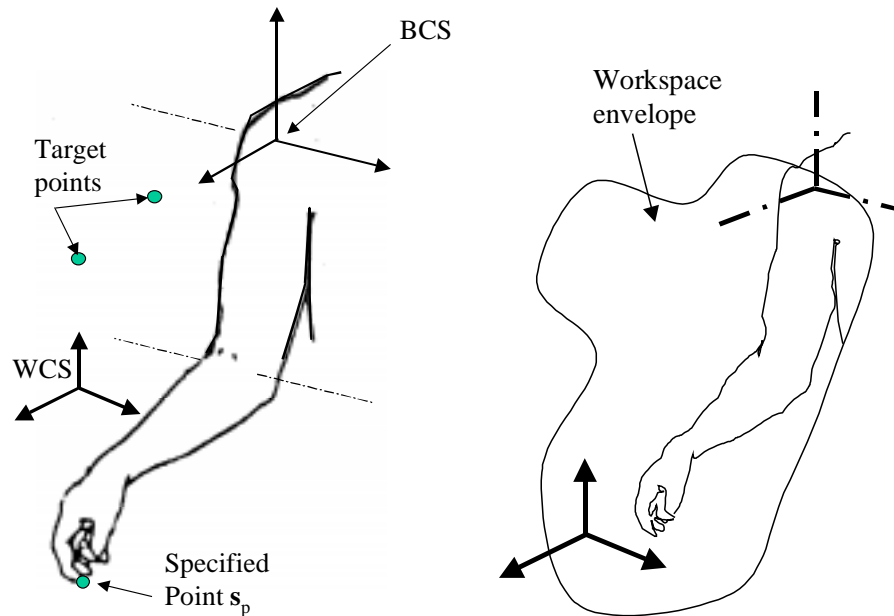


Fig. 1 (a) The point  $s_p$ , BCS, WCS, and target points (b) The workspace envelope of the arm

## Problem Definition

The reader must remember that this formulation is implemented in computer code and is intended for use by a designer to ergonomically place a human with respect to target points (e.g., handles, levers, gauges, etc.). Given a set of target points, a cost function, and a complete definition of an extremity (e.g., dimensions and ranges of motion of an arm), it is required to place the human with respect to these target points in such a way that the given cost function is optimized. This cost function characterizes human performance measures such as reachability, dexterity, stress, effort, discomfort, or potential energy. In this work, we will focus on reachability and stress at joints.

We present a rigorous mathematical formulation based on the following steps.

- (1) Locate the target points  $\mathbf{p}_i$  (e.g., instruments/devices/levers) at specific positions in 3D space (each target point is defined by a triplet- $xyz$ ).
- (2) Identify the workspace of the limb. The workspace is characterized by a number of surface patches on the boundary envelope.
- (3) Define a suitable cost function. Depending on the intended application, an appropriate cost function can be defined such as maximizing the reachability to all points, maximizing the dexterity at a target point, or minimizing the effort needed to move from one point to another.

(4) Numerically position and orient the workspace envelope to reach a configuration where all target points are inside the workspace while optimizing the given cost function. This step will guarantee that all target points (instruments/devices/levers) can be reached. The cost function is used here to drive (move, push, or impel) the workspace envelope.

In the following section, we will establish mathematical constraints whereby the ergonomic design problem will be formulated as an optimization problem. Constraints that assure the inclusion of the target points inside the limb's workspace are necessary. The goal is to pose the problem in a manner that can be addressed using well-established mature optimization techniques but yet that does not need to compute the inverse kinematics of the limb. Inverse kinematics for limb models of high number of DOF are very difficult to achieve. The general algorithm for ergonomic design using the proposed cost functions is presented in Fig. 2, where  $\mathbf{w}$  denotes the six generalized coordinates (explained in detail below) that will be used to track the location of the placement.

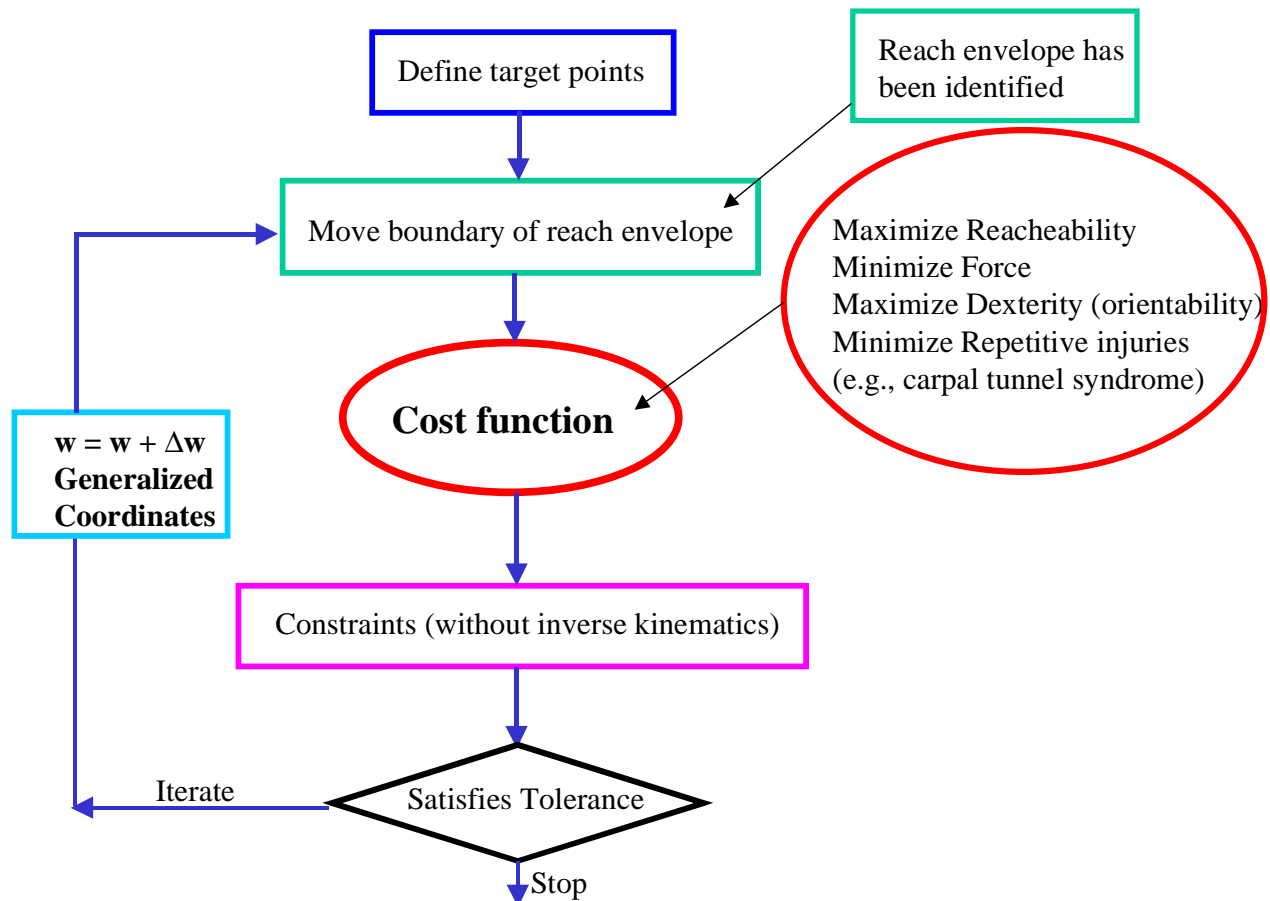


Fig. 2 Algorithm for ergonomic design

## Delineating the Envelope of the Limb's Workspace

In previous work applied to robotic manipulators (Abdel-Malek and Yeh 1997), we have introduced a method for delineating surface patches that define the exact reach envelope of a kinematic chain in closed form. In this section, we review this method, apply the methodology to human extremities, and define the needed surface patches.

Consider, for example, a specified point  $s_p$  on the tip of a finger as shown in Fig. 3. The position vector generated by this point with respect to the BCS for an arm represented by  $n$ -DOF is expressed by

$$\Phi(\mathbf{q}) = [x(\mathbf{q}) \quad y(\mathbf{q}) \quad z(\mathbf{q})]^T \quad (1)$$

where  $\mathbf{q} = [q_1, q_2, \dots, q_n]^T \in \mathbf{R}^n$  are the joint variables and  $\Phi(\mathbf{q})$  characterizes the vector set of all points that belong to the workspace (i.e., the volume touched by  $s_p$ ) and shown in Fig. 3.

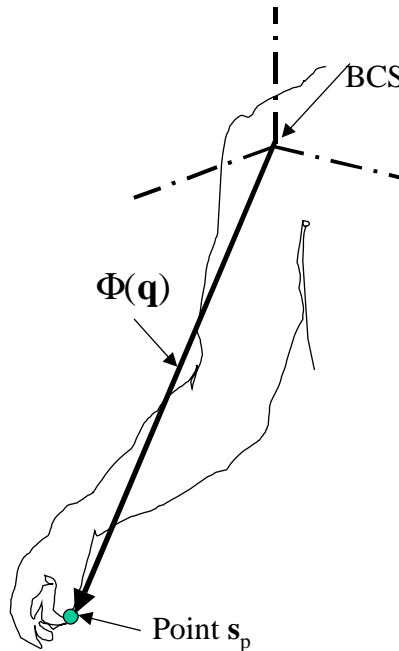


Fig. 3 Description of the  $\Phi(\mathbf{q})$  vector

In order to take into consideration the ranges of motion of each joint usually given in terms of an inequality constraint as

$$q_i^L \leq q_i \leq q_i^U \quad (2)$$

we convert the joint ranges expressed by the inequalities of Eq. (2) to equalities, where the vector of joints variables is now a  $\mathbf{q} = \mathbf{q}(\lambda)$ , where  $\lambda = [\lambda_1 \ \dots \ \lambda_n]^T$  are the new variables such that (see Appendix A for the parameterization)

$$\Phi = \Phi(\mathbf{q}(\lambda)) \quad (3)$$

This analysis was first demonstrated for the visualization of the workspace of robot manipulators (Abdel-Malek and Yeh 1997 and Abdel-Malek, *et al.* 1999a). Surfaces enveloping the workspace are analytically determined by studying the  $(3 \times n)$  Jacobian matrix (from the German Mathematician Jacobi), which represents the differentiation of  $\Phi$  with respect to  $\mathbf{q}$  as

$$\Phi_{\mathbf{q}} = \left[ \frac{\partial \Phi_i}{\partial q_j} \right]; i = 1, \dots, 3 \text{ and } j = 1, \dots, n \quad (4)$$

where  $\Phi_i$  are the components of  $\Phi$  and  $q_j$  are the components of  $\mathbf{q}$ . An expression for the velocity of the point  $\mathbf{s}_p$  on the finger is obtained by differentiating  $\Phi(\mathbf{q})$  with respect to time and using the chain rule as

$$\dot{\Phi} = \Phi_{\mathbf{q}} \mathbf{q}_{\lambda} \dot{\lambda} \quad (5)$$

where the terminology  $\mathbf{q}_{\lambda} = \partial \mathbf{q} / \partial \lambda$  and  $\dot{\lambda} = \partial \lambda / \partial t$ .

It can be shown (Abdel-Malek and Yeh 1997, Abdel-Malek, *et al.* 1997; and Abdel-Malek, *et al.* 1999) that the limb's workspace boundary is delineated by studying the rank deficiency of  $\Phi_{\mathbf{q}} \mathbf{q}_{\lambda}$ . Three rank deficiency conditions were derived to delineate singular sets, denoted  $\mathbf{s}^{(i)}$ ;  $i = 1, \dots, w$ , where  $w$  is the number of singular sets. Substituting  $\mathbf{p}^{(i)}$  into  $\Phi(\mathbf{q})$  yields a number of surfaces  $\xi^{(i)}(\mathbf{u}^{(i)})$ , where  $\mathbf{u}^{(i)}$  is the new vector of generalized coordinates such that

$$\xi^{(i)}(\mathbf{u}^{(i)}) = \Phi(\mathbf{u}^{(i)}, \mathbf{s}^{(i)}) \quad (6)$$

where  $\mathbf{s}^{(i)}$  are obtained by solving a set of analytic functions called varieties (Lu 1978 and Spivak 1968). Furthermore, surface patches in closed-form on the boundary of the envelope are a subset of Eq. (6) defined as  $\xi^{(i)}(\mathbf{u}^{(i)})$ ;  $i = 1, \dots, m$ , where  $m$  is the number of patches enveloping the workspace as illustrated in Fig. 4b for the arm shown in Fig. 4a. Let  $[\mathbf{x} \ \mathbf{y} \ \mathbf{z}]$  be the WCS and let  $[\mathbf{x}_o \ \mathbf{y}_o \ \mathbf{z}_o]$  be the BCS associated with the shoulder (i.e., embedded in the torso) and that is initially coinciding with the WCS (Fig. 4a). Let there be  $m$  number of surface patches on the boundary as illustrated in Fig. 4b (i.e., we have a complete definition of the workspace) as

$$\eta = \left\{ \xi^{(i)}(\mathbf{u}^{(i)}); i = 1, \dots, m \text{ for } \mathbf{u}_{La}^{(i)} \leq \mathbf{u}^{(i)} \leq \mathbf{u}_{Ua}^{(i)} \right\} \quad (7)$$

where  $\mathbf{u}_{La}^{(i)}$  and  $\mathbf{u}_{Ua}^{(i)}$  are the lower and upper limits, respectively, of the surface patch and  $\eta$  characterizes the envelope of the workspace.

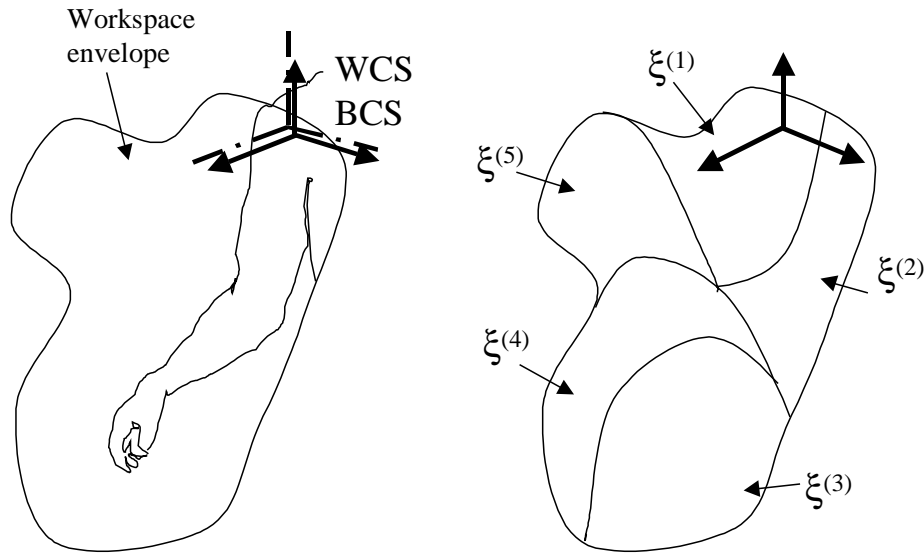


Fig. 4 (a) A human limb in the initial configuration with BCS and WCS coinciding (b) The workspace envelope defined by the  $\xi^{(i)}(\mathbf{u}^{(i)})$  surface patches

## Tracking the Reach Envelope

As both coordinate systems are initially coincident, effecting a translation and a rotation on the workspace of the limb will move the BCS triad  $[\mathbf{x}_o \ \mathbf{y}_o \ \mathbf{z}_o]$  to a new location  $O_1$  with coordinates  $[x_w \ y_w \ z_w]$  and a new orientation defined by a set of Euler angles  $(\alpha \ \beta \ \gamma)$  as shown in Fig. 5a and its placement of the BCS as shown in Fig. 5b.

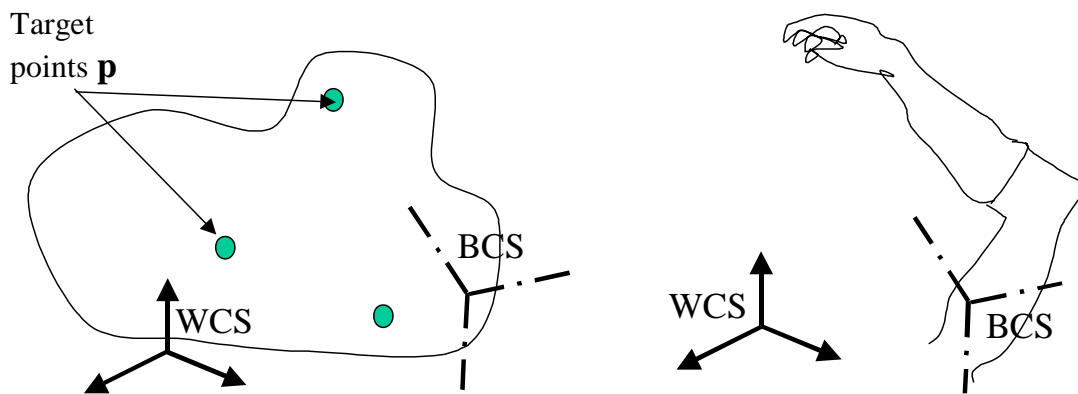


Fig. 5 (a) Workspace envelope after moving to include the target points (b) The new configuration of the BCS

We define the vector  $\mathbf{w} = [x_w \ y_w \ z_w \ \alpha \ \beta \ \gamma]^T$  as the vector of generalized coordinates that defines the translation and rotation of the BCS with respect to WCS and characterizes the configuration of the workspace. Motion of the workspace to a new configuration will be tracked using these six coordinates where the rotation matrix  $\mathbf{R}$  and the position vector  $\mathbf{v}$ , are defined as

$$\mathbf{v} = [x_w \ y_w \ z_w]^T \quad (8)$$

and

$$\mathbf{R}(\alpha, \beta, \gamma) = \begin{bmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta \sin \gamma - \sin \alpha \cos \gamma & \cos \alpha \sin \beta \cos \gamma + \sin \alpha \sin \gamma \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \alpha \sin \beta \cos \gamma - \cos \alpha \sin \gamma \\ -\sin \beta & \cos \beta \sin \gamma & \cos \beta \cos \gamma \end{bmatrix} \quad (9)$$

In order to track the motion of the envelope defined by surface patches  $\xi^{(i)}(\mathbf{u}^{(i)})$ ,  $i = 1, \dots, m$ , we multiply the surface patches by the rotation matrix and position vector as a function of  $\mathbf{w}$ .

$$\Psi^{(i)}(\mathbf{w}, \mathbf{u}) = \mathbf{R}(\alpha, \beta, \gamma) \cdot \xi^{(i)}(\mathbf{u}) + \mathbf{v}(x_w, y_w, z_w) \quad (10)$$

The aim of this work is to determine the vector  $\mathbf{w}$  that defines the new position and orientation of the BCS (therefore of the workspace envelope and consequently of the human) after moving to a new configuration whereby enclosing the specified target points, and driven by the cost function. The formulation that follows is developed to set forth the conditions for calculating a new configuration of the BCS via driving the workspace towards the target points.

## Specifying the Cost Function

### Reacheability

Criteria for establishing whether a point is inside the workspace must be defined (i.e., is the point reachable?). In order to have a rigorous formulation, we develop a number of generalized constraints that are *not based on the inverse kinematics* of the arm. In order to track motion of the workspace envelope, the translated and rotated workspace is given by

$$\Gamma(\mathbf{w}, \mathbf{q}) = \mathbf{R}(\alpha, \beta, \gamma) \Phi(\mathbf{q}) + \mathbf{v}(x_w, y_w, z_w) \quad (11)$$

Consider a number of target points  $\mathbf{p}_i$ ,  $i = 1, \dots, \ell$ , each with coordinates  $(x_p, y_p, z_p)_i$  in the space and a reach envelope  $\eta$  of a given human limb (note that not only a limb, but a torso or combination thereof can be considered). In order to insure that the target points will be located within the limb's workspace volume, we define a distance constraint such that the distance between the target points and the workspace is minimized as

$$\min \|\mathbf{p}^{(i)} - \Gamma(\mathbf{w}, \mathbf{q})\| = 0 \quad \text{for } i = 1, \dots, \ell \quad (12)$$

To ascertain that a target point  $\mathbf{p}$  is inside the workspace, additional conditions are needed. The absolute value of the distance between a target point and the boundary should be greater than a specified value  $\varepsilon$  (depth inside the workspace). This is done to insure that the target points are reachable, i.e. inside the workspace. The distance between all target points  $\mathbf{p}^{(j)}$  and all surface patches  $\Psi^{(i)}(\mathbf{u})$  should be greater than a specified minimum value such as

$$\|\mathbf{p}^{(j)} - \Psi^{(i)}(\mathbf{u}^{(i)}, \mathbf{w})\| \geq \varepsilon_j \quad \text{where } j = 1, \dots, \ell \quad \text{and } i = 1, \dots, m \quad (13)$$

where  $\varepsilon_j > 0$  are specified constants and  $m$  is the number of surface patches. If a target point satisfies both conditions of Eq. (12) and (13), then this point is internal to the workspace (i.e. can be reached). Those conditions are enough to guarantee that the point will be reached but do not provide a *driver* function (or force) to move the workspace.

Therefore, we must introduce an objective function (also called a cost function) that drives the motion of the reach envelope. This cost function can be defined by the designer. For example, if only reachability is of interest, this cost function is defined as the total sum of distances from the target points to the surface patches (we will use this concept in the examples below).

### **Torque on Joints**

$$\boldsymbol{\tau} = \mathbf{F}^T \mathbf{J}(\mathbf{q}) \quad (14)$$

where  $\boldsymbol{\tau}$  denotes the  $(n \times 1)$  vector of joint torques and  $\mathbf{F}^T$  is the  $(m \times 1)$  vector of end-effector forces, where  $m$  is the dimension of the operational space of interest, and where  $\mathbf{J}$  is the Jacobian of the limb. Note that the analysis of forces and torques on serial chains can be found in many references (Tsai 2000, Sciavicco and Siciliano 1996).

### **Discomfort**

We first define the neutral position of a given joint regardless of the motion as  $q_i^N$ . It is well known that a joint experiences significant amount of stress as it is displaced from its most neutral position. Although different people have different neutral positions for the same joint, this approximate measure provides a viable and simple measure for qualitatively identifying stress. To quantify this displacement over all degrees of freedom, we sum the variation from neutral position

for all joints. Again, we use  $w$  as weight values for each joint to emphasize that some joints' stress is deemed higher than others.

$$S = \sum_{i=1}^n w_i |q_i^N - q_i| \quad (14)$$

### **Moving the Workspace**

For  $\ell$  target points and  $m$  surface patches, the problem is posed as a constrained optimization problem where the cost function is defined as a minimization of the distance between all target points and all surface patches

$$D(\mathbf{w}, \mathbf{u}) = \min \left[ \sum_{i=1}^m \sum_{j=1}^{\ell} \left\| \Psi^{(i)}(\mathbf{w}, \mathbf{u}^{(i)}) - \mathbf{p}^{(j)} \right\| \right] \quad (15)$$

subject to the constraints defined above and repeated as follows:

- (1) Workspace envelope at least covering the target points (shortest distance between target points):

$$g_i \equiv \min \left\| \mathbf{p}^{(i)} - \Gamma(\mathbf{q}, \mathbf{w}) \right\| \leq \beta \quad \text{for } i = 1, \dots, \ell \quad (16)$$

where  $\beta$  is a small positive number and subject to the inequality constraints on joint ranges as

$$q_k^L \leq q_k \leq q_k^U \quad \text{for } k = 1, \dots, n \quad (17)$$

- (2) Embedding the target points inside the workspace volume (a minimum distance between target points and surface patches).

$$g_k \equiv \left\| \mathbf{p}^{(j)} - \Psi^{(i)}(\mathbf{u}, \mathbf{w}) \right\| \geq \varepsilon_j \quad \text{for } i = 1, \dots, m \text{ and } j = 1, \dots, \ell, k = 1, \dots, (\ell \times m) \quad (18)$$

where  $\varepsilon_j$  is the depth of the target point inside the workspace volume. There are  $\ell + (\ell \times m) + n$  total number of constraints.

### **Numerical Method for Calculating the Configuration of BCS**

The aim is to define a new configuration for the manipulator base such that all target points are included. As will be shown in this section, several sub-problems utilize optimization methods to obtain a solution, but the general problem will focus on obtaining a single configuration where all target points are reached. An iterative method for obtaining a solution to the above constrained optimization problem is implemented (Arora 1989). Before initiating the algorithm, several

constraints are usually violated and others inactive. In order to obtain a viable configuration, we will initiate an iterative procedure, where for each iteration, two independent constrained optimization sub-problems (Eqs. 15 and 17) are solved to obtain a new set of variables  $\mathbf{w}$ . The step value and direction are obtained using a commercial package (DOT 1999) and used in this iterative procedure. (Note that any other efficient optimization algorithm could have been used).

### Example 1: A 2-Joint Shoulder-Elbow Illustrative Example

This example will be used to illustrate the formulation and to demonstrate the concept of a cost function to drive the workspace. Consider the simplified arm model on a planar surface (surface of a table), where only two joints are active as shown in Fig. 6. The arm is modeled as two revolute joints also shown in Fig. 5. The specified point  $\mathbf{s}_p$  is on the tip of the thumb and there are two target points that need be reached at  $\mathbf{p}_1 = (-35, 12)$  and  $\mathbf{p}_2 = (-25, -8)$ , i.e.,  $\ell = 2$ . It is required to specify the placement of the human with respect to these points while maximizing reachability.

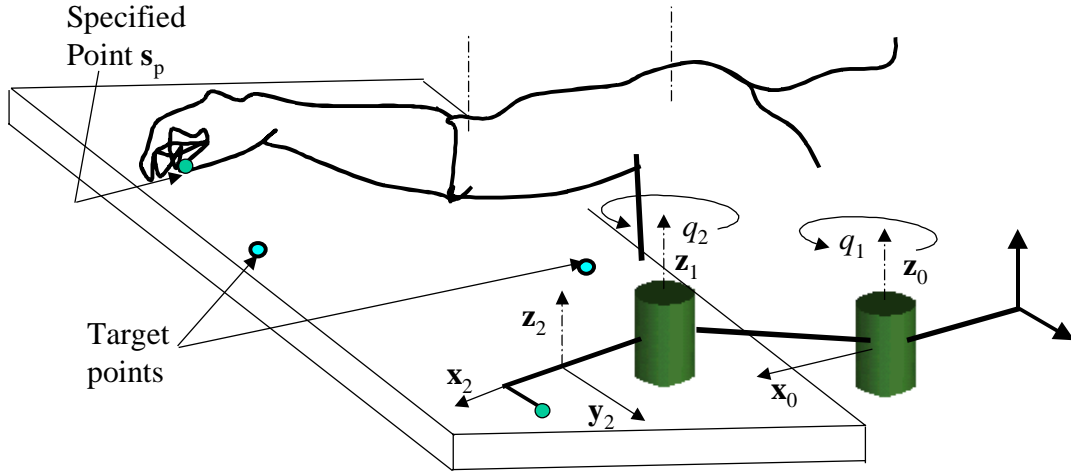


Fig. 6 Planar motion of the arm modeled as a two DOF system and target points specified

Coordinates of the specified point  $\mathbf{s}_p$  located at the tip of the thumb can be written as

$$\Phi(\mathbf{q}) = [-5\cos(q_1 + q_2) - 10\sin q_1 - 10\sin(q_1 + q_2) \quad 10\cos q_1 + 10\cos(q_1 + q_2) - 5\sin(q_1 + q_2)]^T \quad (19)$$

with joint ranges imposed as  $-120^\circ \leq q_1 \leq 70^\circ$  and  $0^\circ \leq q_2 \leq 120^\circ$ , where the thumb is located at

$\mathbf{s}_p = [5 \quad 5 \quad 0]^T$  with respect to the third coordinate system. The Jacobian is calculated as

$$\xi_{\mathbf{q}} = \begin{bmatrix} -5(2\cos q_1 + 2\cos(q_1 + q_2) - \sin(q_1 + q_2)) & 5(-2\cos(q_1 + q_2) + \sin(q_1 + q_2)) \\ -5(2\cos(q_1 + q_2) + 2(\sin q_1 + \sin(q_1 + q_2))) & -5(\cos(q_1 + q_2) + 2\sin(q_1 + q_2)) \end{bmatrix} \quad (20)$$

and 
$$\mathbf{q}_\lambda = \begin{bmatrix} -0.436 \cos \lambda_1 & 0 \\ 0 & 1.658 \cos \lambda_2 \end{bmatrix} \quad (21)$$

Using the rank deficiency criteria, there are four curve segments shown in Fig. 7 (exact reach curves) and that are determined in closed form as:

$$\xi^{(1)} = [-5 \cos q_1 - 20 \sin q_1 \quad 20 \cos q_1 - 5 \sin q_1]^T \text{ (at } q_1 = q_1^L \text{) for } 120^\circ \leq q_1 \leq 70^\circ$$

$$\xi^{(2)} = [-8.66 \cos q_1 + 5(0.5 \cos q_1 + 0.866 \sin q_1) - 5 \sin q_1 \quad 5 \cos q_1 + 5(-0.866 \cos q_1 + 0.5 \sin q_1) - 8.66 \sin q_1]^T$$

(at  $q_2 = 120^\circ$ ) for  $120^\circ \leq q_1 \leq 70^\circ$

$$\xi^{(3)} = \begin{bmatrix} 8.66 + 8.66 \cos q_2 + 5(0.5 \cos q_2 - 0.866 \sin q_2) + 5 \sin q_2 \\ -5 - 5 \cos q_2 + 5(0.866 \cos q_2 + 0.5 \sin q_2) + 8.66 \sin q_2 \end{bmatrix}, \text{ (at } q_1 = q_1^U \text{) for } 0 \leq q_2 \leq 120^\circ$$

$$\xi^{(4)} = \begin{bmatrix} -9.39 - 9.39 \cos q_2 + 5(-0.34 \cos q_2 + 0.94 \sin q_2) - 3.42 \sin q_2 \\ 3.42 + 3.42 \cos q_2 + 5(-0.94 \cos q_2 - 0.34 \sin q_2) - 9.39 \sin q_2 \end{bmatrix} \text{ (at } q_1 = q_1^U \text{) for } 0 \leq q_2 \leq 120^\circ$$

Curves are shown in Fig. 7 and characterize the exact reach envelope of the 2-DOF arm.

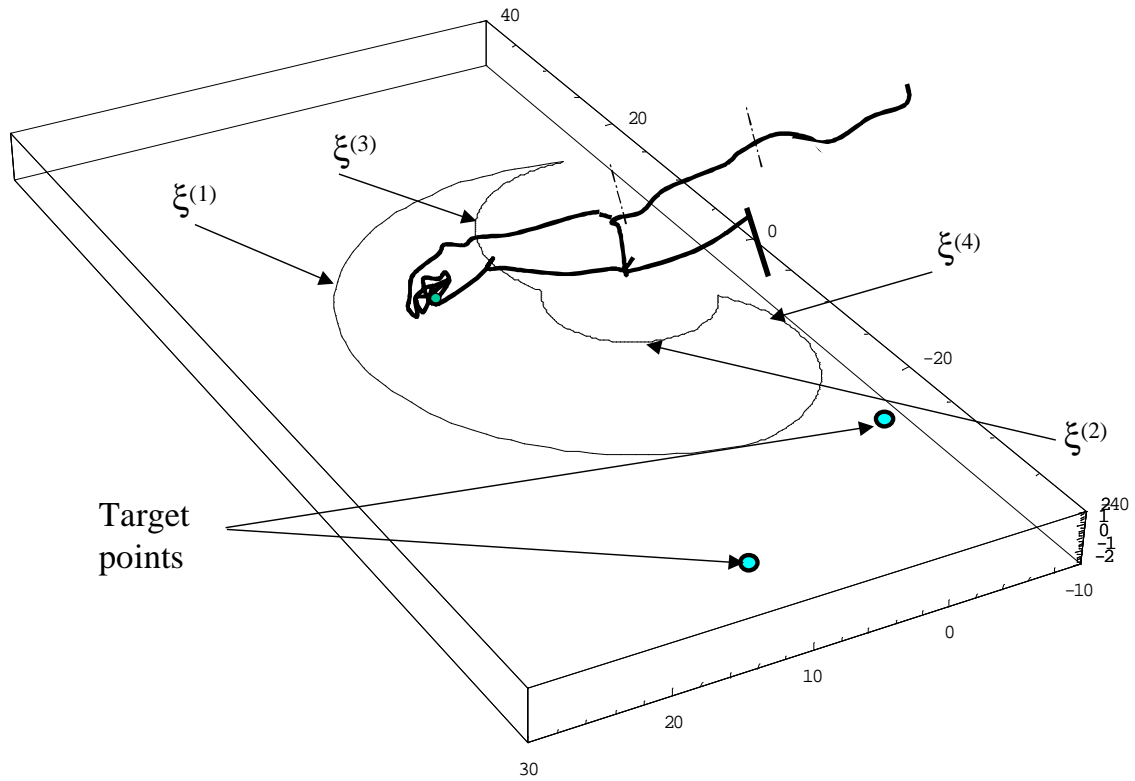


Fig. 7 The workspace envelope of the planar arm and the target

We define the cost function in this simple case as the distance subtended from the target points to the curve segments  $\xi^{(i)}$ . Implementing the numerical algorithm presented above such that the

workspace envelope is driven towards the target points with the generalized variables  $\mathbf{w} = [x_w \ y_w \ \alpha]^T$ , the initial and final configurations are presented in Table 1.

Table 1: Initial and final configuration of the manipulator's base

Parameters	$x_w$	$y_w$	$\alpha$
Initial	0.00	0.00	0.00°
Final	-19.3	6.8	134.5°

The initial and final configurations of the workspace envelope are shown in Fig. 8.

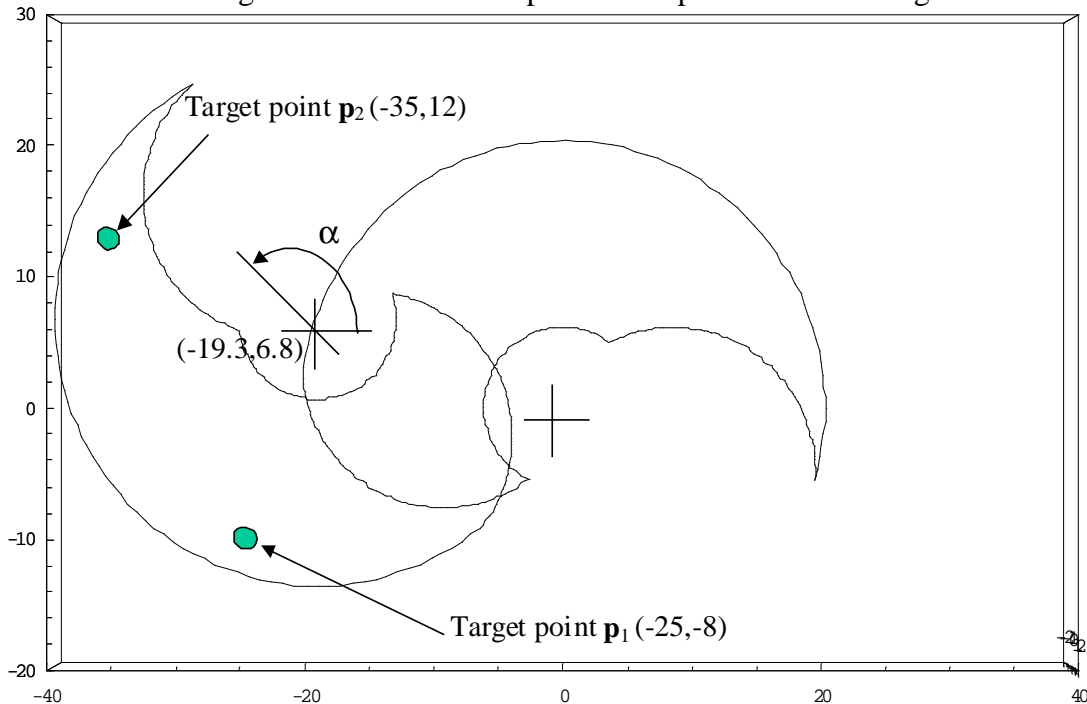


Fig. 8 Driving the workspace envelope to enclose the target points

The numerical results obtained from this algorithm are significant in many respects. They demonstrate that it is possible to rigorously define the reachability criteria and numerically design (or change the layout) to best obtain an ergonomic setup subject to a predefined cost function. This algorithm implemented in computer code is iteratively used to obtain designs based on visual and exact limb workspace analyses. Furthermore, researchers in other fields may elect to use different cost functions. It is conceivable that orthopedic surgeons may define cost functions to reduce repetitive injuries!

## Example 2: Upper extremity

Consider a model of the glenohumeral, elbow, and wrist joints shown in Fig. 9a and modeled as shown in Fig. 9b. The glenohumeral joint will be limited to 2DOF, the elbow to 1DOF, and the wrist to 2DOF. Typical joint ranges are imposed as  $-90^\circ \leq q_1 \leq 90^\circ$ ,  $-110^\circ \leq q_2 \leq 120^\circ$ ,  $0 \leq q_3 \leq 150^\circ$ ,  $-20^\circ \leq q_4 \leq 40^\circ$ , and  $-70^\circ \leq q_5 \leq 80^\circ$ .

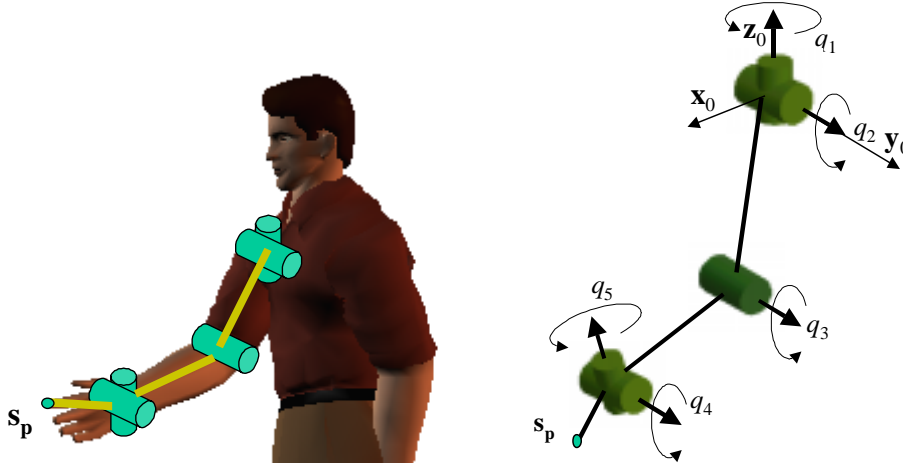


Fig. 9 (a) The specified point on the index finger (b) The arm modeled as a 5-joint system

A point on the end-effector is characterized by the position vector

$$\Phi(\mathbf{q}) = \begin{bmatrix} 5(c_5 s_1 s_4 + c_1(-s_2(2s_3 + c_4 c_5 s_3 + c_3 s_5) + c_2(2 + 2c_3 + c_3 c_4 c_5 - s_3 s_5))) \\ 5(c_1 c_5 s_4 - s_1 s_2(2s_3 + c_4 c_5 s_3 + c_3 s_5) + c_2 s_1(2 + c_3(2 + c_4 c_5) - s_3 s_5)) \\ -5(c_2((2 + c_4 c_5)s_3 + c_3 s_5) + s_2(2 + c_3(2 + c_4 c_5) - s_3 s_5)) \end{bmatrix} \quad (22)$$

where  $s_i$  denotes  $\sin q_i$  and  $c_i$  denotes  $\cos q_i$ . Analysis of the Jacobian  $\Phi_{\mathbf{q}} \mathbf{q}_k$  yields the following singular sets:

### Type I singularities

$$\mathbf{s}_1 = \{q_2 = \pi/2, q_3 = 0, q_5 = 0\}, \quad \mathbf{s}_2 = \{q_2 = -\pi/2, q_3 = 0, q_5 = 0\}, \quad \mathbf{s}_3 = \{q_3 = 0, q_4 = 0, q_5 = 0\}$$

### Type II singularities

(a) Type II with one joint that has reached a limit

$$\mathbf{s}_4 = \{q_2 = q_2^L, q_4 = 0, q_5 = 0\}, \quad \mathbf{s}_5 = \{q_2 = q_2^U, q_4 = 0, q_5 = 0\}, \quad \mathbf{s}_6 = \{q_3 = q_3^L, q_4 = 0, q_5 = 0\},$$

$$\mathbf{s}_7 = \{q_3 = q_3^U, q_4 = 0, q_5 = 0\}$$

(b) Type II with two joints that have reached their limits (listed in Appendix B)

### Type III Singular sets

Additional 80 singular sets are due to rank deficiency 3 (i.e., at least three joints simultaneously at

their limits). For example,  $\mathbf{s}_{33} = \{q_1 = q_1^L, q_2 = q_2^L, q_3 = q_3^L\}$ ,  $\mathbf{s}_{34} = \{q_1 = q_1^L, q_2 = q_2^L, q_3 = q_3^U\}$ .

In order to visualize these surfaces, we substitute the singular set into Eq. (23). Substituting other singularities into Eq. (23) yields equations of surfaces that are shown in Fig. 10(a-e).

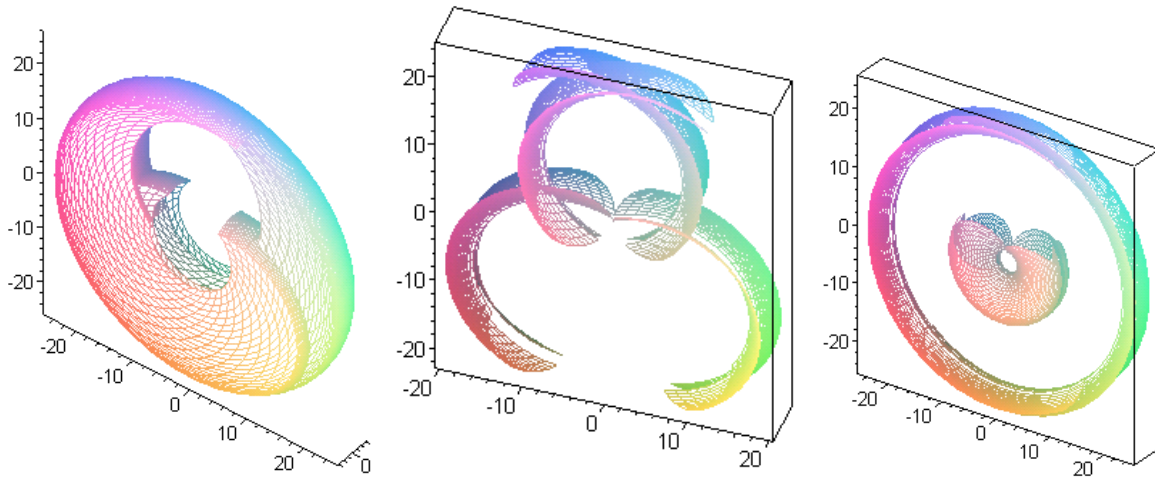


Fig. 10 Type II singularities due to (a)  $s_{8-16}$  (b)  $s_{40-48}$  (c)  $s_{64-72}$

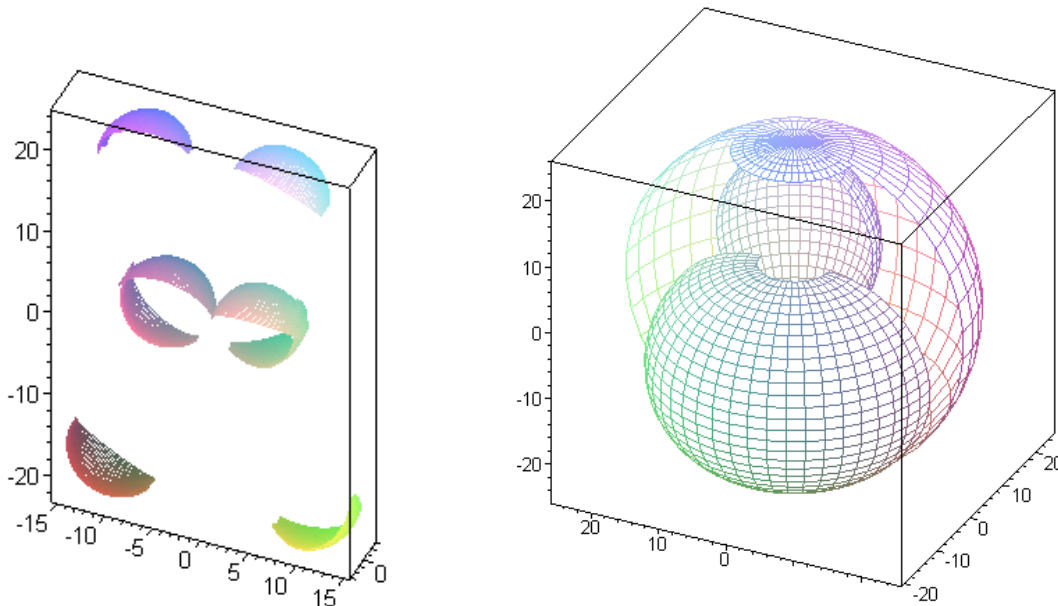


Fig. 10 (d)  $s_{32-80}$  (e)  $s_{4-8}$

The complete workspace envelope of the 5-DOF model of the arm is shown in Fig. 11.

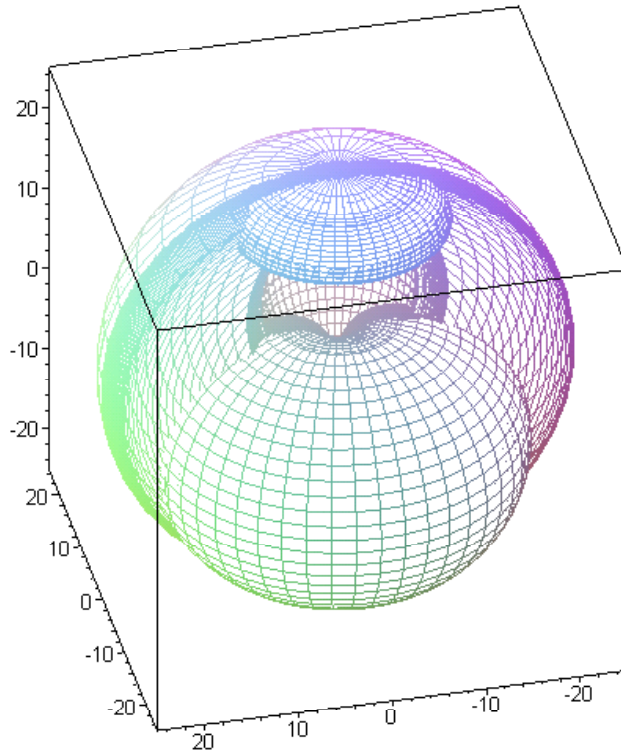


Fig. 11 The reach envelope of the 5-DOF arm model

The configuration of the BCS is described by the vector  $\mathbf{w} = [x_w \ y_w \ z_w \ \alpha \ \beta \ \gamma]^T$ . Because the resulting workspace is indeed complex in nature with 112 surfaces, we introduce cross-sectional cuts through the reach envelope (the reader must note that this is only possible because of the closed form formulation). Consider, for example, the cross section of an  $xy$ -plane through the workspace of the arm at  $z = 0$  (plan view) and the cross section of an  $xz$  plane at  $y = 0$  (side and elevation views) as shown in Fig. 12 (a) and (b).

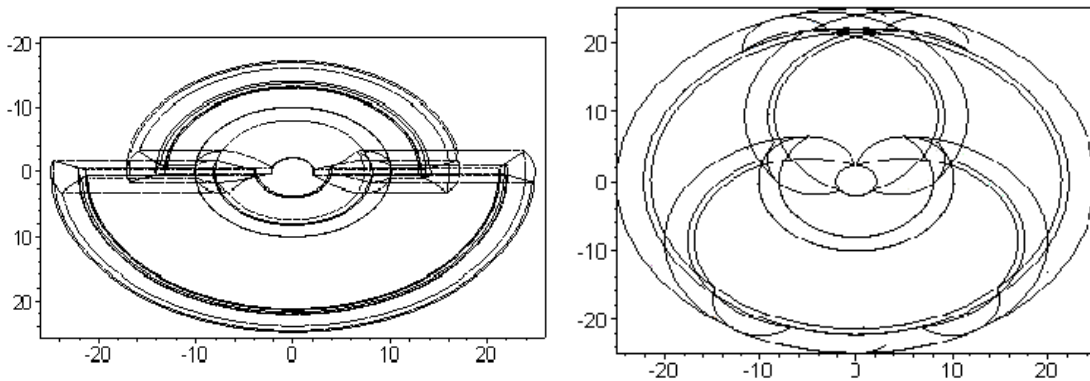


Fig. 12 (a) An  $xy$  cross-sectional plane through the workspace of the arm (b) An  $xz$  cross-sectional plane through the workspace of the arm

Consider four target points located in space at known coordinates as illustrated in Fig. 13, where the person is to be placed into the environment. It is therefore required to move the workspace in such a way to maximize reachability (i.e., maximize the ability to reach all four points). The result of the algorithm is shown in Fig. 14, where the human has been relocated (placed) in the environment.

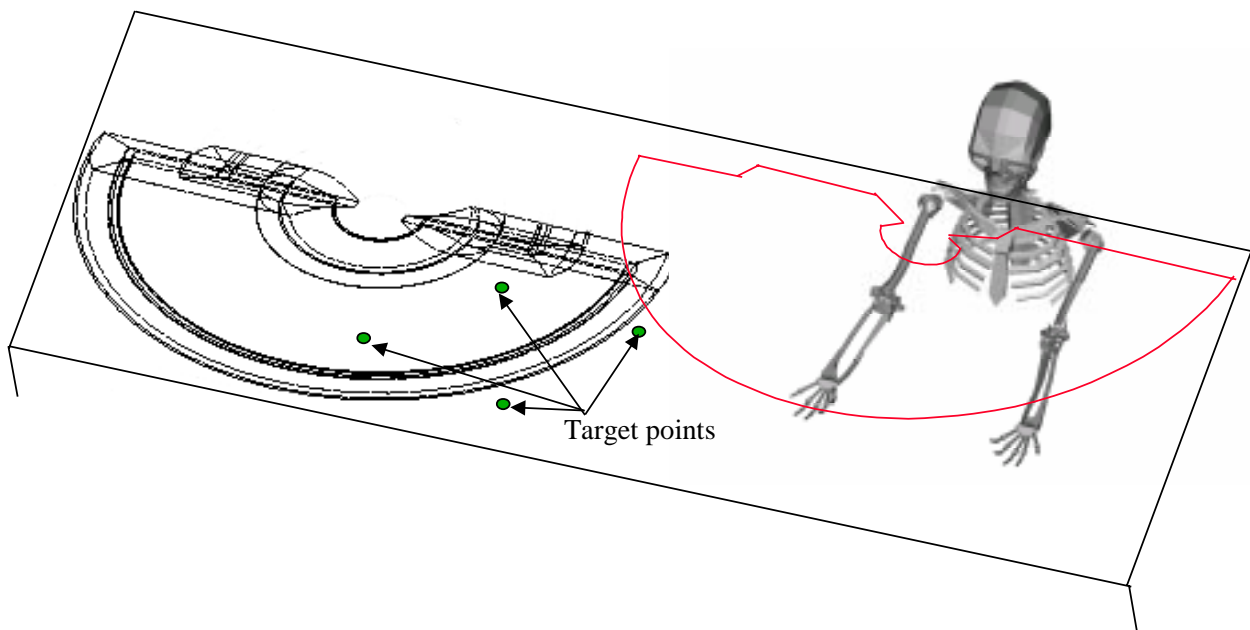


Fig. 13 Moving the workspace towards reaching all target points with minimum distance

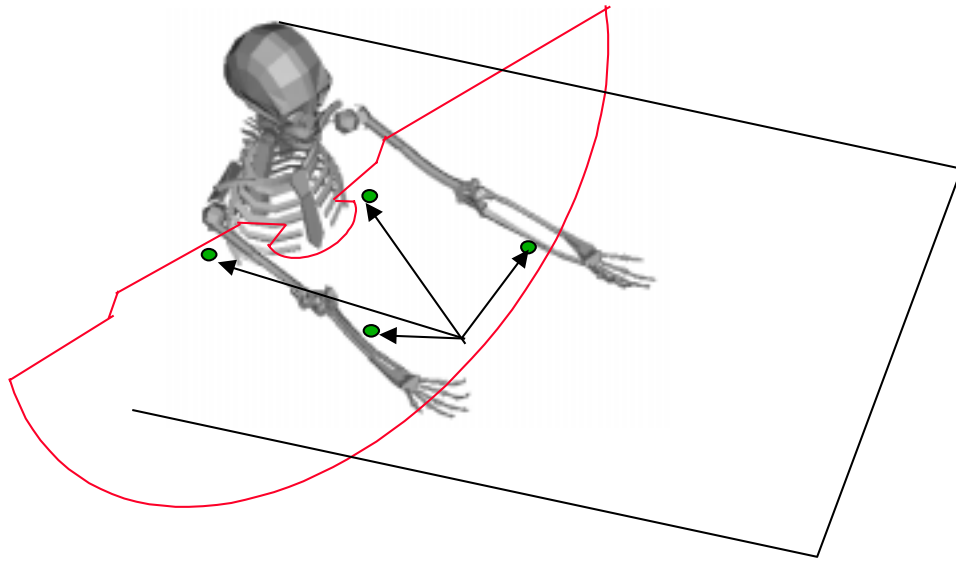


Fig. 14 The new configuration of the seating

### Example 3: Lower Extremity For Minimum Torque

Consider the lower extremity of a human modeled as four revolute joints shown in Fig. 15a, with the following joint constraints:  $-\pi/2 \leq q_4 \leq \pi/2$ ,  $-\pi \leq q_3 \leq \pi/6$ ,  $0 \leq q_2 \leq 330^\circ$ , and  $0 \leq q_1 \leq 210^\circ$ . The workspace envelope of a point  $s_p$  (at the tip of the foot) with respect to the hip is shown in Fig. 15.

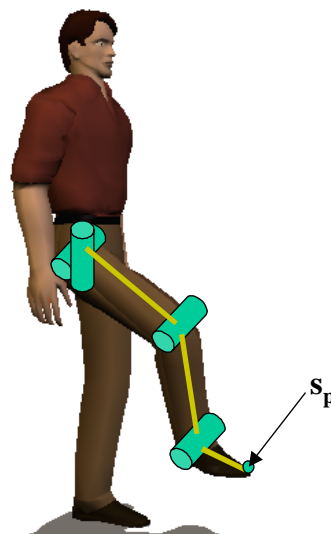


Fig. 15 Modeling of the lower extremity as a 4-revolute kinematic chain

Consider a situation where it is required to design the interior of a vehicle such that the driver is seated in a way that would allow his/her foot to exert a given force on a number of pedals yet while minimizing stress on each joint. The designer will use a human modeling and simulation code to perform the analysis and finally select a design.

The position vector function characterizing the workspace is given by:

$$\Phi(\mathbf{q}) = \begin{bmatrix} \cos q_1 (10 + 5 \cos q_2 + 3 \cos(q_2 + q_3) + \cos(q_2 + q_3 + q_4)) \\ \sin q_1 (10 + 5 \cos q_2 + 3 \cos(q_2 + q_3) + \cos(q_2 + q_3 + q_4)) \\ 7 + 5 \sin q_2 + 3 \sin(q_2 + q_3) + \sin(q_2 + q_3 + q_4) \end{bmatrix} \quad (23)$$

Jacobian rank-deficiency conditions applied to  $\Phi_{\mathbf{q}} \mathbf{q}_i$  yield a number of singular sets, and surface patches on the boundary are delineated and characterized by the following sets:

$$\begin{aligned} & \mathbf{x}^{(17b)}(q_1, q_3); 0 \leq q_1 \leq 210^\circ \text{ and } -\pi \leq q_3 \leq -175^\circ \\ & \mathbf{x}^{(9)}(q_1, q_4); 0 \leq q_1 \leq 210^\circ \text{ and } -17^\circ \leq q_4 \leq 0^\circ \\ & \mathbf{x}^{(15)}(q_1, q_2); 0 \leq q_1 \leq 210^\circ \text{ and } 0^\circ \leq q_2 \leq 330^\circ \\ & \mathbf{x}^{(19)}(q_1, q_2); 0 \leq q_1 \leq 210^\circ \text{ and } 0 \leq q_2 \leq 330^\circ \\ & \mathbf{x}^{(17c)}(q_1, q_3); 0 \leq q_1 \leq 210^\circ \text{ and } 34^\circ \leq q_3 \leq 7^\circ \\ & \mathbf{x}^{(18)}(q_1, q_3); 0 \leq q_1 \leq 210^\circ \text{ and } 0^\circ \leq q_3 \leq 30^\circ \\ & \mathbf{x}^{(10)}(q_1, q_4); 0 \leq q_1 \leq 210^\circ \text{ and } -20^\circ \leq q_4 \leq 20^\circ \\ & \mathbf{x}^{(5)}(q_1, q_2); 0 \leq q_1 \leq 210^\circ \text{ and } 276^\circ \leq q_2 \leq 283^\circ, \\ & \mathbf{x}^{(17a)}(q_1, q_3); 0 \leq q_1 \leq 210^\circ \text{ and } -154^\circ \leq q_3 \leq -70^\circ, \\ & \mathbf{x}^{(8)}(q_1, q_2); 0 \leq q_1 \leq 210^\circ \text{ and } 293^\circ \leq q_2 \leq 312^\circ, \text{ and} \\ & \mathbf{x}^{(11)}(q_1, q_3); 0 \leq q_1 \leq 210^\circ \text{ and } -142^\circ \leq q_3 \leq -30^\circ. \end{aligned}$$

Substituting these sets into Eq. (23) yields parametric equations of singular surfaces shown in Fig. 16 (a cross section of the workspace is shown).

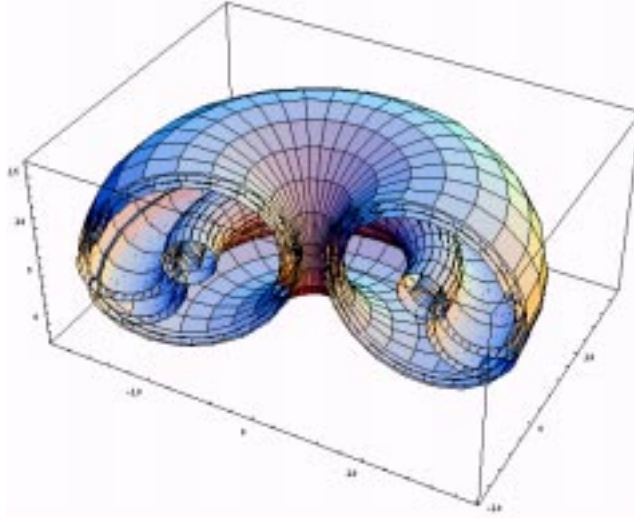


Fig. 16 Workspace envelope of the lower extremity

In order to perform the iterative algorithm while minimizing torque on joints, it is necessary to develop an analytic expression for the Jacobian of the 4-revolute model relating the linear and angular velocity to joint rates as:

$$\mathbf{J}^{4dof}(\mathbf{q}) = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33} & J_{34} \\ J_{41} & J_{42} & J_{43} & J_{44} \\ J_{51} & J_{52} & J_{53} & J_{54} \\ J_{61} & J_{62} & J_{63} & J_{64} \end{bmatrix}_{6 \times 4}$$

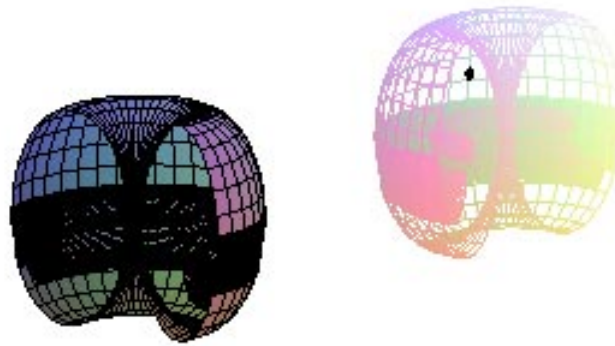
Where

$$\begin{aligned}
J_{14} &= -\cos(q_1) * \sin(q_2 + q_3 + q_4) \\
J_{24} &= -\sin(q_1) * \sin(q_2 + q_3 + q_4) \\
J_{34} &= \cos(q_2 + q_3 + q_4) \\
J_{13} &= -\cos(q_1) * (3.0 * \sin(q_2 + q_3) + \sin(q_2 + q_3 + q_4)) \\
J_{23} &= -\sin(q_1) * (3.0 * \sin(q_2 + q_3) + \sin(q_2 + q_3 + q_4)) \\
J_{33} &= 3.0 * \cos(q_2 + q_3) + \cos(q_2 + q_3 + q_4) \\
J_{12} &= -\cos(q_1) * (5.0 * \sin(q_2) + 3.0 * \sin(q_2 + q_3) + \sin(q_2 + q_3 + q_4)) \\
J_{22} &= -\sin(q_1) * (5.0 * \sin(q_2) + 3.0 * \sin(q_2 + q_3) + \sin(q_2 + q_3 + q_4)) \\
J_{32} &= 5.0 * \cos(q_2) + 3.0 * \cos(q_2 + q_3) + \cos(q_2 + q_3 + q_4) \\
J_{11} &= -\sin(q_1) * (10.0 - \sin(q_2) * ((3 + \cos(q_4)) * \sin(q_3) + \cos(q_3) * \sin(q_4)) + \\
&\quad \cos(q_2) * (5 + \cos(q_3) * (3 + \cos(q_4)) - \sin(q_3) * \sin(q_4))) \\
J_{21} &= \cos(q_1) * (10.0 + 5.0 * \cos(q_2) + 3.0 * \cos(q_2 + q_3) + \cos(q_2 + q_3 + q_4)) \\
J_{31} &= 0.0 \\
J_{41} &= 0.0 \\
J_{42} &= \sin(q_1) \\
J_{43} &= \sin(q_1) \\
J_{44} &= \sin(q_1) \\
J_{51} &= 0.0 \\
J_{52} &= -\cos(q_1) \\
J_{53} &= -\cos(q_1) \\
J_{54} &= -\cos(q_1) \\
J_{61} &= 1.0 \\
J_{62} &= 0.0 \\
J_{63} &= 0.0 \\
J_{64} &= 0.0
\end{aligned} \tag{24}$$

For the given target point at  $\mathbf{P}(30,30,10)$  and a given force vector at the foot of  $\mathbf{F}(1, 1, 1, 1, 1, 1)$ , we implement the proposed method. The iterative optimization algorithm yields the coordinates of the hip at  $(22.6, 20.6, 1.25)$  with the following orientation angles  $(2.76, -3.14, -3.14)$  (units in radians). The corresponding posture for minimum torque on all joints is when

$$\begin{aligned}
q_1 &= 2.3284 \\
q_2 &= 2.2723 \\
q_3 &= 0.3451 \\
q_4 &= 1.0154
\end{aligned}$$

The initial and final positions of the workspace envelope are shown in Fig. 17.



## Conclusions

Ergonomic design has depended upon trial and error and vastly on empirical data, whereby experience and rules of thumb are used. Nevertheless, it is widely understood that ergonomic design is indeed an optimization problem with many parameters. This paper presents a mathematical approach to ergonomic design using well-established methods in optimization and workspace analysis.

It was shown that the workspace envelope of any human limb is rigorously identified using criteria developed from the row rank deficiency of the Jacobian, while taking into consideration ranges of motion. These ranges are imposed in terms of inequality constraints that are converted into equality constraints. The boundary surfaces/curves are determined in closed form and readily visualized. Boundary surface patches (or curve segments) were delineated. Placement of a person in the environment is performed by driving the workspace towards the target points using the cost function and subject to a set of constraints. The result is an ability to visualize the limb's workspace envelope in 3D and to mathematically explore operations that can be conducted on this workspace.

Because of the closed-form nature of these surfaces, it was shown that an ergonomic design is achieved based on a cost function and constraints defined by the user. For example, if the objective is to have the arm reach all points in the workspace, then the distance function to target points can be used. If the goal is to place the human such that he/she can reach a number of points while minimizing stress on each joint, a different cost function is imposed. The result is a placement of the workspace envelope (and hence the human) such that the cost function is optimized.

The motivation for such work stems from a need for accurately designing furniture, workstations, assembly lines, vehicle interiors, and general ergonomic workspaces. Whether for the design of control rooms, the placement of instrumentation and gauges, or for the design of furniture, this method will yield accurate mathematically-based results.

The potential for this work to play a key role in other fields is relevant. Future implementation of this work is aimed at designing the workspace to minimize repetitive motions (leading to injuries), and to place humans while minimizing low back pain. This group is currently developing analytic models of such cost functions. The formulation developed herein was illustrated through examples of the upper and lower extremities with various cost functions.

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### Appendix A (Parameterization)

A convenient parameterization of constraints imposed on  $\mathbf{q}$  was presented such that joint inequality constraints  $q_i^L \leq q_i \leq q_i^U$  are given by

$$\mathbf{q}(\lambda) = (\mathbf{q}^U + \mathbf{q}^L)/2 + [(\mathbf{q}^U - \mathbf{q}^L)/2] \sin \lambda \quad (\text{A.1})$$

where  $\mathbf{q}^U = [q_1^U \ \dots \ q_n^U]^T$  and  $\mathbf{q}^L = [q_1^L \ \dots \ q_n^L]^T$  are the upper and lower joint limits, respectively, and  $\lambda = [\lambda_1 \ \dots \ \lambda_n]^T$  are the new variables that have been introduced by adding as many equations as the number of variables without reducing the dimensionality of the problem ( $\lambda$  are usually called slack variables in the field of optimization).

### Appendix B (Type II Singular sets)

$$\begin{aligned} \mathbf{s}_8 &= \{q_1 = q_1^L, q_2 = q_2^U, q_5 = 0\}, \quad \mathbf{s}_9 = \{q_1 = q_1^U, q_2 = q_2^L, q_5 = 0\}, \\ \mathbf{s}_{10} &= \{q_1 = q_1^U, q_2 = q_2^U, q_5 = 0\}, \quad \mathbf{s}_{11} = \{q_1 = q_1^L, q_3 = q_3^L, q_5 = 0\}, \quad \mathbf{s}_{12} = \{q_1 = q_1^U, q_3 = q_3^L, q_5 = 0\} \\ \mathbf{s}_{13} &= \{q_1 = q_1^L, q_4 = q_4^L, q_5 = 0\}, \quad \mathbf{s}_{14} = \{q_1 = q_1^L, q_4 = q_4^U, q_5 = 0\}, \quad \mathbf{s}_{15} = \{q_1 = q_1^U, q_4 = q_4^L, q_5 = 0\}, \\ \mathbf{s}_{16} &= \{q_1 = q_1^U, q_4 = q_4^U, q_5 = 0\}, \quad \mathbf{s}_{17} = \{q_2 = q_2^L, q_3 = q_3^L, q_4 = 0\}, \quad \mathbf{s}_{18} = \{q_2 = q_2^L, q_3 = q_3^U, q_4 = 0\}, \\ \mathbf{s}_{19} &= \{q_2 = q_2^U, q_3 = q_3^L, q_4 = 0\}, \quad \mathbf{s}_{20} = \{q_2 = q_2^U, q_3 = q_3^U, q_4 = 0\}, \quad \mathbf{s}_{21} = \{q_2 = q_2^L, q_4 = q_4^L, q_5 = 0\}, \\ \mathbf{s}_{22} &= \{q_2 = q_2^L, q_4 = q_4^U, q_5 = 0\}, \quad \mathbf{s}_{23} = \{q_2 = q_2^U, q_4 = q_4^L, q_5 = 0\}, \quad \mathbf{s}_{24} = \{q_2 = q_2^U, q_4 = q_4^U, q_5 = 0\}, \\ \mathbf{s}_{25} &= \{q_2 = q_2^L, q_4 = 0, q_5 = q_5^L\}, \quad \mathbf{s}_{26} = \{q_2 = q_2^L, q_4 = 0, q_5 = q_5^U\}, \quad \mathbf{s}_{27} = \{q_2 = q_2^U, q_4 = 0, q_5 = q_5^U\}, \\ \mathbf{s}_{28} &= \{q_3 = q_3^L, q_4 = q_4^L, q_5 = 0\}, \quad \mathbf{s}_{29} = \{q_3 = q_3^L, q_4 = q_4^U, q_5 = 0\}, \quad \mathbf{s}_{30} = \{q_3 = q_3^L, q_5 = q_5^L, q_4 = 0\}, \\ \mathbf{s}_{31} &= \{q_3 = q_3^L, q_5 = q_5^U, q_4 = 0\}, \quad \text{and } \mathbf{s}_{32} = \{q_1 = q_1^L, q_2 = q_2^L, q_5 = 0\}. \end{aligned}$$