Crossable Surfaces of Robotic Manipulators with Joint Limits

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Abstract

A broadly applicable formulation for determining the crossability of singular surfaces and curves in the workspace of serial robotic manipulators is presented. Singular surfaces and curves are analytically determined using nullspace rank-deficiency criteria of the mechanism's constraint position Jacobian. Imposed joint limits in terms of inequality constraints are taken into account in the formulation. Directions of admissible normal movements on a surface or curve are established from the analysis of the normal curvature of singular surfaces. The normal curvature of a parametric surface used in this formulation, is determined from the first and second fundamental forms adapted from differential geometry. Definiteness properties of a quadratic form developed from the acceleration analysis determine admissible normal movements. For singular surfaces resulting from active joint constraints, definiteness properties are not enough and supplementary criteria are necessary. Two additional criteria are derived. This paper is a complete treatment of the problem of determining whether a singular surface/curve in the workspace is crossable. Planar and spatial numerical examples are presented to illustrate the formulation.

Introduction

The problem of determining singular behavior of mechanical manipulators has received considerable attention. The reciprocal screw method for workspace generation, for example, is based on the fact that when the hand reference point of the manipulator is on the workspace boundary when the manipulator has a spherical wrist, all the joint axes of a manipulator are reciprocal to a zero pitch wrench axis (Sugimoto 1982). For each degree-of-freedom (DOF) lost, there exists one reciprocal screw that, if applied as a wrench to the end-effector, produces no virtual work for the manipulator joints. Wang and Waldron (1986), based upon earlier work (Waldron, *et al.* 1985), stated that as the Jacobian of the manipulator becomes singular, its columns, which are screw quantities, do not span the full rank of the matrix, therefore, the Jacobian rank decreases by at least one. Proof of the nullspace criteria applied to the Jacobian was presented by Spanos and Kohli (1985).

Other methods that are based on Jacobian singularity can be found in (Shu, *et al.* 1983, Litvin, *et al.* 1986a, and 1986b). An enumeration of singular configurations due to the vanishing of the determinant of the Jacobian and the Jacobian's minors is presented by Lipkin and Pohl (1991). Shamir (1990) provided an analytical tool to determine if the singularities are avoidable for three

DOF manipulators. Gorla (1981) was able to get expressions for the set of singular points by assuming that link twists were multiple of $\pi/2$. Geometric approaches to the study of singular configurations of a manipulator arm were addressed by Lai and Yang (1986), Ahmad and Luo (1988), and Tourassis and Ang (1992). Other earlier important studies that discussed manipulator singularities include Soylu and Duffy (1988) and Lai and Yang (1984).

Early studies that have addressed difficulties in the control of manipulators due to the appearance of interior curves and surfaces were reported by Waldron (1987) and Nielsen, *et al.* (1991). In the latter work, the controllability of a mechanical arm is discussed from a differential geometry point of view. Nielsen, *et al.* showed control difficulties based upon vector fields and their consecutive Lie brackets spanning a state space. A criterion to define possible motion (so-called feasible trajectory) from a singularity was presented by Chevallereau and Daya (1994) and Chevallereau (1996).

The fundamental concept of crossable and noncrossable surfaces inside a manipulator's workspace was addressed by Oblak and Kohli (1988). Although their report has touched upon a crossable surface using the singularity criteria of the Jacobian, the paper does not present a unified method for identifying a crossable surface or curve, neither does it determine directions of admissible movement. Further work on the issue of singularity determination and the structure of singular regions inside the workspace was presented by Pai and Leu (1992), Burdick (1991; 1992), Karger (1995), and Merlet (1989).

Recently, Haug, *et al.* (1995; 1996) presented a *numerical* algorithm for identifying and analyzing barriers to output control of manipulators using first and second-order Taylor approximations of the output in selected directions. Haug, *et al.* showed that the output velocity in the direction normal to such curves and surfaces must be zero. The work was extended to the investigation of the domains of mobility for a planar body moving among obstacles (Haug, *et al.* 1998) and to the study the workspace (called operational envelope) of the Stewart platform (Adkins and Haug 1997). It should be noted that this method is completely numerical, yields only curves on the boundary that must be manually meshed to depict the envelope. Similar work based on the Euler-Rodrigues parameters and aimed at determining the workspace of serial and platform manipulators through the occurrence of singular behavior (Bulca, *et al.* 1999).

Recently, Abdel-Malek and Yeh (1997a) presented a general method by which all singular surfaces and curves can be analytically determined for a 3DOF serial manipulator and hence obtained a closed form of the surface patches enveloping the workspace. The method and algorithm were extended to the concept of swept volumes whereby the volume generated by the motion of a geometric entity in space is numerically determined (Abdel-Malek and Yeh 1997b) and was shown to apply to solid modeling. The concept of singular curves and surfaces was then used to determine feasible path trajectories where the end-effector can cross a given barrier (Abdel-Malek and Yeh 1997c). The formulation therein was compared with the results of the numerical method by Haug, *et al.* (1996) in a joint paper (Abdel-Malek, *et al.* 1997). The same theory implemented for the determination of the serial 5-DOF manipulator was presented by

Abdel-Malek, *et al.* (1999) and reported singular surfaces that are boundary to the workspace but that can be further stratified.

We first present a rigorous formulation for identifying singular curves and surfaces internal and on the boundary to the workspace envelope. We then adapt methods from differential geometry to investigate properties associated with a proposed acceleration function on these curves and surfaces. A number of examples are presented to demonstrate the theory and formulation.

Formulation

It has been shown (Abdel-Malek and Yeh 1997a) that singular surfaces/curves comprising the workspace of serial robotic manipulators, can analytically be determined by studying the analytic Jacobian of the mechanism. For a position vector $\mathbf{x} = \Phi(\mathbf{q})$, where $\mathbf{q} \in \mathbf{R}^n$ is the vector of generalized coordinates of an n -DOF manipulator, the dimension of the nullspace of the Jacobian $\Phi_q = [\partial\Phi_i/\partial q_j]$ is associated with singular behavior. Define the subvector of $\mathbf{q}$ as $\mathbf{p}_i \in \mathbf{R}^m$, where $m \leq n-1$ such that $\mathbf{q} \equiv \mathbf{u} \cup \mathbf{p}_i$, where $\mathbf{u} \in \mathbf{R}^{n-m}$.

Generalized coordinates called singular sets, are determined from

$$\mathbf{S} = \{\mathbf{p}_i; \dim[Null(\Phi_q^T(\mathbf{q}))] \geq 1, \mathbf{q} = [\mathbf{u} \ \mathbf{p}_i], \text{ for some } \mathbf{p}_i\} \quad (1)$$

where $\mathbf{p}_i$ are joint variables that are either constant values of joints or defined as functions of other variables. The set of analytic functions generated by a subset of a Euclidean space defined by zeros of a finite number of differentiable functions (sub-determinants) are called varieties (singular sets). The nullspace of the Jacobian corresponds to joint velocities that do not produce any end-effector velocity for a given manipulator configuration $\mathbf{p}_i$. Singular configurations $\mathbf{p}_i$ are obtained from $\mathbf{S}$ and from the reduced-order manipulator condition discussed in Abdel-Malek, *et al.* (1999), which occurs when joint constraints are active. It was shown that substituting $\mathbf{p}_i$ into $\Phi(\mathbf{q})$ yields singular surfaces and curves parameterized by $\Psi(\mathbf{u})$ where $\mathbf{u}$ is the new vector of generalized coordinates containing the remaining variables such that

$$\Phi(\mathbf{u}, \mathbf{p}_i) = \Psi(\mathbf{u}) \quad (2)$$

A proof was presented showing that the basis of the null space of Φ_q^T is a normal vector to singular surfaces/curves in the workspace. i.e., if a vector $\mathbf{n}_o$ evaluated at $\mathbf{q}_o$ satisfies

$$\Phi_q^T \mathbf{n}_o = \mathbf{0} \quad (3)$$

where $\mathbf{n}_o = [\gamma_1 \ \gamma_2 \ \gamma_3]^T$, it is the vector normal to singular surfaces/curves at $\mathbf{q}_o$. This important result was used in the velocity analysis of the end-effector on singular surfaces/curves to show that the end-effector on a singular surface may have only tangential velocities. Acceleration analysis of the end-effector revealed a quadratic form that can be used to establish possible directions of admissible movements. The normal acceleration is derived as

$$a_n = \mathbf{n}_o^T \ddot{\mathbf{x}} = \dot{\mathbf{q}}^T [\mathbf{n}_o^T \Phi]_{qq} \dot{\mathbf{q}} \quad (4)$$

where $\ddot{\mathbf{x}}$ is the acceleration of the end-effector, $\dot{\mathbf{q}}$ is the vector of joint velocities and two subscripts indicate a second derivative of $[\mathbf{n}_o^T \Phi]$. Definiteness properties of the quadratic form

in Eq. (4) predict the direction of admissible normal acceleration, but do not *guarantee* whether the end-effector can admit movement across a singular surface/curve (crossability analysis).

Consider the end-effector at a point on a singular surface with radius of curvature ρ_o , with a specified normal acceleration a_n , and a tangential velocity v_t . This manipulator will admit motion in one normal direction or another subject to

$$a_n - \frac{|v_t^2|}{\rho_o} \quad (5)$$

where $v_t = \dot{\mathbf{x}} = \Phi_q \dot{\mathbf{q}}$ is the tangential velocity and $1/\rho_o$ is the normal curvature of the singular surface *with respect to the tangent direction* of v_t (ρ_o is the radius of curvature). The sign of (5) above establishes the admissible direction of motion.

From the theory of differential geometry (Farin 1993), the *First and Second Fundamental Forms* (denoted by $\mathbf{I}_p$ and $\mathbf{II}_p$, respectively) of a parametric geometric entity $\Psi(\mathbf{u})$, where

$\mathbf{u} = [u \ v]^T$, is defined as

$$\mathbf{I}_p \equiv \delta \mathbf{u}^T \Psi_u^T \Psi_u \delta \mathbf{u} \quad (6)$$

where $\Psi_u = \partial \Psi / \partial u$ and $\Psi_v = \partial \Psi / \partial v$, and

$$\mathbf{II}_p \equiv \delta \mathbf{u}^T [\mathbf{n}^T \Psi]_{uu} \delta \mathbf{u} \quad (7)$$

or expanded as

$$\mathbf{II}_p = \mathbf{n}^T \Psi_{uu} du^2 + 2\mathbf{n}^T \Psi_{uv} du dv + \mathbf{n}^T \Psi_{vv} dv^2 \quad (8)$$

where $\mathbf{n}$ is the normal vector to the singular surface and $\Psi_{uv} = \partial^2 \Psi / \partial u \partial v$, $\Psi_{vv} = \partial^2 \Psi / \partial v \partial v$, and $\Psi_{uu} = \partial^2 \Psi / \partial u \partial u$. The Normal Curvature K_o of a parametric singular surface at a configuration $\mathbf{q}_o$, can then be defined as the ratio

$$K_o = \frac{1}{\rho_o} = \frac{\mathbf{II}_p}{\mathbf{I}_p} \quad (9)$$

Define the *time-Modified First and Second Fundamental Forms* as

$$\mathbf{I}'_p \equiv \dot{\mathbf{u}}^T \Psi_u^T \Psi_u \dot{\mathbf{u}} \quad (10a)$$

$$\mathbf{II}'_p \equiv \dot{\mathbf{u}}^T [\mathbf{n}^T \Psi]_{uu} \dot{\mathbf{u}} \quad (10b)$$

such that the normal curvature can still be defined as

$$K_o = \frac{1}{\rho_o} = \frac{\mathbf{II}_p}{\mathbf{I}_p} = \frac{\mathbf{II}'_p}{\mathbf{I}'_p} \quad (11)$$

The tangent velocity in terms of Ψ or Φ can be written as

$$\Psi_u \dot{\mathbf{u}} = \Phi_q \dot{\mathbf{q}} \quad (12)$$

Hence, it can be shown that the time-Modified First Fundamental Form is indeed the square of the tangential velocity

$$\mathbf{I}'_p = |v_t|^2 \quad (13)$$

Therefore, the difference in acceleration components can be written as

$$\begin{aligned}
a_n - \frac{|v_t|^2}{\rho_o} &= a_n - |v_t|^2 \frac{\mathbf{\Pi}'_p}{\mathbf{I}'_p} \\
&= a_n - \mathbf{\Pi}'_p
\end{aligned} \tag{14}$$

Since a_n is in terms of $\dot{\mathbf{q}}$ and to express $\mathbf{\Pi}'_p$ in terms of $\dot{\mathbf{q}}$, it was shown that the velocity vector on a singular surface can be written as

$$\dot{\mathbf{u}} = [\mathbf{E}\Psi_u]^{-1} \mathbf{E}\Phi_q \dot{\mathbf{q}} \tag{15}$$

where

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{if the first and second rows of } \Psi_u \text{ are independent} \tag{16A}$$

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{if the first and third rows of } \Psi_u \text{ are independent} \tag{16b}$$

$$\mathbf{E} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{if the second and third rows of } \Psi_u \text{ are independent} \tag{16c}$$

For an end-effector on a singular surface, the crossability criteria was then expanded into a quadratic form written as

$$a_n - \frac{|v_t|^2}{\rho_o} = a_n - \mathbf{\Pi}'_p = \dot{\mathbf{q}}^T \mathbf{Q} \dot{\mathbf{q}} \tag{17a}$$

where

$$\mathbf{Q} = [\mathbf{n}^T \Phi]_{qq} - \Phi_q^T \mathbf{B}^T [\mathbf{n}^T \Psi]_{uu} \mathbf{B} \Phi_q \tag{17b}$$

and $\mathbf{B}$ is a generalized inverse of Ψ_u defined in Eq. (15) by

$$\mathbf{B} \equiv [\mathbf{E}\Psi_u]^{-1} \mathbf{E} \tag{18}$$

Definiteness properties of the quadratic form in Eq. (17) indicate the crossability of a singular surface/curve. Since singular surfaces/curves that are non-crossable are impediments to motion, controllability issues are directly related to this formulation.

In this paper, we extend the formulation to manipulators with joint limits. The formulation includes joint parameters that are subject to mobility constraints. Since additional singular surfaces/curves may appear in the workspace due to active joint constraints, velocities of the joints that reached their limits on those surfaces are nulled. The nullification of joint velocities will give rise to an apparent loss of information from the matrix of the quadratic form (Eq. 17). Therefore, additional (supplementary) criteria are developed to determine admissible normal movements on those surfaces/curves.

Extending the Formulation to Include Joint Limits

Joint limits imposed in the form of inequality constraints such as $q_i^{\min} \leq q_i \leq q_i^{\max}$, are parameterized into an equality using the following transformation such that new generalized coordinates λ_i are introduced

$$q_i = a_i + b_i \sin \lambda_i \tag{19}$$

where $a_i = (q_i^{\max} + q_i^{\min})/2$ and $b_i = (q_i^{\max} - q_i^{\min})/2$ are the mid point and half range of the inequality constraint (Haug, *et al.* 1994). The constraint function is then written in terms of the extended vector $\mathbf{s} = [\lambda_1 \ \lambda_2 \ \dots \ \lambda_n]^T$ such that

$$\mathbf{x} = \Phi(\mathbf{s}) = [\Phi(\mathbf{q}(\lambda_i))] \quad (20)$$

Differentiating with respect to time and using the chain rule, the velocity of the end-effector is

$$\dot{\mathbf{x}} = \Phi_{\mathbf{q}} \mathbf{q}_s \dot{\mathbf{s}} \quad (21)$$

where $\mathbf{q}_s = \partial \mathbf{q}_i / \partial \lambda_j$ and $\dot{\mathbf{s}} = d\mathbf{s}/dt$. On a singular surface, the term $\Phi_{\mathbf{q}} \mathbf{q}_s|_{\mathbf{q}_o, \mathbf{s}_o}$ is rank-deficient.

Therefore, the rows of $\Phi_{\mathbf{q}} \mathbf{q}_s$ are dependent and there exists a set of constants $\mathbf{n}_o = [n_1 \ n_2 \ n_3]^T$ that satisfy

$$[\Phi_{\mathbf{q}} \mathbf{q}_s]^T \mathbf{n}_o = \mathbf{0} \quad (22)$$

where $\mathbf{n}_o$ is the vector normal to a singular surface at $(\mathbf{q}_o, \mathbf{s}_o)$. In addition, the normal acceleration of the end-effector can be written as:

$$\ddot{x}_i = \sum_{j=1}^n \left\{ \left[\frac{d}{dt} \left(\frac{\partial \Phi_i}{\partial q_j} \right) \right] \left(\frac{\partial q_j}{\partial \lambda_j} \right) \dot{\lambda}_j + \left(\frac{\partial \Phi_i}{\partial q_j} \right) \left[\frac{d}{dt} \left(\frac{\partial q_j}{\partial \lambda_j} \right) \right] \dot{\lambda}_j + \left(\frac{\partial \Phi_i}{\partial q_j} \right) \left(\frac{\partial q_j}{\partial \lambda_j} \right) \ddot{\lambda}_j \right\} \quad (23)$$

where $\frac{d}{dt}(\partial \Phi_i / \partial q_j) \equiv (\partial^2 \Phi_i / \partial q_\ell \partial q_j)(\partial q_\ell / \partial \lambda_\ell) \dot{\lambda}_\ell$

where $\ell = 1, \dots, n$ and $\frac{d}{dt}(\partial q_j / \partial \lambda_j) \equiv (\partial^2 q_j / \partial \lambda_\ell \partial \lambda_j) \dot{\lambda}_\ell$

Expanding terms in the form of joint rates and collecting on similar terms yields

$$\ddot{x}_i = \sum_{j=1}^n \left\{ \dot{\lambda}_k \left[\left(\frac{dq_k}{d\lambda_k} \right) \left(\frac{\partial^2 \Phi_i}{\partial q_k \partial q_j} \right) \left(\frac{\partial q_j}{\partial \lambda_k} \right) + \left(\frac{\partial \Phi_i}{\partial q_j} \right) \left(\frac{\partial^2 q_j}{\partial \lambda_k \partial \lambda_j} \right) \right] \dot{\lambda}_j + \left(\frac{\partial \Phi_i}{\partial q_j} \right) \left(\frac{\partial q_j}{\partial \lambda_j} \right) \ddot{\lambda}_j \right\} \quad (24)$$

where $k = 1, \dots, n$. Written in matrix form, the acceleration is

$$[\ddot{\mathbf{x}}]_i = \dot{\lambda}^T \mathbf{q}_{\lambda}^T [\Phi_i]_{\mathbf{q}\mathbf{q}} \mathbf{q}_{\lambda} \dot{\lambda} + \dot{\lambda}^T \sum_{i=1}^n \left(\frac{d\Phi_i}{dq_i} [q_i]_{\lambda\lambda} \right) \dot{\lambda} + [\Phi_i]_{\mathbf{q}} \mathbf{q}_{\lambda} \ddot{\lambda} \quad (25)$$

where $[\Phi_i]_{\mathbf{q}\mathbf{q}} = \begin{bmatrix} \partial \Phi_i / \partial q_1 \partial q_1 & \dots & \partial \Phi_i / \partial q_1 \partial q_n \\ \vdots & \ddots & \vdots \\ \partial \Phi_i / \partial q_n \partial q_1 & \dots & \partial \Phi_i / \partial q_n \partial q_n \end{bmatrix}$ and $[q_i]_{\lambda\lambda} = \begin{bmatrix} \partial q_i / \partial \lambda_1 \partial \lambda_1 & \dots & \partial q_i / \partial \lambda_1 \partial \lambda_n \\ \vdots & \ddots & \vdots \\ \partial q_i / \partial \lambda_n \partial \lambda_1 & \dots & \partial q_i / \partial \lambda_n \partial \lambda_n \end{bmatrix}$.

To obtain the normal acceleration, $\ddot{\mathbf{x}}$ is projected onto the normal $\mathbf{n}_o$. Multiplying both sides of Eq. (25) by the vector $\mathbf{n}_o^T$ eliminates the last term of the right hand side (definition of the normal as the basis of the null space in Eq. 22).

The component of the normal acceleration is then

$$a_n = \mathbf{n}_o^T \ddot{\mathbf{x}} = \dot{\mathbf{s}}^T \mathbf{H}^* \dot{\mathbf{s}} \quad (26)$$

where

$$\mathbf{H}^*(\mathbf{q}_o, \mathbf{s}_o) = \mathbf{q}_s^T [\mathbf{n}_o^T \Phi]_{\mathbf{q}\mathbf{q}} \mathbf{q}_s + \sum_{i=1}^n \frac{d(\mathbf{n}_o^T \Phi)}{dq_i} \cdot [q_i]_{ss} \quad (27)$$

In the formulation above (not considering joint limits), definiteness properties of Eq. (17) depended on the $\mathbf{Q}$ matrix. In this extended formulation (considering joint limits), the quadratic form is written in terms of the $\mathbf{Q}^*$ matrix as

$$a_n - \mathbf{\Pi}'_p = \dot{\mathbf{s}}^T \mathbf{Q}^* \dot{\mathbf{s}} \quad (28)$$

where

$$\mathbf{Q}^* = \mathbf{H}^* - \mathbf{q}_s^T \Phi_q^T \mathbf{B}^T [\mathbf{n}^T \Psi]_{\text{uu}} \mathbf{B} \Phi_q \mathbf{q}_s \quad (29)$$

Definiteness properties of the quadratic form of Eq. (28) defines the crossability of a singular surface/curve.

Singular Surfaces Due to Active Joint Constraints

For a singular surface that is due to singularities that are at the upper or lower bounds of an inequality constraint, the surface will definitely be crossable if $\mathbf{Q}^*$ has both positive and negative eigenvalues. However, when $\mathbf{Q}^*$ is either positive semi-definite or negative semi-definite, the singular surface/curve may still be crossable. On those surfaces, the matrix of the quadratic form loses some terms because joint velocities are zero. Joint velocities at a singular configuration derived from Eq. (19) (i.e., $\dot{q}_i = [q_i^{\max} - q_i^{\min}/2] \cos \lambda_i$) are zero at an upper or lower bound ($\lambda_i = -\pi/2$ or $\pi/2$, i.e., $q_i = q_i^{\max}$ or $q_i = q_i^{\min}$). Therefore, some terms in $\mathbf{Q}^*$ are lost. Supplementary criteria are necessary. To motivate the discussion, consider the two DOF (one prismatic and one revolute) planar manipulator shown in Fig. (1a) with joint limits given by $0 \leq q_1 \leq 4$ and $(-\pi/4) \leq q_2 \leq \pi$.

Figure 1. (a) A two DOF manipulator and (b) Some singular curves due to joint limits (c) All singular curves (d) Workspace envelope

Since q_2 has its lower and upper bounds as $-\pi/4$ and π , the lines Ψ_1 and Ψ_2 are the two corresponding singular curves. For the two cases when the end-effector is at points $\mathbf{p}_1$ and $\mathbf{p}_2$ as shown in Fig. (1b), the joint velocity of q_2 will always be zero (i.e., $\dot{q}_2 = (5\pi/8) \cos(-\pi/2)$). Therefore, there will not be any component of normal acceleration at point $\mathbf{p}_1$ nor at $\mathbf{p}_2$. Definiteness properties of $\mathbf{Q}^*$ are, therefore, not sufficient to determine all possible directions of motion for these two cases. It should be noted that determination of the workspace is readily carried out first by plotting all singular curves (Fig. 1c) and identifying the boundary (Fig. 1d).

To address this case, we propose the projection of a variational movement $\delta \mathbf{x} = \mathbf{x}_{q_i} \delta q_i$ due to δq_i onto the normal direction $\mathbf{n}$ such that the normal component

$$\sigma = \mathbf{n}^T \mathbf{x}_{q_i} \delta q_i \quad (30)$$

determines admissible normal movement, where δq_i is given a magnitude of ± 1 as follows

$$\delta q_i = \begin{cases} +1 & \text{if } q_i \text{ is at lower bound} \\ -1 & \text{if } q_i \text{ is at upper bound} \end{cases} \quad (31)$$

Positive values of σ in Eq. (30) indicate that the end-effector can admit movement in the positive direction of $\mathbf{n}$.

Thus, for a singular surface that is due to an active joint constraint, Eq. (30) provides supplementary information in lieu of terms lost due to the enforced zero joint velocity. For example, consider the case at point $\mathbf{p}_1$ again. Since $\mathbf{Q}^*$ at this point has only zero eigenvalues, it is necessary to use Eq. (30) with $\partial q_2 = +1$ since joint 2 is at the lower limit. Equation (30) yields a positive value, which indicates that the end-effector admits motion into the positive direction of $\mathbf{n}_1$.

Another situation arises when the normal vector $\mathbf{n}$ is perpendicular to $\mathbf{x}_{q_i}$. In this case, σ in Eq. (30) evaluates to zero. Therefore, the direction of normal movement cannot be determined. The normal curvature of the singular surface with respect to the tangent direction of $\mathbf{x}_{q_i} \delta q_i$ is compared with the normal curvature of the trajectory curve when only q_i is varying (other joint variables are held constant).

To determine the normal direction of admissible movement, it is necessary to evaluate the sign of the difference in the normal curvature. Substituting for the first and second fundamental forms (Eqs. 6 and 7) into the normal curvature (Eq. 9) yields

$$K_2 - K_1 = \frac{\delta q_i [\mathbf{n}^T \mathbf{x}]_{q_i q_i} \delta q_i}{\delta q_i \mathbf{x}_{q_i}^T \mathbf{x}_{q_i} \delta q_i} - \frac{\delta \mathbf{u}^T [\mathbf{n}^T \Psi]_{\mathbf{u}\mathbf{u}} \delta \mathbf{u}}{\delta \mathbf{u}^T \Psi_{\mathbf{u}}^T \Psi_{\mathbf{u}} \delta \mathbf{u}} \quad (32)$$

where K_1 is the normal curvature of Ψ_2 with respect to the tangent direction $\mathbf{x}_{q_i} \partial q_i$, and K_2 is the normal curvature of the trajectory curve when q_i is varying while other joints are held constant.

The case at point $\mathbf{p}_2$ in Fig. (1b), for example, exhibits a zero normal curvature of Ψ_2 . The curvature of the curve due to varying only q_2 is positive with respect to the normal vector $\mathbf{n}_2$. Therefore, the end-effector can move into the side of positive direction of $\mathbf{n}_2$ by decreasing q_2 .

Both curves of the right-hand side of Eq. (32) must have the same tangent vector given by Eq. (12) such that

$$\Psi_{\mathbf{u}} \delta \mathbf{u} = \mathbf{x}_{q_i} \delta q_i \quad (33)$$

where $\mathbf{u}$ is the vector of parametric coordinates for the corresponding singular surface/curve. To simplify the calculation, δq_i can be determined from Eq. (31).

Rearranging Eq. (33) and using the inverse defined in Eq. (18), the vector $\delta \mathbf{u}$ can be written as

$$\delta \mathbf{u} = \mathbf{B} \mathbf{x}_{q_i} \delta q_i \quad (34)$$

Substituting Eq. (34) into Eq. (32) and simplifying yields

$$K_2 - K_1 = \frac{\delta q_i [\mathbf{n}^T \mathbf{x}]_{q_i q_i} \delta q_i - \delta q_i \mathbf{x}_{q_i}^T \mathbf{B}^T [\mathbf{n}^T \mathbf{x}]_{\mathbf{uu}} \mathbf{B} \mathbf{x}_{q_i} \delta q_i}{\delta q_i^T \mathbf{x}_{q_i}^T \mathbf{x}_{q_i} \delta q_i} \quad (35)$$

Since the denominator of the right-hand side of Eq. (35) is always positive, the sign is dependent on the numerator. Define the numerator as μ where

$$\mu = \delta q_i [\mathbf{n}^T \mathbf{x}]_{q_i q_i} \delta q_i - \delta q_i \mathbf{x}_{q_i}^T \mathbf{B}^T [\mathbf{n}^T \mathbf{x}]_{\mathbf{uu}} \mathbf{B} \mathbf{x}_{q_i} \delta q_i \quad (36)$$

which can be written as

$$\mu = \left\{ [\mathbf{n}^T \mathbf{x}]_{q_i q_i} - \mathbf{x}_{q_i}^T \mathbf{B}^T [\mathbf{n}^T \mathbf{x}]_{\mathbf{uu}} \mathbf{B} \mathbf{x}_{q_i} \right\} \delta q_i^2 \quad (37)$$

Since δq_i^2 is always positive, the sign of $K_1 - K_2$ depends on the term inside the brackets. Define

$$\mu = \bar{K} \delta q_i^2 \quad (38)$$

where

$$\bar{K} = \mathbf{x}_{q_i}^T \mathbf{B}^T [\mathbf{n}^T \mathbf{x}]_{\mathbf{uu}} \mathbf{B} \mathbf{x}_{q_i} - [\mathbf{n}^T \mathbf{x}]_{q_i q_i} \quad (39)$$

Therefore, if $\bar{K} > 0$, the end-effector can admit movement into the positive direction of $\mathbf{n}$. If $\bar{K} < 0$, the end-effector admits movement into the negative direction of $\mathbf{n}$.

From the above discussion, it can be seen that Eq. (30) and (38) are used to supplement the criteria of the definiteness properties of $\mathbf{Q}^*$ to determine crossability of a singular surface with joint limits. Crossability criteria are summarized as follows.

Crossability Criteria

- (1) If $\mathbf{Q}^*$ is indefinite (has both positive and negative eigenvalues), the singular surface/curve is **crossable** for each joint q_i which is at its limit.
- (2) If $\mathbf{Q}^*$ is either positive semi-definite or negative semi-definite, the following additional criteria must be evaluated.
 - (a) If $\mathbf{n}_o$ is not perpendicular to $\mathbf{x}_{q_i}$, then σ in Eq. (30) must be evaluated
 - (b) If $\mathbf{n}_o$ is perpendicular to $\mathbf{x}_{q_i}$, then $\bar{K}$ in Eq. (38) must be evaluated.
 - (i) If any of σ or $\bar{K}$ has a different sign than the nonzero eigenvalues of $\mathbf{Q}^*$, the singular surface/curve is crossable.
 - (ii) If σ and $\bar{K}$ have the same sign as the nonzero eigenvalues of $\mathbf{Q}^*$, the singular surface/curve is non-crossable.

The direction along which the end-effector can move, either in the positive or negative direction of $\mathbf{n}$, will be in the same sense of signs of σ and $\bar{K}$.

Example 1: A Planar 3-bar Linkage

This example is presented to demonstrate the implementation of the presented formulation and to validate the results in comparison with those reported by Haug, *et al.* (1994) for the planar three-bar linkage shown in Fig. 2 with the following imposed limits $-\frac{\pi}{3} \leq q_i \leq \frac{\pi}{3}; i = 1...3$.

Figure 2. A planar three DOF manipulator

The position vector of a point on the end-effector is defined by

$$\Phi(\mathbf{q}) = \begin{bmatrix} 4 \cos q_1 + 2 \cos(q_1 + q_2) + \cos(q_1 + q_2 + q_3) \\ 4 \sin q_1 + 2 \sin(q_1 + q_2) + \sin(q_1 + q_2 + q_3) \end{bmatrix} \quad (40)$$

The (2×3) analytic Jacobian matrix is computed as

$$\Phi_q = \begin{bmatrix} -4 \sin q_1 - 2 \sin(q_1 + q_2) - \sin(q_1 + q_2 + q_3) & -2 \sin(q_1 + q_2) - \sin(q_1 + q_2 + q_3) & -\sin(q_1 + q_2 + q_3) \\ 4 \cos q_1 + 2 \cos(q_1 + q_2) + \cos(q_1 + q_2 + q_3) & 2 \cos(q_1 + q_2) + \cos(q_1 + q_2 + q_3) & \cos(q_1 + q_2 + q_3) \end{bmatrix} \quad (41)$$

and $\mathbf{q}_s = \begin{bmatrix} (2\pi/6) \cos \lambda_1 & 0 & 0 \\ 0 & (2\pi/6) \cos \lambda_2 & 0 \\ 0 & 0 & (2\pi/6) \cos \lambda_3 \end{bmatrix}.$

Only one singular set results from the $\dim[Null(\Phi_q^T(\mathbf{q}))] \geq 1$ defined by $\mathbf{p}_1 = \{q_2 = 0, q_3 = 0\}$.

Singularities resulting from the reduced order manipulator (Abdel-Malek and Yeh 1997a) are $S_2 = \{\mathbf{p}_i, i = 2, \dots, 7\}$ where $\mathbf{p}_2 = \{q_1 = -\pi/3, q_3 = 0\}$, $\mathbf{p}_3 = \{q_1 = \pi/3, q_3 = 0\}$,

$$\mathbf{p}_4 = \{q_2 = -\pi/3, q_3 = \tan^{-1}(\sqrt{3}/2)\}, \mathbf{p}_5 = \{q_2 = \pi/3, q_3 = -\tan^{-1}(\sqrt{3}/2)\},$$

$$\mathbf{p}_6 = \{q_3 = -\pi/3, q_2 = \tan^{-1}(\sqrt{3}/5)\}, \mathbf{p}_7 = \{q_3 = \pi/3, q_2 = -\tan^{-1}(\sqrt{3}/5)\}.$$

Singularity sets resulting from a combination of joint limits are $\{\mathbf{p}_i = \{q_i^o, q_j^o\}, i \neq j, i, j = 8...19\}$ where q_i^o and q_j^o are the limits of the inequality constraints. Singular curves are shown in Fig. (3a).

Figure 3. (a) Singular curves of the planar 3-bar linkage

Substituting $\mathbf{p}_i$'s into Eq. (40) yields singular curves Ψ_i part of which are shown in Fig. (3b).

Figure 3. (b) Identifying singular curves

To determine crossable curves, it is necessary to evaluate the criteria above for each independent segment, i.e., curves that cross other curves are segmented. For example, consider a point $\mathbf{p}_1$ on curve Ψ_{19} at $q_1 = \pi/4$, $q_2 = \pi/3$, and $q_3 = \pi/3$. Evaluating the normal $\mathbf{n}$ from the basis of the null space of

$$[\Phi_q \mathbf{q}_s]^T = \begin{bmatrix} -3.4765 & 0 & 0 \\ 0.9315 & 0 & 0 \end{bmatrix}$$

at $\mathbf{p}_1$ yields the normal $\mathbf{n} = [0.2588 \quad 0.9659]^T$. Evaluating the matrix $\mathbf{H}^*$ of Eq. (27) yields

$$\mathbf{H}^* = \begin{bmatrix} -2.4930 & 0 & 0 \\ 0 & 2.0944 & 0 \\ 0 & 0 & 1.0472 \end{bmatrix}$$

Since this is a planar example, the singular curve is parameterized by only one joint variable (i.e., $\Psi_{19}(q_1)$), therefore, $\mathbf{x}_{q_1}$ is a (2×1) matrix. The matrix $\mathbf{E}$ is therefore a $\mathbf{E} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ vector and $\mathbf{B} = [\mathbf{E}\Psi_u]^{-1}\mathbf{E} = \begin{bmatrix} -0.1992 & 0 \end{bmatrix}$. The matrix $\mathbf{Q}^*$ of the quadratic form is evaluated as

$$\mathbf{Q}^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2.094 & 0 \\ 0 & 0 & 1.047 \end{bmatrix}$$

The eigenvalues of $\mathbf{Q}^*$ are evaluated $\{0, 1.047, \text{ and } 2.094\}$, which indicates a positive semi-definite quadratic form. Since the constraints of joints 2 and 3 are active (upper bounds) and $\mathbf{x}_{q_2}$ and $\mathbf{x}_{q_3}$ are not perpendicular to $\mathbf{n}$, the additional value of σ is evaluated. For q_2 at its upper limit, $\delta q_2 = -1$, and $\sigma = \mathbf{n}^T \mathbf{x}_{q_2} \delta q_2 = 2$ which is positive. For q_3 , choose $\delta q_3 = -1$ and evaluate $\sigma = \mathbf{n}^T \mathbf{x}_{q_3} \delta q_3 = 1$ which is also positive. Both values for σ are positive (same signs as the eigenvalues of $\mathbf{Q}^*$). Therefore, surface Ψ_{19} only admits movement into the positive direction of the vector normal $\mathbf{n}$ (i.e., only towards the internal of the workspace) as expected.

To further demonstrate, consider point $\mathbf{p}_2$ on Ψ_6 at $q_1 = \pi/8$, $q_2 = \tan^{-1}(\sqrt{3}/5)$, and $q_3 = -\pi/3$. The matrix $\mathbf{Q}^*$ of the quadratic form is indefinite (both negative and positive eigenvalues) indicating a crossable curve at $\mathbf{p}_2$. Similar analysis is performed for all curve segments. For example, the curve Ψ_8 is crossable for the segment of $0.7137 \leq q_3 \leq \pi/3$; non-crossable for the segment $0 \leq q_3 \leq 0.7137$; and crossable for $-\pi/3 \leq q_3 \leq 0$. The curve Ψ_9 is non-crossable for the segment of $0 \leq q_3 \leq \pi/3$; crossable for the segment $-0.7137 \leq q_3 \leq 0$; and non-crossable for $-\pi/3 \leq q_3 \leq -0.7137$. Crossable curves are shown dotted in Fig. (4a) while non-crossable lines are shown solid. Directions of admissible motion are shown by arrows on each curve segment in Fig. (4b).

Figure 4. (a) Singular curves (b) Admissible normal movement directions

Example 2: A Four DOF Manipulator

The manipulator shown in Fig. 5 comprises two revolute and two prismatic joints. Joint limits are imposed as $0 \leq q_1 \leq 2\pi$, $20 \leq q_2 \leq 50$, $(-\pi/4) \leq q_3 \leq \pi$, and $10 \leq q_4 \leq 20$.

Figure 5. A four DOF manipulator with prismatic joints

For a point on the end-effector, the position vector is

$$\Phi(\mathbf{q}) = [q_4 \cos q_1 \cos q_3 + 30 \cos q_1 \quad q_4 \sin q_1 \cos q_3 + 30 \sin q_1 \quad q_4 \sin q_3 + q_2]^T \quad (42)$$

Inequality constraints are parameterized as $q_1 = \pi + \pi \sin \lambda_1$, $q_2 = 35 + 15 \sin \lambda_2$, $q_3 = \frac{3\pi}{8} + \frac{5\pi}{8} \sin \lambda_3$, and $q_4 = 15 + 5 \sin \lambda_4$. The Jacobian is derived as

$$\Phi_{\mathbf{q}} = \begin{bmatrix} -q_4 \sin q_1 \cos q_3 - 30 \sin q_1 & 0 & -q_4 \cos q_1 \sin q_3 & \cos q_1 \cos q_3 \\ q_4 \cos q_1 \cos q_3 + 30 \cos q_1 & 0 & q_4 \sin q_1 \sin q_3 & \sin q_1 \cos q_3 \\ 0 & 1 & q_4 \cos q_3 & \sin q_3 \end{bmatrix} \quad (43)$$

The rank-deficiency criteria of the Jacobian yields *no singular* sets since singularities obtained from the determinants of the sub-Jacobians do not satisfy the inequality constraints. The reduced-order manipulator set (substituting a joint limit value) yields four singular sets as $\mathbf{p}_1 = \{q_3 = 0, q_4 = 10\}$, $\mathbf{p}_2 = \{q_3 = \pi, q_4 = 10\}$, $\mathbf{p}_3 = \{q_3 = 0, q_4 = 20\}$, $\mathbf{p}_4 = \{q_3 = \pi, q_4 = 20\}$.

Other singular sets result from active joint constraints: $\mathbf{p}_5 = \{q_2 = 20, q_3 = -\pi/4\}$, $\mathbf{p}_6 = \{q_2 = 20, q_3 = \pi\}$, $\mathbf{p}_7 = \{q_2 = 20, q_4 = 10\}$, $\mathbf{p}_8 = \{q_2 = 20, q_4 = 20\}$, $\mathbf{p}_9 = \{q_2 = 50, q_3 = -\pi/4\}$, $\mathbf{p}_{10} = \{q_2 = 50, q_3 = \pi\}$, $\mathbf{p}_{11} = \{q_2 = 50, q_4 = 10\}$, $\mathbf{p}_{12} = \{q_2 = 50, q_4 = 20\}$, $\mathbf{p}_{13} = \{q_3 = -\pi/4, q_4 = 10\}$, and $\mathbf{p}_{14} = \{q_3 = -\pi/4, q_4 = 20\}$. Substituting each singularity set into Eq. (42) yields parametric equations of singular surfaces in $\mathbf{R}^3$. Figure 6 is a cross-section of the workspace volume depicting all singular surfaces.

Figure 6. A cross-section of the workspace depicting all singular surfaces

Figure 7a depicts a cross section (a longitudinal slice) due to $q_1 = \pi/2$. To study the crossability of singular surfaces, consider surface Ψ_{11} . Let $\mathbf{p}_1$ be a point on Ψ_{11} at $q_1 = \pi/2$, $q_2 = 50$, $q_3 = 3\pi/4$, and $q_4 = 10$. The vector normal at $\mathbf{p}_1$ is calculated using Eq. (22) as $\mathbf{n} = [0 \quad -\sqrt{2}/2 \quad -\sqrt{2}/2]^T$. The Eigenvalues of $\mathbf{Q}^*$ are $\{0, 0, -10.6, 5.0\}$. Since the quadratic form is indefinite, the singular surface is crossable at this point.

For another point $\mathbf{p}_2$ on surface Ψ_{11} at $q_1 = \pi/2$, $q_2 = 50$, $q_3 = -\pi/6$, and $q_4 = 10$. The vector normal is $\mathbf{n} = [0 \quad -\sqrt{3}/2 \quad 1/2]^T$. Eigenvalues of $\mathbf{Q}^*$ are $\{0, 0, -7.5, -5.0\}$. Since the quadratic form is negative semidefinite, the additional criteria should be evaluated. For this surface, the joint variables q_2 and q_4 are at their limits. The additional criteria are also negative indicating a non-crossable surface at this point. The end-effector can move only in the direction of the negative normal vector. In fact, the region of surface Ψ_{11} for which $-\pi/4 \leq q_3 \leq 0$, is non-crossable. The region for which $0 \leq q_3 \leq \pi$ is crossable. Similarly, for surface Ψ_{12} , the region for which $-\pi/4 \leq q_3 \leq 0$ is crossable and for which $0 \leq q_3 \leq \pi$ is non-crossable. Figure 7b depicts crossable (solid) and noncrossable (dotted) traces of surfaces of a cross-section of the workspace. Arrows indicating admissible normal movements are shown in Fig. (7c). An arrow pointing in one direction indicates the admissible motion of the end-effector in that direction.

Figure 7. (a) A cross-section of the workspace at $q_1 = \pi/2$, (b) crossable and noncrossable surfaces, and (c) Admissible directions of normal movement

Conclusions

The formulation presented in this paper and illustrated using planar and spatial manipulators with joint limits, demonstrates a complete solution to the problem of determining singular behavior in the workspace of open-chain mechanisms. New results regarding the admissibility of movement of the end-effector across singular surfaces are presented. Validation of these

It was shown that the crossability of a singular surface can be determined from the definiteness properties of a quadratic form based on acceleration analysis. It was also shown that singular surfaces that are due to active joint constraints require additional criteria to determine their crossability. The additional criterion is needed because of the loss of information in the quadratic term due to zero enforced joint velocity on the singular surface. Two supplementary criteria were derived based on the direction of the normal curvature for surfaces that have active constraints. Two conditions were shown to exist. The first is based upon the projection of a differential motion on the normal to the singular surface. The second method is based upon comparing the normal curvatures of the singular surface with that of a trajectory due to the variation of the joint that reached the limit.

The planar and spatial examples introduced represent the first complete treatment of crossability analysis of singular surfaces in an analytical formulation. Although control problems and manipulability measures are a natural extension of this work, they have not been addressed in this paper, but are the subject of current studies.

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