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CONTROL BARRIERS OF POSITIONING MANIPULATORS

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ABSTRACT

Broadly applicable formulations for identifying control difficulties in positioning serial manipulators are developed based on the Implicit Function Theorem and acceleration analyses on singular surfaces. As a result, voids in the workspace are also identified. Rank deficiency conditions are applied to the positioning Jacobian with joint constraints included in the formulation. Geometric entities upon which the manipulator exhibits control difficulties are identified. These difficulties are associated with a admissible motion along a vector normal to the singular geometric entities. Definiteness properties of a quadratic form that is as a result of normal acceleration define control difficulties. Numerical examples are illustrated.

1. INTRODUCTION

Control barriers are those surfaces inside the workspace upon which a difficulty in the manipulator's output control is observed. These barriers are associated with restrictions on the output motion control of the end-effector. The purpose of this paper is to *analytically* identify barriers to output control of serial manipulators, conditions under which difficulties may occur, and to identify voids in the workspace.

Singular behavior of manipulators has been addressed by many researchers Wang and Waldron (1987) and Waldron, *et al.* (1985), stated that as the Jacobian of the manipulator becomes singular, its columns, which are screw quantities, do not span the full rank of the matrix, therefore, the Jacobian rank decreases by at least one. The proof of the null-space

criteria applied to the Jacobian was presented by Spanos and Kohli (1985).

Methods based on Jacobian singularity are found in (Shu *et al.* 1986, Litvin *et al.* 1986a, and 1986b). An enumeration of singular sets was presented by Lipkin and Pohl (1991). Shamir (1990) provided tools to determine avoidable singularities for 3-DOF manipulators. Geometric approaches to the study of singular configurations of a manipulator arm were addressed by Lai and Yang (1986) and Ahmad and Luo (1988). A general method for determining the manipulator's singularities was presented by Tourassis and Ang (1992). Pai and Leu (1992) report a characterization of orientation singularities of manipulator arms based upon a mathematical singularity theory. Others that have discussed singularities include Soylu and Duffy (1988), Lai and Yang (1984), Mohammadi *et al.* (1995), and Merlet (1989).

Early studies that have addressed difficulties in the control of manipulators due to the appearance of interior curves and surfaces (barriers to control) were reported by Nielsen *et al.* (1991) and Waldron (1987). The fundamental concept of crossable and noncrossable surfaces inside a manipulator's workspace was addressed by Oblak and Kohli (1988).

Haug *et al.* (1995) presented a numerical algorithm for identifying and analyzing barriers to output control of manipulators using first- and second-order Taylor approximations of the output in selected directions. Haug *et al.* showed that the output velocity in the direction normal to such curves and surfaces must be zero (Haug *et al.* 1996). More recently, a criterion to define possible motion (so-called

feasible trajectory) from a singularity was presented by Chevallereau (1996). A recent comparison of numerical and analytical methods for workspace generation can be found in Abdel-Malek *et al.* (1997).

In this paper, singular curves and surfaces, inside and outside the workspace, are *analytically* determined. The background for this method is introduced in Sec. 2. Singular sets used in the identification of barriers of the manipulator's end-effector are developed in Sec. 3. The formulation for determining admissible motion in the normal direction is presented in Sec. 4, while criteria for crossing a barrier and for void identification are developed in Sec. 5. The formulation is verified through examples of planar and spatial manipulators.

2. BACKGROUND

A kinematic map of a manipulator $\mathbf{k} \rightarrow K$ is a transformation that maps a configuration $\mathbf{q} = [q_1, q_2, \dots, q_n]^T$ of an n -DOF manipulator to the position of the end-effector reference frame. The position vector of a point on the end-effector of a serial manipulator written as a multiplication of rotation matrices and position vectors can be expressed by

$$\mathbf{k}(\mathbf{q}) = [x(\mathbf{q}) \quad y(\mathbf{q}) \quad z(\mathbf{q})]^T = \sum_{i=1}^n \left[\prod_{j=1}^{j=i-1} ({}^{j-1}\mathbf{R}_j) \right] ({}^{i-1}\mathbf{p}_i) \quad (1)$$

where both ${}^i\mathbf{p}_j$ and ${}^i\mathbf{R}_j$ are defined using the D-H representation method (Fu, *et al.* 1987). Inequality constraints such as $q_i^{\min} \leq q_i \leq q_i^{\max}$; $i = 1, \dots, n$ are transformed to an equation by introducing a new set of generalized coordinates λ_i such that $q_i = c_i + b_i \sin \lambda_i$ $i = 1, \dots, n$, where $c_i = (q_i^{\max} + q_i^{\min})/2$ and $b_i = (q_i^{\max} - q_i^{\min})/2$ are the mid-point and half-range, respectively. For an assembled configuration at $\mathbf{p}^E$, a vector constraint function is developed

$$\Phi(\mathbf{q}^*) = \begin{bmatrix} \mathbf{k}(\mathbf{q}) - \mathbf{p}^E \\ q_1 - c_1 - b_1 \sin \lambda_1 \\ \vdots \\ q_n - c_n - b_n \sin \lambda_n \end{bmatrix} = \mathbf{0} \quad (2)$$

where $\mathbf{q}^*$ is now the extended vector $\mathbf{q}^* \equiv [\mathbf{q}^T \quad \lambda^T]^T$, $\lambda \equiv [\lambda_1 \quad \lambda_2 \quad \dots \quad \lambda_n]^T$, and $\Phi(\mathbf{q}^*)$ is a $(m+n) \times 1$ constraint vector, where $\dim(\mathbf{k}) = m$, such that $m = 3$ for spatial and $m = 2$ for planar manipulators. The implicit function theorem for Φ dictates that if the Jacobian $\Phi_{\mathbf{q}^*} = [\partial\Phi/\partial\mathbf{q}^*]$ is nonsingular at a point, then there exists a

unique solution in a neighborhood of a point within a tolerance. The Jacobian of the constraint function is

$$\Phi_{\mathbf{q}^*} = \begin{bmatrix} \mathbf{k}_{\mathbf{q}} & \mathbf{0} \\ \mathbf{I} & \mathbf{q}_{\lambda} \end{bmatrix} \quad (3)$$

where $x_{q_i} = \partial x / \partial q_i$ for $i = 1, \dots, n$, $\mathbf{k}_{\mathbf{q}} = \partial \mathbf{k} / \partial \mathbf{q}$ is a $m \times n$ block submatrix, $\mathbf{q}_{\lambda} = \partial \mathbf{q} / \partial \lambda$ is a $n \times n$ block submatrix, $\mathbf{I}$ is $n \times n$ identity matrix, and $\mathbf{0}$ is a $m \times n$ zero matrix. End-effector velocities can be determined using the analytic Jacobian from

$$\dot{\mathbf{k}} = [\dot{x} \quad \dot{y} \quad \dot{z}]^T = \mathbf{k}_{\mathbf{q}} \mathbf{q}_{\lambda} \dot{\lambda} \quad (4)$$

where $\dot{\lambda}$ is $(n \times 1)$. Define the Null space of $\Phi_{\mathbf{q}^*}(\mathbf{q}^*)$ as the subspace that does not produce any end-effector velocity for a given manipulator configuration. A singular surface is identified in the workspace by a row-rank deficiency condition as

$$\partial W \subset \left\{ \text{Rank}[\Phi_{\mathbf{q}^*}(\mathbf{q}^*)] \leq (n+m-1), \right. \\ \left. \text{for some } \mathbf{q}^* \text{ with } \Phi(\mathbf{q}^*) = \mathbf{0} \right\} \quad (5)$$

The set ∂W identifies singular behavior of the manipulator.

3. RANK DEFICIENCY OF THE JACOBIAN

Using the Implicit Function Theorem, and for an n -DOF system, the Jacobian $\Phi_{\mathbf{q}^*}$ is row-rank deficient if and only if one of the following conditions are satisfied.

Condition (i) The upper left $(m \times n)$ corner matrix of $\Phi_{\mathbf{q}^*}$ is row rank deficient at least one. The rank of the $(m \times n)$ matrix is defined to be the order of the largest non-singular square submatrix that can be formed by selecting rows and columns of the upper corner matrix $\mathbf{k}_{\mathbf{q}}$. Equating the determinants to zero yields η -number of equations to be solved simultaneously. The solutions to these η equations are the singular sets.

Condition (ii) If the matrix $\mathbf{q}_{\lambda}$ is row-rank deficient (i.e., $b_i \cos \lambda_i$ is zero for either $i = 1, \dots, n$ then q_i has reached a limit and the corresponding $\mathbf{k}_{\mathbf{q}}$ is studied for row-rank deficiency. Therefore, the corresponding $\frac{\partial}{\partial \mathbf{q}} [\Phi(q_i, \mathbf{q})]$ is subjected to the rank-deficiency conditions. For example, if $q_i = q_i^{\min}$ (or $q_i^{\max}$), the diagonal element of $\mathbf{q}_{\lambda}$ will be zero. Thus, the row-rank deficiency of $\Phi_{\mathbf{q}^*}$ depends on the following submatrix

$$\mathbf{k}_q \sim \begin{bmatrix} \mathbf{k}_{q_1} & \dots & \mathbf{k}_{q_i} & \dots & \mathbf{k}_{q_n} \\ 0 & \dots & 1 & \dots & 0 \end{bmatrix} \quad (6)$$

which is a $((m+1) \times n)$ matrix. The solution to this submatrix rank deficiency condition is equivalent to solving the rank deficiency for $[\mathbf{k}_q \setminus \mathbf{k}_{q_i}]$, with $q_i = q_i^{\text{limit}}$, where the notation of $\setminus$ represents the exclusion of the right column from the left matrix. Sub-jacobians are evaluated and their determinants are set to zero. Two generalized coordinates that have reached their limits are identified and denoted by $\mathbf{p}_i$ characterized by

$$\mathbf{p}_i = \left\{ \begin{bmatrix} \det(\mathbf{h}_i \mathbf{h}_j \mathbf{h}_k)_1 \\ \vdots \\ \det(\mathbf{h}_i \mathbf{h}_j \mathbf{h}_k)_\eta \end{bmatrix} = \mathbf{0}, \text{ for } i, j, k = 1, \dots, n \text{ and } i \neq j \neq k \right\} \quad (7)$$

$i = 1, 2, \dots, \beta$

where $\mathbf{h}_i$ denotes a column of the matrix $\mathbf{k}_q = [\mathbf{h}_1 \dots \mathbf{h}_n]$, β is the total number of singular sets, and $\mathbf{p}_i$ is a subset of $\mathbf{q}$ such that $\mathbf{q}^T = [\mathbf{p}_i^T \quad \mathbf{s}^T]$, i.e., $\mathbf{s}$ contains the remaining q 's.

It is noted that some singularities may be identical. Substituting the set $\mathbf{p}_i$ into $\mathbf{k}(\mathbf{q}^*)$ yields a *parametric geometric entity* (i.e., an equation of a geometric entity). Depending on the number of DOF, the substitution yields a **singular surface** denoted by

$$\Gamma(\mathbf{s}) = \mathbf{k}(q_1^c, \dots, q_{i-1}^c, q_{i+1}^c, \dots, q_{j-1}^c, q_{j+1}^c, \dots, q_n^c) \quad (8)$$

where the superscript c denotes a constant where $\mathbf{s} = [q_i \quad q_j]^T$. It was shown by Abdel-Malek and Yeh (1998) that these entities envelop the workspace where perturbation and continuation methods were introduced to identify the boundary.

4. MOTION IN THE NORMAL DIRECTION

We investigate the admissible normal motion of the end-effector while positioned at a singular entity. At a specific singular configuration of the manipulator $\mathbf{q}_o$, the velocity of the end-effector is given by

$$\dot{\mathbf{k}}_i = \sum_{j=1}^n \left(\frac{\partial \mathbf{k}_i}{\partial q_j} \right) \dot{q}_j \quad (9)$$

and in matrix form

$$\dot{\mathbf{k}} = \mathbf{k}_q \mathbf{q}_\lambda \dot{\lambda} \quad (10)$$

Due to the implicit function theorem and on a singular surface, the term $\mathbf{k}_q \mathbf{q}_\lambda \big|_{\mathbf{q}_o, \lambda_o}$ is rank-deficient. Multiplying both sides

by ξ_o^T (the basis of the null space of $\mathbf{k}_q \mathbf{q}_\lambda$) yields

$$\xi_o^T \dot{\mathbf{k}} = \xi_o^T \mathbf{k}_q(\mathbf{q}_o) \mathbf{q}_\lambda \dot{\lambda} = 0 \quad (11)$$

Since ξ_o^T is a constant vector at $\mathbf{q}_o$, the left hand side of Eq. (11) characterizes the equation of a plane in $\mathcal{R}^3$ as $\gamma_1 \dot{x} + \gamma_2 \dot{y} + \gamma_3 \dot{z} = 0$ where $\xi_o = [\gamma_1 \quad \gamma_2 \quad \gamma_3]^T$ is indeed a vector **normal** to the tangent plane of the singular surface at $\mathbf{q}_o$. Indeed, for any value of $\dot{\mathbf{q}}$, all resulting velocity vectors $\dot{\mathbf{k}}$ will lie on a plane which has ξ_o^T as its normal. Two observations can be made: (i) The basis of the null space of $[\mathbf{k}_q \mathbf{q}_\lambda]^T$ is the vector normal to the singular surface (or curve) at $\mathbf{q}_o$. (ii) For any given joint velocity vector $\dot{\mathbf{q}}$, the velocity of the end-effector is either tangent to the singular surface or zero, which indicates that the normal component of the end-effector velocity is always zero, i.e.,

$$v_n = \xi_o^T \dot{\mathbf{k}} = 0 \quad (12)$$

Consider the acceleration of a point on the end-effector at a configuration $\mathbf{q}_o$ which can be obtained by differentiating Eq. (9) with respect to time and written in index notation

$$\ddot{\mathbf{k}}_i = \sum_{j=1}^n \left\{ \left[\frac{d}{dt} \left(\frac{\partial \mathbf{k}_i}{\partial q_j} \right) \right] \dot{q}_j + \left(\frac{\partial \mathbf{k}_i}{\partial q_j} \right) \left[\frac{d}{dt} \left(\frac{dq_j}{d\lambda_j} \right) \right] \dot{\lambda}_j + \left(\frac{\partial \mathbf{k}_i}{\partial q_j} \right) \left(\frac{dq_j}{d\lambda_j} \right) \ddot{\lambda}_j \right\} \quad (13)$$

$$\text{where } \frac{d}{dt} \left(\frac{\partial \mathbf{k}_i}{\partial q_j} \right) \equiv (\partial^2 \mathbf{k}_i / \partial q_i \partial q_j) (\partial q_i / \partial \lambda_i) \dot{\lambda}_i \quad (14)$$

$$\text{and } \frac{d}{dt} \left(\frac{dq_j}{d\lambda_j} \right) \equiv (\partial^2 q_j / \partial \lambda_j \partial \lambda_j) \dot{\lambda}_j \quad (15)$$

Expanding terms in the form of joint rates and collecting on similar terms yields

$$\ddot{\mathbf{k}}_i = \dot{\lambda}_n \left\{ \frac{dq_j}{d\lambda_n} \left[\frac{d^2 \mathbf{k}_i}{dq_m dq_j} \right] \frac{dq_j}{d\lambda_k} + \frac{d\mathbf{k}_i}{dq_j} \frac{d^2 q_j}{d\lambda_n d\lambda_k} \right\} \dot{\lambda}_k + \left[\frac{d\mathbf{k}_i}{dq_j} \frac{dq_j}{d\lambda_k} \right] \ddot{\lambda}_k \quad (16)$$

Written in matrix form, the acceleration is

$$[\ddot{\mathbf{k}}]_i = \dot{\lambda}^T \mathbf{q}_\lambda^T [k_i]_{\mathbf{q}\mathbf{q}} \mathbf{q}_\lambda \dot{\lambda} + \dot{\lambda}^T \sum_{i=1}^n \left(\frac{d\mathbf{k}_i}{dq_i} [q_i]_{\lambda\lambda} \right) \dot{\lambda} + [k_i]_{\mathbf{q}} \mathbf{q}_\lambda \ddot{\lambda} \quad (17)$$

Multiplying both sides of Eq. (17) by the vector ξ_o^T (the basis of the null space of $[\Phi_q \mathbf{q}_\lambda]^T$), the last term of the right-hand side vanishes. The normal component of the acceleration is

$$a_n = \xi_o^T \ddot{\mathbf{k}} = \dot{\lambda}^T \left[\mathbf{q}_\lambda^T [\xi_o^T \mathbf{k}]_{\mathbf{q}\mathbf{q}} \mathbf{q}_\lambda + \sum_{i=1}^n \left(\frac{d(\xi_o^T \mathbf{k})}{dq_i} [q_i]_{\lambda\lambda} \right) \right] \dot{\lambda} \quad (18)$$

and written as a quadratic form, the normal component is

$$a_n = \xi_o^T \ddot{\mathbf{k}} = \dot{\lambda}^T \mathbf{H}^* \dot{\lambda} \quad (19)$$

which is a quadratic form, where

$$\mathbf{H}^*(\mathbf{q}_o, \lambda_o) = \mathbf{q}_\lambda^T [\xi_o^T \mathbf{k}]_{\mathbf{q}\mathbf{q}} \mathbf{q}_\lambda + \sum_{i=1}^n \frac{d(\xi_o^T \mathbf{k})}{dq_i} \cdot [q_i]_{\lambda\lambda} \quad (20)$$

It is important to observe that on a singular surface Γ_i , any joint acceleration $\ddot{\mathbf{q}}$ does not contribute towards the normal component of the end-effector acceleration. Moreover, since the component of the normal end-effector velocity is zero ($v_n = 0$), with nonzero a_n , the end-effector may have a component along ξ_o^T consistent with a positive or negative a_n .

Consider the planar 2-DOF manipulator shown in Fig. 1a. At a singularity at $q_2 = \pi$ which characterizes a singular curve Ψ_2 shown in Fig. 1b. When the end-effector is located at Ψ_2 , both directions of the normal acceleration (a_n^+ and a_n^-) may be possible depending on the joint velocity vector $[\dot{q}_1 \ \dot{q}_2]^T$.

Fig. 1. (a) Planar 2 degree-of-freedom manipulator (b) Singular curve and normal accelerations

It is clear, however, that the end-effector cannot admit motion towards the inside of Ψ_2 . It admits motion towards the outside of Ψ_2 . The reason is, regardless what values of $[\dot{q}_1 \ \dot{q}_2]^T$ are specified, there exists a_n^- (where a_n^- is the component towards the center) that always satisfies

$$a_n^- - \frac{v_t^2}{r} \leq 0 \quad (21)$$

where v_t is the velocity of the end-effector tangent to the singular curve and r is the radius of Ψ_2 . This observation leads to a generalization in the following section.

5. MOTION ALONG THE NORMAL TO A SINGULAR ENTITY

Consider the end-effector at a point on a singular surface (with radius of curvature ρ_o), the normal acceleration a_n , and a tangential velocity v_t . This arm admits motion such that

$$\mu = a_n - \frac{|v_t|^2}{\rho_o} = \begin{cases} > 0 & \text{moving away from the center} \\ = 0 & \text{moving along the singular surface} \\ < 0 & \text{moving toward the center} \end{cases} \quad (22)$$

Note that only using the formulation presented above, it will be possible to determine the expression in Eq. (22) and only because singular surfaces are identified in closed-form.

An expression for (v_t^2/ρ_o)

For a parametric entity $\Gamma(\mathbf{s}) \in \mathcal{R}^3$, where $\mathbf{s} = [q_i \ q_j]^T$, two fundamental forms are used to characterize a point on the surface. The *First Fundamental Form* denoted by $\mathbf{I}_p$, as

$$\mathbf{I}_p \equiv \delta \mathbf{s}^T \Gamma_s^T \Gamma_s \delta \mathbf{s} \quad (23)$$

where $\Gamma_s = \partial \Gamma / \partial \mathbf{s}$ and the *Second Fundamental Form* as

$$\mathbf{II}_p = \xi_o^T \Gamma_{q_i q_i} dq_i^2 + 2 \xi_o^T \Gamma_{q_i q_j} dq_i dq_j + \xi_o^T \Gamma_{q_j q_j} dq_j^2 \quad (24)$$

where the notation $\Gamma_{q_i q_i} = \partial^2 \Gamma / \partial q_i^2$, $\Gamma_{q_j q_j} = \partial^2 \Gamma / \partial q_j^2$, and $\Gamma_{q_i q_j} = \partial^2 \Gamma / \partial q_i \partial q_j$, and ξ_o^T is the vector normal to a parametric surface at a point (see Farin 1993 for a complete treatment of parametric surfaces). The Normal Curvature K_o of a parametric surface at a configuration $\mathbf{q}_o$, in the direction of dq_i/dq_j , can then be defined as the ratio

$$K_o = 1/\rho_o = \mathbf{II}_p / \mathbf{I}_p \quad (25)$$

We introduce the *Time-Modified First and Second Fundamental Forms* as

$$\mathbf{I}'_p \equiv \dot{\mathbf{s}}^T \Gamma_s^T \Gamma_s \dot{\mathbf{s}} \quad (26)$$

$$\mathbf{II}'_p \equiv \dot{\mathbf{s}}^T [\xi^T \Gamma]_{ss} \dot{\mathbf{s}} \quad (27)$$

respectively, such that the normal curvature can still be defined as

$$K_o = 1/\rho_o = \mathbf{II}'_p / \mathbf{I}'_p \quad (28)$$

In general, the tangential velocity in terms of Γ or $\mathbf{k}$ at any point on the surface is

$$\mathbf{v}_t = \Gamma_s \dot{\mathbf{s}} = \mathbf{k}_q \mathbf{q}_\lambda \dot{\lambda} \quad (29)$$

where $\dot{\mathbf{q}} = \mathbf{q}_\lambda \dot{\lambda}$. The squared norm of the velocity is

$$|\mathbf{v}_t|^2 = \mathbf{v}_t^T \mathbf{v}_t = \dot{\mathbf{s}}^T \Gamma_s^T \Gamma_s \dot{\mathbf{s}} \quad (30)$$

which is equal to the Modified First Fundamental Form $\mathbf{I}'_p$ of

Eq. (26). Therefore, $\mathbf{I}'_p$ can be written as $\mathbf{I}'_p = |v_t|^2$

Substituting for $1/\rho_o$ into η yields

$$\mu = a_n - |v_t|^2 \frac{\mathbf{II}'_p}{\mathbf{I}'_p} \quad (32)$$

Using Eq. (31), the indicator reduces to

$$\mu = a_n - \mathbf{II}'_p \quad (33)$$

which suggests that

$$|v_t|^2 / \rho_o = \mathbf{II}'_p \quad (34)$$

The objective is to derive an expression for μ for any singular surface at a specified point.

6. AN EXPRESSION FOR THE TIME-MODIFIED SECOND FUNDAMENTAL FORM

Since a_n is in terms of $\dot{\lambda}$ and to express $\mathbf{II}'_p$ in terms of $\dot{\lambda}$, the velocity vector *on a singular surface* is extracted from Eq. (29). Since Γ_s is not square, a generalized inverse for Γ_s must be developed. Let $\mathbf{B}$ be a generalized inverse for Γ_s such that

$$\mathbf{B} = [\mathbf{E}\Gamma_s]^{-1}\mathbf{E} \quad (35)$$

where $\mathbf{E}$ is a (2×3) matrix defined by $\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ if the

first and second rows of Γ_s are independent $\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ if

the first and third rows of Γ_s are independent $\mathbf{E} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

if the second and third rows of Γ_s are independent such that

$[\mathbf{E}\Gamma_s]$ will become a (2×2) non-singular matrix.

Premultiplying Eq. (29) by the generalized inverse yields

$$\dot{\mathbf{s}} = [\mathbf{E}\Gamma_s]^{-1}\mathbf{E}\mathbf{k}_q\mathbf{q}_\lambda\dot{\lambda} = \mathbf{B}\mathbf{k}_q\mathbf{q}_\lambda\dot{\lambda} \quad (36)$$

Substituting for $\dot{\mathbf{s}}$ into Eq. (27) yields

$$\mathbf{II}'_p = \dot{\lambda}^T (\mathbf{q}_\lambda)^T (\mathbf{k}_q)^T \mathbf{B}^T [\xi_o^T \Gamma]_{ss} \mathbf{B} \mathbf{k}_q \mathbf{q}_\lambda \dot{\lambda} \quad (37)$$

Substituting a_n (Eq. 19) and $\mathbf{II}'_p$ (Eq. 37) into Eq. (33) yields

$$a_n - \frac{|v_t|^2}{\rho_o} = \dot{\lambda}^T \left[\mathbf{H}^* - (\mathbf{q}_\lambda)^T \mathbf{k}_q^T \mathbf{B}^T [\xi_o^T \Gamma]_{ss} \mathbf{B} \mathbf{k}_q \mathbf{q}_\lambda \right] \dot{\lambda} \quad (38)$$

If joint limits are considered, the above equation can be written in terms of the new generalized variables as

$$a_n - \mathbf{II}'_p = \dot{\lambda}^T \mathbf{Q}^* \dot{\lambda} \quad (39)$$

where

$$\mathbf{Q}^* = \mathbf{H}^* - (\mathbf{q}_\lambda)^T \mathbf{k}_q^T \mathbf{B}^T [\xi_o^T \Gamma]_{ss} \mathbf{B} \mathbf{k}_q \mathbf{q}_\lambda \quad (40)$$

which represents a quadratic form.

Parameter at a Limit: In the case of a parameter at its limit, Eq. (39) yields a *semi-definite* quadratic form (imposed zero velocities). For this case, we propose the projection of a variational movement $\delta \mathbf{k} = \mathbf{k}_{q_i} \delta q_i$ due to δq_i onto the normal direction ξ_o^T such that the normal component is given by

$$\sigma = \xi_o^T \mathbf{k}_{q_i} \delta q_i \quad (41)$$

where δq_i is (+1) if q_i is at a lower bound and (-1) if at an upper bound. Positive values of σ indicate that the entity admits motion in the positive direction of ξ_o^T .

Lemma 1 (crossing a barrier): For a singular configuration $\mathbf{q}_o$, which makes the Jacobian Φ_q row rank deficient, the end-effector is located on a singular surface Γ . The crossability of a barrier is characterized by the following criteria.

- (i) If $\mathbf{Q}^*$ is definite, i.e., has both positive and negative eigenvalues, the end-effector can admit normal movements along either direction of ξ_o . The end-effector can cross this barrier.
- (ii) If $\mathbf{Q}^*$ is negative or positive semi-definite, the value for σ must be computed. If σ has a different sign than the nonzero eigenvalues of $\mathbf{Q}^*$, the singular surface is crossable. If the *all* values of σ for joints at their limits have the same sign as the nonzero eigenvalues of $\mathbf{Q}^*$, the singular surface is non-crossable. The admissible normal direction is determined by this common sign.

Lemma 2 (Void Criterion)

An enclosure in the workspace of a spatial manipulator or an area in the workspace of a planar manipulator that is bound by geometric entities that do not admit motion towards this enclosure or region is defined as a void. This also includes the definition of the external boundary as it will also exhibit a connected region or surfaces that do not admit motion towards the external.

7 A 4R-MANIPULATOR

Consider the 4R serial manipulator shown in Fig. 2 with joint constraints as follows $-\pi/2 \leq q_4 \leq \pi/2$, $-\pi \leq q_3 \leq \pi/6$, $0 \leq q_2 \leq 330^\circ$, and $0 \leq q_1 \leq 210^\circ$.

Fig. 2 A 4R manipulator

The position vector function characterizing its workspace is given by

$$\mathbf{k}(\mathbf{q}) = \begin{bmatrix} \cos q_1 (10 + 5 \cos q_2 + 3 \cos(q_2 + q_3) + \cos(q_2 + q_3 + q_4)) \\ \sin q_1 (10 + 5 \cos q_2 + 3 \cos(q_2 + q_3) + \cos(q_2 + q_3 + q_4)) \\ 7 + 5 \sin q_2 + 3 \sin(q_2 + q_3) + \sin(q_2 + q_3 + q_4) \end{bmatrix} \quad (42)$$

Using the criteria presented in Abdel-Malek and Yeh (1997), Jacobian rank-deficiency conditions resulting from the row rank deficiency of $[\Phi_q]^T$ and from the reduced-order set yield singular sets. Substituting these singular sets into Eq. (42) yields parametric equations of singular surfaces shown in Fig. 3a, and its cross section in Fig. 3b.

Fig. 3 (a) The workspace depicting all singular surfaces (b) A cross section of the workspace

To determine any voids, it is necessary to evaluate the criteria above for all surface patches. To illustrate the formulation, we shall demonstrate the algorithm for a the point $\mathbf{p}_o$ shown in

Fig. 4 on surface $\Gamma^{(15)}$, for which the singular set is $\mathbf{p}^{(15)} = \{q_3 = -\pi, q_4 = 0\}$, and the point is evaluated at $\mathbf{s}_o^{(15)} = \{q_1 = 0, q_2 = 200^\circ\}$, hence $\mathbf{q}_o^{(15)} = [0 \ 200^\circ \ -180^\circ \ 0]^T$.

Substituting $\mathbf{p}^{(15)}$ into Eq. (33) yields

$$\Gamma^{(15)}(\mathbf{u}^{(15)}) = [10 \cos q_1 + \cos q_1 \cos q_2 \quad 10 \sin q_1 + \cos q_2 \sin q_1 \quad 7 + \sin$$

To compute the normal at $\mathbf{p}_o$, the matrix

$$[\mathbf{k}_q(\mathbf{q}_o^{(15)}) \mathbf{q}_s(\mathbf{s}_o^{(15)})]^T = \begin{bmatrix} 0 & 0 & 0 \\ 0.962 & 0 & -2.644 \\ 0 & 0 & 0 \\ -0.0537 & 0 & 1.476 \end{bmatrix} \text{ is computed, and its}$$

nullspace is readily determined as

$$\{\{0.939 \ 0 \ 0.342\}, \{0 \ 1 \ 0\}\}, \text{ Since the second}$$

vector is trivial, the normal is indeed

$$\xi_o^T = [0.939 \ 0 \ 0.342]. \text{ The matrix } \mathbf{H} \text{ is calculated as}$$

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 7.920 & 0 & -4.421 \\ 0 & 0 & 0 & 0 \\ 0 & -4.421 & 0 & -2.467 \end{bmatrix} \text{ and the generalized inverse is}$$

$$\text{calculated as } \mathbf{B} = [\mathbf{E}_1 \Gamma_s(\mathbf{s}_o^{(15)})]^{-1} \cdot \mathbf{E}_1 = \begin{bmatrix} 0 & 0.110 & 0 \\ 2.924 & 0 & 0 \end{bmatrix},$$

where the matrix was used. The matrix is computed as and.

The final (4×4) $\mathbf{Q}^*$ matrix is computed and its eigenvalues

are computed as $\{-4.935, 0, 0, 0\}$. Since this is a

negative semi definite form, the algorithm must compute a

value for σ . Since $q_3 = -\pi$ is a lower limit, $\delta q_3 = +1$ and $\sigma = \xi_o^T \mathbf{k}_{q_3} (+1) = 0$. Since the normal is along the same direction as the derivative, a value for $\bar{K}$ must be computed. The terms are evaluated as follows: $[\xi_o^T \Gamma]_{q_3 q_3} = -4.0$, $\mathbf{k}_{q_3}^T \mathbf{B}^T = [0 \quad -4]$, and $[\xi_o^T \Gamma]_{uu} \mathbf{B} \mathbf{k}_{q_3} = [0 \quad -4]^T$. Therefore, $\bar{K} = -20.0056$. A negative sign indicates that the singular surface admits movement into the negative direction of ξ_o^T .

Fig. 4 Part of the swept area showing curve segments

At point $\mathbf{p}_1$ on singular surface $\Gamma^{(17)}(\mathbf{u})$ shown on Fig. 4, where $\mathbf{s}^{(17)} = \{q_1, q_3\}$ at $\mathbf{p}^{(17)} = \{q_2^L, q_4 = 0\}$ and $q_1 = 0$, $q_3 = -100^\circ$. Evaluating the normal yields $\xi_o = [-0.1736 \quad 0 \quad -0.9848]^T$. The eigenvalues of $\mathbf{Q}^*$ are evaluated $\{-14.1802, -1.8505, 0, 0\}$, which indicates a negative semi-definite quadratic form. Since q_2 is at its lower bound (q_2^L), the additional value of σ is evaluated with $\delta q_2 = +1$. The value for $\sigma = \xi_o^T \mathbf{k}_{q_2} \delta q_2 = -4.9240$. Since the sign of σ is the same as that of the eigenvalues of $\mathbf{Q}^*$, it indicates that the surface $\Gamma^{(17)}$ at $\mathbf{p}_1$ only admits movement into the opposite direction of the normal $\mathbf{n}_o$ (as shown in Fig. 4). This indicates that this surface patch is a boundary.

Similarly, for the same singular surface at point $\mathbf{p}_2$ $q_1 = 0, q_3 = -10^\circ$ (which is an external boundary), the normal is $\xi_o = [-0.985 \quad 0 \quad 0.17]^T$ and the eigenvalues are computed as $\{2.5, 1.8505, 0, 0\}$ and $\sigma = 0.868$; same sign as the eigenvalues. This indicates an admissible motion in that direction (but not the opposite direction, i.e., a boundary).

On singular surface $\Gamma^{(8)}$ at point $\mathbf{p}_3$ where $\mathbf{s}^{(8)} = \{q_1, q_2\}$, and at $\mathbf{p}^{(8)} = \{q_3^U, q_4^U\}$ and $q_1 = 0$, $q_2 = 305^\circ$, the normal is $\xi_o = [-0.8032 \quad 0 \quad 0.5957]^T$ and the eigenvalues are computed as $\{-2.8974, -1.5389, 0, 0\}$ and $\sigma = -1.5810$.

On singular surface $\Gamma^{(3)}$ at point $\mathbf{p}_4$ with $\mathbf{s}^{(3)} = \{q_1, q_3\}$, at $\mathbf{p}^{(4)} = \{q_2^L, q_4^U\}$ at $q_1 = 0$, $q_3 = 0^\circ$, and $q_4 = q_4^L$, the normal is $\xi_o = [-0.9487 \quad 0 \quad 0.3160]^T$ and the eigenvalues are computed as $\{4.5529, -1.4901, 0, 0\}$ which is an indefinite quadratic form, i.e., it does not represent a boundary. It is evident that those surface patches that do not admit motion are indeed a boundary. For this example, there are three sets of boundary surface patches shown in Fig. 5 which represent the exact boundary to the workspace including voids.

The external boundary is characterized by the following surface regions

$$\begin{aligned} &\Gamma^{(19)}(q_1, q_2); 0 \leq q_1 \leq 210^\circ \text{ and } 0 \leq q_2 \leq 330^\circ, \\ &\Gamma^{(17c)}(q_1, q_3); 0 \leq q_1 \leq 210^\circ \text{ and } 34^\circ \leq q_3 \leq 7^\circ, \\ &\Gamma^{(10)}(q_1, q_4); 0 \leq q_1 \leq 210^\circ \text{ and } -20^\circ \leq q_4 \leq 20^\circ \\ &\Gamma^{(18)}(q_1, q_3); 0 \leq q_1 \leq 210^\circ \text{ and } 0^\circ \leq q_3 \leq 30^\circ \end{aligned}$$

The inner boundary is characterized by

$$\begin{aligned} &\Gamma^{(15)}(q_1, q_2); 0 \leq q_1 \leq 210^\circ \text{ and } 0^\circ \leq q_2 \leq 330^\circ, \\ &\Gamma^{(9)}(q_1, q_4); 0 \leq q_1 \leq 210^\circ \text{ and } -17^\circ \leq q_4 \leq 0^\circ, \\ &\Gamma^{(17b)}(q_1, q_3); 0 \leq q_1 \leq 210^\circ \text{ and } -\pi \leq q_3 \leq -175^\circ. \end{aligned}$$

The boundary to the void is characterized by the following surface patches:

$$\begin{aligned} &\Gamma^{(5)}(q_1, q_2); 0 \leq q_1 \leq 210^\circ \text{ and } 276^\circ \leq q_2 \leq 283^\circ, \\ &\Gamma^{(17a)}(q_1, q_3); 0 \leq q_1 \leq 210^\circ \text{ and } -154^\circ \leq q_3 \leq -70^\circ, \\ &\Gamma^{(8)}(q_1, q_2); 0 \leq q_1 \leq 210^\circ \text{ and } 293^\circ \leq q_2 \leq 312^\circ, \quad \text{and} \\ &\Gamma^{(11)}(q_1, q_3); 0 \leq q_1 \leq 210^\circ \text{ and } -142^\circ \leq q_3 \leq -30^\circ. \end{aligned}$$

Fig. 5 Identification of boundary curves

8. A 4DOF RPRP MANIPULATOR

Consider the manipulator shown in Fig. 6 with two revolute and two prismatic joints. Joint limits are imposed as $0 \leq q_1 \leq 2\pi$, $20 \leq q_2 \leq 50$, $(-\pi/4) \leq q_3 \leq \pi$, and $10 \leq q_4 \leq 20$.

Fig. 6 A four degree-of-freedom manipulator with prismatic joints

For a point on the end-effector, the position vector is given by $\mathbf{k}(\mathbf{q}) = [q_4 \cos q_1 \cos q_3 + 30 \cos q_1$

$$q_4 \sin q_1 \cos q_3 + 30 \sin q_1 \quad q_4 \sin q_3 + q_2]^T \quad (35)$$

Singular sets are entered into Table 1.

Table 1: Singular sets of the spatial manipulator

$\mathbf{p}_1$	$q_3 = 0, q_4 = 10$	$\mathbf{p}_2$	$q_3 = \pi, q_4 = 10$
$\mathbf{p}_3$	$q_3 = 0, q_4 = 20$	$\mathbf{p}_4$	$q_3 = \pi, q_4 = 20$
$\mathbf{p}_5$	$q_2 = 20, q_3 = -\pi/4$	$\mathbf{p}_6$	$q_2 = 20, q_3 = \pi$
$\mathbf{p}_7$	$q_2 = 20, q_4 = 10$	$\mathbf{p}_8$	$q_2 = 20, q_4 = 20$
$\mathbf{p}_9$	$q_2 = 50, q_3 = -\pi/4$	$\mathbf{p}_{10}$	$q_2 = 50, q_3 = \pi$
$\mathbf{p}_{11}$	$q_2 = 50, q_4 = 10$	$\mathbf{p}_{12}$	$q_2 = 50, q_4 = 20$
$\mathbf{p}_{13}$	$q_3 = -\pi/4, q_4 = 10$	$\mathbf{p}_{14}$	$q_3 = -\pi/4, q_4 = 20$

Substituting each singularity set into Eq. (35) yields parametric equations of surfaces in $\mathcal{R}^3$ since each equation is parametrized in terms of two variables. Some of these surfaces will indeed act as barriers to output control. Figure 7 is a section of the workspace depicting singular surfaces, and Fig. 7 is a cut at $q_1 = \pi/2$ where each surface is identified.

Fig. 7 A section of the workspace depicting singular surfaces

Similar analyses are performed at other points on each region, and those regions that admit output normal movements are indicated as dotted lines in Fig. 8a.

Fig. 8 (a) Admissible output normals (b) a cross-section identifying noncrossable surfaces

Each crossable region of a surface is identified by a dotted line in the cross-section of the workspace shown in Fig. 8a, and only non-crossable regions are shown in Fig. 8b.

9. CONCLUSIONS

A formulation for determining output control difficulties and barrier crossability of serial manipulators is presented. Analytic singular entities were determined using conditions imposed on the manipulator Jacobian and on the reduced-order manipulator when a joint constraint is active (joint limits are considered). Three types of singular behavior of a manipulator were identified. The analytic formulation developed in this paper and illustrated using planar and spatial controlled manipulators demonstrates the validity of a general method for determining workspace-internal singular surfaces, their control difficulties, and barrier crossability. It has been shown that all singular entities (curves and surfaces) inside the workspace can be determined.

It was shown that the basis of the null space of the Jacobian at a singular configuration is indeed the normal to a singular surface. This is a significant result that contributes to the understanding of manipulator control behavior at singularities. It also clarifies the significance of a Jacobian singularity with respect to the movement of the manipulator across the surface which was referred to by Oblak and Kohli (1988) as a crossable surface, and more recently by Chevallereau (1996) as a feasible trajectory at a singularity.

From the velocity and acceleration analyses, it was shown that the motion across a singular surface is dependent upon the definiteness properties of the Hessian matrix of the basis to the null space, the constraint function, and the fundamental forms of the singular surface. It was also shown that on a singular surface, the joint acceleration vector has no contribution towards the normal component of the end-effector's acceleration.

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