Interior and Exterior Boundaries to the Workspace of Mechanical Manipulators

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Abstract

Analytical methods for identifying the boundary to the workspace of serial mechanical manipulators and the boundary to voids in the workspace are presented. The determination of parametric equations of surface patches that envelop the workspace of serial manipulators was presented elsewhere and is extended in this paper to an analytical method for void identification. Because of the ability to identify closed-form surface patches that exist internal and external to the workspace, a mathematical formulation based on the concept of a normal acceleration function is introduced. Admissible motion in the normal direction to a point on a singular surface is delineated and characterized by definiteness properties of a quadratic form. An enclosure bound by surface patches that do not admit normal motion is identified as a void. Several examples are treated using this formulation to illustrate the method.

Keywords: manipulator workspace, voids, singular surfaces, workspace boundary, robotics

1 INTRODUCTION

Numerical methods for determining workspace boundaries of serial manipulators have been developed in recent years. Exact computation of the workspace and its boundary is of significant importance because of its impact on manipulator design, manipulator placement in an environment, and manipulator dexterity.

Early investigations of manipulator workspaces were reported by Kumar and Waldron (1981), Tsai and Soni (1981), Gupta and Roth (1982), Sugimoto and Duffy (1982), Spanos and Kohli (1985), Kumar (1985), Gupta (1986), and Davidson and Hunt (1987). The consideration of joint limits in the study of manipulator workspaces was presented in a recent report by Delmas and Bidard (1995). Other works that have dealt with manipulator workspace are reported by Agrawal (1990), Gosselin and Angeles (1990), and Emiris (1993).

Pennock and Kassner (1993) presented a numerical algorithm for the general study of a planar 3DOF manipulator. More recently, Cecarelli (1995) used an algebraic formulation of a workspace boundary to formulate design equations of three-revolute manipulators and Zhang, *et al.* (1996) presented the graphical representation of kinematic workspaces.

Recently, Haug *et al.* (1996) formulated numerical criteria to find the workspace (called the accessible output set) of a general multi-degree-of-freedom system using a continuation method to trace boundary curves suitable for the study of both open- and closed-loop manipulators. The initial criteria for this computational method were presented by Haug *et al.* (1992) and Wang and Wu (1993).

Although there has been many reports in this field, there are only a few that have addressed the existence of voids in the workspace. The work by Cecarelli and Vinciguerra (1995) using an algebraic formulation to determine the workspace of 4DOF-revolute manipulators exhibits a capability for determining holes and voids in the workspace. Two types of voids were identified, those inside the boundary (called ring voids) and those formed by the rotational motion of the boundary and include the main $\mathbf{z}_o$ -axis (so-called apple voids).

It is noted here that the importance of the analysis of voids in manipulator workspaces extends to many fields, most importantly to computer-aided design (Abdel-Malek and Yeh 1997), where swept volume techniques can be used to generate solid models. Moreover, identification of voids is necessary in the determination of swept trajectories generated by a manipulator link to avoid collisions. The computation of swept volumes is also important in manufacturing where the NC verification of machining processes requires the computation of the workspace generated by the tool path on the workpiece.

In recent years, this group has made significant contributions towards the understanding of the workspace of serial manipulators (Abdel-Malek and Yeh 1997; 1997b; 2000, Abdel-Malek (1997), Abdel-Malek, et al. 1997; 1999a; 1999b). The main contribution of this formulation is that it yields the exact workspace in closed form. Furthermore, barriers inside the workspace where control difficulties may be encountered are also identified.

In this paper, we further expand the above formulation towards better understanding the identification of internal and external boundaries.

Criteria for determining singular surfaces are established in Section 2. Motion in the normal direction to a singular surface is addressed in Section 3. A quadratic form independent of acceleration vectors is derived in Section 4 and the criteria are stated in Sections 5 and 6. Numerous examples are illustrated.

2. REVIEW OF SINGULAR SURFACE DETERMINATION

In order to mathematically address the volume in space generated by the motion of the end-effector, it is necessary to obtain a representation for that motion in terms of joint coordinates (i.e., joint variables). The position vector generated by a point on the end-effector of a n -DOF arm is expressed by

$$\mathbf{G}(\mathbf{q}) = [x(\mathbf{q}) \quad y(\mathbf{q}) \quad z(\mathbf{q})]^T \quad (1)$$

where $\mathbf{G}(\mathbf{q}) \in \mathbf{R}^3$, and $\mathbf{q} = [q_1, q_2, \dots, q_n]^T$. In fact, $\mathbf{G}(\mathbf{q})$ characterizes the set of points that belongs to the workspace. Singular surfaces/curves comprising the workspace were analytically determined by studying the Jacobian. For the set $\mathbf{G}(\mathbf{q})$, rank deficiency criteria applied to the Jacobian

$$\mathbf{G}_q = [\partial G_i / \partial q_j] \quad (2)$$

where G_i , $i = 1, 2, 3$ are the components of $\mathbf{G}$, yield singular sets $\mathbf{p}^{(i)}$ ($\mathbf{p}$ have been defined by Abdel-Malek and Yeh 1999) in terms of joint variables. It was shown that substituting $\mathbf{p}^{(i)}$ into $\mathbf{G}(\mathbf{q})$ yields $\mathbf{f}^{(i)}(\mathbf{u}^{(i)})$ where $\mathbf{u}^{(i)}$ is the new vector of generalized coordinates such that

$$\mathbf{f}^{(i)}(\mathbf{u}^{(i)}) = \mathbf{G}(\mathbf{u}^{(i)}, \mathbf{p}^{(i)}) \quad (3)$$

where $\dim(\mathbf{f}) = (3 \times 1)$ and $\mathbf{f}^{(i)}$ characterize a closed form equation of a singular surface. It was also shown that these singular curves are used to verify the planned trajectory of a manipulator (Abdel-Malek and Yeh 1997).

A convenient parametrization of constraints imposed on $\mathbf{q}$ was presented such that joint inequality constraints $q_i^L \leq q_i \leq q_i^U$ are parametrized as

$$q_i(\lambda_i) = \frac{(q_i^U + q_i^L)}{2} + \frac{(q_i^U - q_i^L)}{2} \sin \lambda_i \quad (4)$$

where new parameters were introduced such that $\mathbf{q} = \mathbf{q}(\mathbf{s})$, where $\mathbf{s} = [\lambda_1 \ \lambda_2 \ \dots \ \lambda_n]^T$ and $\mathbf{G} = \mathbf{G}(\mathbf{q}(\mathbf{s}))$ is now a function of the new variables (usually called slack variables). This parametrization is important because it will enable us to include joint limits in the formulation. Note that the limits of a joint are now obtained by varying λ_i (i.e., $\lambda_i = -\pi/2$ or $\lambda_i = \pi/2$ identifies a limit). An expression for the velocity *at any point in the workspace* is obtained by deriving $\mathbf{G}(\mathbf{s})$ with respect to time as

$$\dot{\mathbf{G}} = \mathbf{G}_q \mathbf{q}_s \dot{\mathbf{s}} \quad (5)$$

where the terminology $\mathbf{q}_s = \partial \mathbf{q} / \partial \mathbf{s}$ and $\dot{\mathbf{s}} = \partial \mathbf{s} / \partial t$. Using the Implicit Function Theorem, the manipulator is at a singular configuration $\mathbf{q}_o$ when the rows of $\mathbf{G}_q \mathbf{q}_s$ are dependent and there exists a set of constants $\mathbf{N}_o = [\gamma_1 \ \gamma_2 \ \gamma_3]^T$ that satisfy

$$[\mathbf{G}_q \mathbf{q}_s]^T \mathbf{N}_o = \mathbf{0} \quad (6)$$

where $\mathbf{N}_o$ is the vector normal to a singular surface at $\mathbf{q}_o$.

3. ADMISSIBLE MOTION

In this section, the acceleration normal at a point on a singular surface is formulated as a quadratic form. The void identification condition states that a closed boundary for which the acceleration form yields admissible output normals towards the external of the enclosed space for all surfaces enclosing this region, then that space is a void and cannot be reached by the manipulator's end-effector. A point on a singular surface admits motion normal to the surface in either direction depending on the difference in acceleration components (defined by η), such that

$$\eta = a_n - \frac{v_t^2}{\rho_o} \quad (7)$$

where v_t is the tangential velocity, a_n is the normal acceleration, and $1/\rho_o$ is the normal curvature of the singular surface (Fig. 1) *with respect to the tangent direction* of v_t (ρ_o is the radius of curvature). Although it may be unclear at this time, the need for formulating the problem in terms of velocities and accelerations will become apparent. It will be shown that although crossing a barrier is characterized in terms of velocities and accelerations, the crossability criteria is invariant (i.e., not affected by the values of the said quantities).

Figure 1 Concept of normal acceleration

A point on a singular surface will have no acceleration if the quantity η computes to null. It will be shown that this quantity will indeed identify the boundary to a void since at those singular surfaces, Eq. 7 will indicate no motion into the void, or towards the external of the workspace volume regardless of the value of the velocities or accelerations. Part of a surface enveloping a void in the volume must admit motion only towards the external of that void as shown in Fig. 2. This concept will be generalized in the discussion that follows.

It should be noted that although some barriers may occur in the workspace at which the surface does not admit motion, these barriers do not enclose a region, and therefore, do not form a void. These entities, however, may cause impediments to motion (Haug, *et al.* 1995) and exhibit control difficulties.

Figure 2 Boundary and void identification

For a singular parametric entity $\mathbf{f}^{(i)}(\mathbf{u}^{(i)}) \in \mathbf{R}^3$ (where $\mathbf{u}$ is indeed a vector representing the remaining joint variables—those not fixed), and in the field of differential geometry, *The First Fundamental Form* (Farin 1993) is denoted by $\mathbf{I}_p$, where $\mathbf{u} = [u \ v]^T = [q_i \ q_j]^T$, and is defined as

$$\mathbf{I}_p \equiv \delta \mathbf{u}^T \mathbf{f}_u^T \mathbf{f}_u \delta \mathbf{u} \quad (8)$$

where $\mathbf{f}_u = \partial \mathbf{f} / \partial \mathbf{u}$. The *Second Fundamental Form* is defined as

$$\mathbf{II}_p \equiv \delta \mathbf{u}^T [\mathbf{N}^T \mathbf{f}]_{uu} \delta \mathbf{u} \quad (9)$$

or expanded as

$$\mathbf{II}_p = \mathbf{N}^T \mathbf{f}_{uu} du^2 + 2\mathbf{N}^T \mathbf{f}_{uv} du dv + \mathbf{N}^T \mathbf{f}_{vv} dv^2 \quad (10)$$

where $\mathbf{N}$ is the vector normal to the singular surface and $\mathbf{f}_{uv} = \partial^2 \mathbf{f} / \partial u \partial v$. The Normal Curvature K_o of a parametric surface at a configuration $\mathbf{q}_o$, in the direction of du/dv , can then be defined as the ratio (Farin 1993)

$$K_o = \frac{1}{\rho_o} = \frac{\mathbf{II}_p}{\mathbf{I}_p} \quad (11)$$

In order to define the kinematics quantities, we define the *Time-Modified First and Second Fundamental Forms* as

$$\mathbf{I}'_p \equiv \dot{\mathbf{u}}^T \mathbf{f}_u^T \mathbf{f}_u \dot{\mathbf{u}} \quad (12a)$$

$$\mathbf{II}'_p \equiv \dot{\mathbf{u}}^T [\mathbf{N}^T \mathbf{f}]_{uu} \dot{\mathbf{u}} \quad (12b)$$

such that the normal curvature can still be defined as

$$K_o = \frac{1}{\rho_o} = \frac{\mathbf{II}_p}{\mathbf{I}_p} = \frac{\mathbf{II}'_p}{\mathbf{I}'_p} \quad (13)$$

For a singular surface $\mathbf{f}$ with its vector of variables $\mathbf{u}$ (i.e., $\mathbf{f}(\mathbf{u})$), the derivative using the chain rule is $\mathbf{f}_u \dot{\mathbf{u}}$. Similarly, for the overall description of the workspace $\mathbf{G}(\mathbf{q})$, the derivative is $\mathbf{G}_q \dot{\mathbf{q}}$. Therefore, at an instant of time, the tangential velocity in terms of $\mathbf{f}$ or $\mathbf{G}$ at any point on a singular surface is

$$\mathbf{v}_t = \mathbf{f}_u \dot{\mathbf{u}} = \mathbf{G}_q \dot{\mathbf{q}} \quad (14)$$

where $\dot{\mathbf{u}} = \mathbf{u}_s \dot{\mathbf{s}}$ and $\dot{\mathbf{q}} = \mathbf{q}_s \dot{\mathbf{s}}$. The squared norm of the velocity is

$$|\mathbf{v}_t|^2 = \mathbf{v}_t^T \mathbf{v}_t = \dot{\mathbf{u}}^T \mathbf{f}_u^T \mathbf{f}_u \dot{\mathbf{u}} \quad (15)$$

which is equal to the Time-Modified First Fundamental Form $\mathbf{I}'_p$ of Eq. 10. Therefore, $\mathbf{I}'_p$ can be written as

$$\mathbf{I}'_p = |\mathbf{v}_t|^2 \quad (16)$$

Substituting for $1/\rho_o$ into η yields

$$\eta = a_n - |\mathbf{v}_t|^2 \frac{\mathbf{II}'_p}{\mathbf{I}'_p} = a_n - \mathbf{II}'_p \quad (17)$$

The objective is to derive an expression for η for any singular surface at a specified point. We will first determine an expression for $\mathbf{II}'_p$, followed by an expression for a_n , and finally obtain a quadratic form for η .

4. A QUADRATIC FORM FOR THE NORMAL ACCELERATION

4.1 An Expression for the Time-Modified Second Fundamental Form

Since a_n is in terms of $\dot{\mathbf{q}}$ and to express $\mathbf{II}'_p$ in terms of $\dot{\mathbf{q}}$, the velocity vector *on a singular surface* is extracted from Eq. 14. Since $\mathbf{f}_u$ is not square, the velocity is written as

$$\dot{\mathbf{u}} = \mathbf{B} \mathbf{G}_q \dot{\mathbf{q}} \quad (18)$$

where $\mathbf{B}$ is the generalized inverse defined by $\mathbf{B} = [\mathbf{E} \mathbf{f}_u]^{-1} \mathbf{E}$ and

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{if the first and second rows of } \mathbf{f}_u \text{ are independent} \quad (19a)$$

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{if the first and third rows of } \mathbf{f}_u \text{ are independent} \quad (19b)$$

$$\mathbf{E} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{if the second and third rows of } \mathbf{f}_u \text{ are independent} \quad (19c)$$

Substituting the expression for $\dot{\mathbf{u}}$ into Eq. 12 yields

$$\begin{aligned} \mathbf{II}'_p &= \dot{\mathbf{u}}^T [\mathbf{N}^T \mathbf{f}]_{uu} \dot{\mathbf{u}} \\ &= \dot{\mathbf{s}}^T \mathbf{q}_s \mathbf{G}_q \mathbf{B}^T [\mathbf{N}^T \mathbf{f}]_{uu} \mathbf{B} \mathbf{G}_q \mathbf{q}_s \dot{\mathbf{s}} \end{aligned} \quad (20)$$

where $\dot{\mathbf{u}}$ and $\dot{\mathbf{q}}$ were written in terms of $\dot{\mathbf{s}}$.

4.2 A Form for a_n

Differentiating Eq. 5 again with respect to time yields the acceleration at any point as

$$\ddot{\mathbf{G}} = \frac{d}{dt} (\mathbf{G}_q) \mathbf{q}_s \dot{\mathbf{s}} + \mathbf{G}_q \frac{d}{dt} (\mathbf{q}_s) \dot{\mathbf{s}} + \mathbf{G}_q \mathbf{q}_s \ddot{\mathbf{s}} \quad (21a)$$

or written in expanded form as

$$\ddot{G}_i = \frac{d}{dt} \left[\frac{dG_i}{dq_j} \right] \frac{dq_j}{ds_k} \dot{s}_k + \left[\frac{dG_i}{dq_j} \right] \frac{d}{dt} \left(\frac{dq_j}{ds_k} \right) \dot{s}_k + \left[\frac{dG_i}{dq_j} \right] \frac{dq_j}{ds_k} \ddot{s}_k \quad (21b)$$

Expanding terms in the form of joint rates and collecting similar terms yields

$$\ddot{G}_i = \dot{s}_n \left\{ \frac{dq_j}{ds_n} \left[\frac{d^2 G_i}{dq_m dq_j} \right] \frac{dq_j}{ds_k} + \frac{dG_i}{dq_j} \frac{d^2 q_j}{ds_n ds_k} \right\} \dot{s}_k + \left[\frac{dG_i}{dq_j} \frac{dq_j}{ds_k} \right] \ddot{s}_k \quad (22)$$

In matrix form, Eq. 22 is written as

$$\ddot{\mathbf{G}} = \dot{\mathbf{s}} \left\{ \mathbf{q}_s^T \mathbf{G}_{qq} \mathbf{q}_s + \sum_{i=1}^n \frac{d\mathbf{G}}{dq_i} \cdot [\mathbf{q}_i]_{ss} \right\} \dot{\mathbf{s}} + [\mathbf{G}_q \mathbf{q}_s] \ddot{\mathbf{s}} \quad (23)$$

To obtain the normal acceleration, $\ddot{\mathbf{G}}$ is projected onto the normal $\mathbf{N}_o$. Pre-multiplying both sides of Eq. 23 by $\mathbf{N}_o^T$ eliminates the last term of the right hand side (definition of the normal as the basis of the null space in Eq. 6). The component of the normal acceleration is

$$a_n = \mathbf{N}_o^T \ddot{\mathbf{G}} = \dot{\mathbf{s}}^T \mathbf{H}^* \dot{\mathbf{s}} \quad (24)$$

where

$$\mathbf{H}^*(\mathbf{q}_o, \mathbf{s}_o) = \mathbf{q}_s^T [\mathbf{N}_o^T \mathbf{G}]_{qq} \mathbf{q}_s + \sum_{i=1}^n \frac{d(\mathbf{N}_o^T \mathbf{G})}{dq_i} \cdot [\mathbf{q}_i]_{ss} \quad (25)$$

4.3 A Form for η

In order to obtain an expression for η , substitute for a_n (from Eq. 24) and for $\mathbf{\Pi}'_p$ (from Eq. 20) into Eq. 17, yields

$$\eta = a_n - \mathbf{\Pi}'_p = \dot{\mathbf{s}}^T \mathbf{Q}^* \dot{\mathbf{s}} \quad (26)$$

where

$$\mathbf{Q}^* = \mathbf{H}^* - \mathbf{q}_s^T \mathbf{G}_q^T \mathbf{B}^T [\mathbf{N}_o^T \mathbf{f}]_{uu} \mathbf{B} \mathbf{G}_q \mathbf{q}_s \quad (27)$$

where $\mathbf{Q}^*$ is a quadratic form. Definiteness properties of the quadratic form of Eq. 27 defines whether motion occurs normal to a singular surface in the direction of $\mathbf{N}^T$ at a specified point (i.e., independent of the value of $\dot{\mathbf{s}}$). This form will indicate if a singular surface is a boundary to the workspace or boundary to a void. If $\mathbf{Q}^*$ is indefinite, the singular surface admits motion along either normal direction and is not a boundary to the workspace.

When q_i is at a limit (q_i^L or q_i^U), the form characterized by $\mathbf{Q}^*$ yields semi-definite forms because $\dot{q}_i = [q_i^{\max} - q_i^{\min}/2] \cos \lambda_i = 0$ at an upper or lower bound since at a joint limit, $\lambda_i = -\pi/2$ or $\lambda_i = \pi/2$. Semi-definite forms are addressed in the following section.

5. SEMI-DEFINITE QUADRATIC FORM AT $q_i = q_i^{\text{limit}}$

In the case of a joint at its limit, Eq. 26 yields a semi-definite quadratic form. For this case, project a variational motion $\delta \mathbf{G} = \mathbf{G}_{q_i} \delta q_i$ due to δq_i onto the normal direction $\mathbf{N}_o^T$ such that the normal component is given by

$$\sigma = \mathbf{N}_o^T \mathbf{G}_{q_i} \delta q_i \quad (28)$$

where δq_i is a specified magnitude of ± 1 as follows

$$\delta q_i = \begin{cases} +1 & \text{if } q_i \text{ is at lower bound} \\ -1 & \text{if } q_i \text{ is at upper bound} \end{cases} \quad (29)$$

Positive values of σ in Eq. 29 indicate that the singular surface admits motion in the positive direction of $\mathbf{N}_o^T$. However, if the normal vector $\mathbf{N}_o^T$ is perpendicular to $\mathbf{G}_{q_i}$ the indicator σ

in Eq. 28 evaluates to zero. In this case, the normal curvature of the singular surface with respect to the tangent direction of $\mathbf{G}_{q_i} \delta q_i$ is compared with the normal curvature of the parametric curve when only q_i is varying (other parameters are held constant).

To determine the normal direction of admissible movement, it is necessary to evaluate the sign of the difference in the normal curvature. Substituting for the first and second fundamental forms (Eqs. 9 and 10) into the normal curvature (Eq. 11) yields

$$K_2 - K_1 = \frac{\delta q_i [\mathbf{N}^T \mathbf{G}]_{q_i q_i} \delta q_i}{\delta q_i \mathbf{G}_{q_i}^T \mathbf{G}_{q_i} \delta q_i} - \frac{\delta \mathbf{u}^T [\mathbf{n}^T \mathbf{f}]_{\mathbf{u}} \delta \mathbf{u}}{\delta \mathbf{u}^T \mathbf{f}_{\mathbf{u}}^T \mathbf{f}_{\mathbf{u}} \delta \mathbf{u}} \quad (30)$$

where K_1 is the normal curvature of $\mathbf{f}$ with respect to the tangent direction $\mathbf{G}_{q_i} \delta q_i$, and K_2 is the normal curvature of the parametric curve when q_i is varying while other parameters are held constant. Both curves of the right-hand side of Eq. (30) must have the same tangent vector given by Eq. (14) such that

$$\mathbf{f}_{\mathbf{u}} \delta \mathbf{u} = \mathbf{G}_{q_i} \delta q_i \quad (31)$$

where δq_i is selected as in Eq. 29. Rearranging Eq. 31 and using the generalized inverse defined in Eq. 18, the vector $\delta \mathbf{u}$ can be written as

$$\delta \mathbf{u} = \mathbf{B} \mathbf{G}_{q_i} \delta q_i \quad (32)$$

Substituting Eq. (2) into Eq. (30) and simplifying yields

$$K_2 - K_1 = \frac{\delta q_i [\mathbf{N}_o^T \mathbf{G}]_{q_i q_i} \delta q_i - \delta q_i \mathbf{G}_{q_i}^T \mathbf{B}^T [\mathbf{N}_o^T \mathbf{G}]_{\mathbf{u}} \mathbf{B} \mathbf{G}_{q_i} \delta q_i}{\delta q_i^T \mathbf{G}_{q_i}^T \mathbf{G}_{q_i} \delta q_i} \quad (33)$$

Since the denominator of the right-hand side of Eq. 33 is always positive, the sign is dependent on the numerator. Define the numerator as μ where

$$\mu = \delta q_i [\mathbf{N}_o^T \mathbf{G}]_{q_i q_i} \delta q_i - \delta q_i \mathbf{G}_{q_i}^T \mathbf{B}^T [\mathbf{N}_o^T \mathbf{G}]_{\mathbf{u}} \mathbf{B} \mathbf{G}_{q_i} \delta q_i \quad (34)$$

and can be written as

$$\mu = \left\{ [\mathbf{N}_o^T \mathbf{G}]_{q_i q_i} - \mathbf{G}_{q_i}^T \mathbf{B}^T [\mathbf{N}_o^T \mathbf{G}]_{\mathbf{u}} \mathbf{B} \mathbf{G}_{q_i} \right\} \delta q_i^2 \quad (35)$$

Since δq_i^2 is always positive, the sign of $K_1 - K_2$ depends on the term inside the brackets.

Define

$$\mu = \bar{K} \delta q_i^2 \quad (36)$$

where

$$\bar{K} = \mathbf{G}_{q_i}^T \mathbf{B}^T [\mathbf{N}_o^T \mathbf{G}]_{\mathbf{u}} \mathbf{B} \mathbf{G}_{q_i} - [\mathbf{N}_o^T \mathbf{G}]_{q_i q_i} \quad (37)$$

Therefore, if $\bar{K} > 0$, the singular surface can admit motion into the positive direction of $\mathbf{N}_o^T$.

If $\bar{K} < 0$, the singular surface admits motion into the negative direction of $\mathbf{N}_o^T$.

6. CRITERIA

6.1 Crossability Criteria

- (1) If $\mathbf{Q}^*$ is indefinite (has both positive and negative eigenvalues), the singular surface admits motion for each parameter q_i which is at its limit, and is not a boundary.
- (2) If $\mathbf{Q}^*$ is either positive semi-definite or negative semi-definite, the following additional criteria must be evaluated.

- (a) If $\mathbf{N}_o$ is not perpendicular to $\mathbf{G}_{q_i}$, then σ in Eq. 30 must be evaluated
- (b) If $\mathbf{N}_o$ is perpendicular to $\mathbf{G}_{q_i}$, then $\bar{K}$ in Eq. 38 must be evaluated.
 - (i) If any of σ or $\bar{K}$ has a different sign than the nonzero eigenvalues of $\mathbf{Q}^*$, the singular surface admits motion along its normal.
 - (ii) If σ and $\bar{K}$ have the same sign as the nonzero eigenvalues of $\mathbf{Q}^*$, the singular surface does not admit motion along its normal. The direction for admitting motion in the positive or negative direction of $\mathbf{N}_o$, is determined by the sign of σ and $\bar{K}$.

6.2 Void Criteria

An enclosure in the workspace of a spatial manipulator or an area in the workspace of a planar manipulator that is bound by geometric entities that do not admit motion towards this enclosure or region is defined as a void. This also includes the definition of the external boundary as it will also exhibit a connected region or connected surfaces that enclose a volume and that do not admit motion towards the external.

7. EXAMPLES

7.1 A 4R-Manipulator

Consider the 4R serial manipulator shown in Fig. 3 with joint constraints as follows $-90^\circ \leq q_4 \leq 90^\circ$, $-180^\circ \leq q_3 \leq 30^\circ$, $0 \leq q_2 \leq 330^\circ$, and $0 \leq q_1 \leq 210^\circ$.

Figure 3 A 4R manipulator

The position vector function characterizing its workspace is given by

$$\mathbf{G}(\mathbf{q}) = \begin{bmatrix} \cos q_1 (10 + 5 \cos q_2 + 3 \cos(q_2 + q_3) + \cos(q_2 + q_3 + q_4)) \\ \sin q_1 (10 + 5 \cos q_2 + 3 \cos(q_2 + q_3) + \cos(q_2 + q_3 + q_4)) \\ 7 + 5 \sin q_2 + 3 \sin(q_2 + q_3) + \sin(q_2 + q_3 + q_4) \end{bmatrix} \quad (38)$$

Jacobian rank-deficiency conditions resulting from the row rank deficiency of $[\mathbf{G}_q]^T$ presented in Abdel-Malek and Yeh (1997) yield the following singular sets.

$\mathbf{p}^{(1)} = \{q_2 = 0, q_3 = -\pi\}$, $\mathbf{p}^{(2)} = \{q_2 = 0, q_3 = -\pi/6\}$, $\mathbf{p}^{(3)} = \{q_2 = 0, q_4 = -\pi/2\}$, $\mathbf{p}^{(4)} = \{q_2 = 0, q_4 = \pi/2\}$,
 $\mathbf{p}^{(5)} = \{q_3 = -\pi, q_4 = -\pi/2\}$, $\mathbf{p}^{(6)} = \{q_3 = -\pi, q_4 = \pi/2\}$, $\mathbf{p}^{(7)} = \{q_3 = \pi/6, q_4 = -\pi/2\}$, $\mathbf{p}^{(8)} = \{q_3 = \pi/6, q_4 = \pi/2\}$,
 $\mathbf{p}^{(9)} = \{q_2 = 33\pi/18, q_3 = -\pi\}$, $\mathbf{p}^{(10)} = \{q_2 = 33\pi/18, q_3 = \pi/6\}$, $\mathbf{p}^{(11)} = \{q_2 = 33\pi/18, q_4 = -\pi/2\}$,
 $\mathbf{p}^{(12)} = \{q_2 = 33\pi/18, q_4 = \pi/2\}$, $\mathbf{p}^{(13)} = \{q_3 = 0.3217, q_4 = -\pi/2\}$, $\mathbf{p}^{(14)} = \{q_3 = -0.3217, q_4 = \pi/2\}$,
 $\mathbf{p}^{(15)} = \{q_3 = -\pi, q_4 = 0\}$, $\mathbf{p}^{(16)} = \{q_3 = \pi/6, q_4 = -0.3287\}$, $\mathbf{p}^{(17)} = \{q_2 = 0, q_4 = 0\}$, $\mathbf{p}^{(18)} = \{q_2 = 33\pi/18, q_4 = 0\}$,
 $\mathbf{p}^{(19)} = \{q_3 = 0, q_4 = 0\}$, $\mathbf{p}^{(20)} = \{q_1^L, q_2^L\}$, $\mathbf{p}^{(21)} = \{q_1^L, q_2^U\}$, $\mathbf{p}^{(22)} = \{q_1^L, q_3^L\}$, $\mathbf{p}^{(23)} = \{q_1^L, q_3^U\}$,
 $\mathbf{p}^{(24)} = \{q_1^L, q_4^L\}$, $\mathbf{p}^{(25)} = \{q_1^L, q_2^L\}$, $\mathbf{p}^{(26)} = \{q_1^L, q_4^U\}$, $\mathbf{p}^{(27)} = \{q_1^U, q_2^L\}$, $\mathbf{p}^{(28)} = \{q_1^U, q_2^U\}$, $\mathbf{p}^{(29)} = \{q_1^U, q_3^L\}$,
 $\mathbf{p}^{(30)} = \{q_1^U, q_3^U\}$, $\mathbf{p}^{(31)} = \{q_1^U, q_4^L\}$, and $\mathbf{p}^{(32)} = \{q_1^U, q_4^U\}$. Substituting these singular sets into Eq. 38 yields parametric equations of singular surfaces shown in Fig. 4, and its cross section (at $q_1 = 0$ or xz -plane) in Fig. 5.

Figure 4 The workspace depicting all singular surfaces

Figure 5 A cross section of the workspace of the 4R manipulator

To determine the boundary to any voids, it is necessary to evaluate the criteria above for all surface patches. Consider for example point A_5 shown in Fig. 6 on surface $\mathbf{f}^{(15)}$, for which the singular set is $\mathbf{p}^{(15)} = \{q_3 = -\pi, q_4 = 0\}$, and the point is evaluated at

$\mathbf{u}_o^{(15)} = \{q_1 = 0, q_2 = 200^\circ\}$, hence $\mathbf{q}_o^{(15)} = [0 \quad 200^\circ \quad -180^\circ \quad 0]^T$. Note that Fig. 6 depicts an enlarged section of Fig. 5.

Figure 6 Part of the workspace showing curve segments

Substituting $\mathbf{p}^{(15)}$ into Eq. (38) yield

$\mathbf{f}^{(15)}(\mathbf{u}^{(15)}) = [10\cos q_1 + \cos q_1 \cos q_2 \quad 10\sin q_1 + \cos q_2 \sin q_1 \quad 7 + \sin q_2]^T$, where $\mathbf{u}^{(15)} = [q_1 \quad q_2]^T$. To

compute the normal at A_5 , the matrix $[\mathbf{G}_q(\mathbf{q}_o^{(15)})\mathbf{q}_s(\mathbf{s}_o^{(15)})]^T = \begin{bmatrix} 0 & 0 & 0 \\ 0.962 & 0 & -2.644 \\ 0 & 0 & 0 \\ -0.0537 & 0 & 1.476 \end{bmatrix}$ is

computed, and its nullspace is readily determined as $\{\{0.939 \quad 0 \quad 0.342\}, \{0 \quad 1 \quad 0\}\}$. Since the second vector is trivial, the normal is indeed $\mathbf{n}_o^T = [0.939 \quad 0 \quad 0.342]$. The matrix $\mathbf{H}^*$ is

calculated as $\mathbf{H}^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 7.920 & 0 & -4.421 \\ 0 & 0 & 0 & 0 \\ 0 & -4.421 & 0 & -2.467 \end{bmatrix}$ and the generalized inverse is calculated as

$\mathbf{B} = [\mathbf{E}_1 \mathbf{f}_u(\mathbf{u}_o^{(15)})]^{-1} \cdot \mathbf{E}_1 = \begin{bmatrix} 0 & 0.110 & 0 \\ 2.924 & 0 & 0 \end{bmatrix}$. The final (4×4) $\mathbf{Q}^*$ matrix is evaluated and its eigenvalues are computed as $\{-4.935, 0, 0, 0\}$. Since this is a negative semi-definite form, the algorithm must compute a value for σ . Since $q_3 = -\pi$ is a lower limit, $\delta q_3 = +1$ and $\sigma = \mathbf{N}_o^T \mathbf{G}_{q_3} (+1) = 0$. Since the normal is along the same direction as the derivative, a value

for $\bar{K}$ must be computed. The terms are evaluated as follows: $[\mathbf{N}_o^T \mathbf{G}]_{q_3 q_3} = -4.0$,

$\mathbf{G}_{q_3}^T \mathbf{B}^T = [0 \quad -4]$, and $[\mathbf{N}_o^T \mathbf{G}]_{\text{uu}} \mathbf{B} \mathbf{G}_{q_3} = [0 \quad -4]^T$. Therefore, $\bar{K} = -20.0056$. A negative sign indicates that the singular surface admits movement into the negative direction of $\mathbf{N}_o^T$. The reader is reminded that the curves shown in Fig. 6 are traces of the surfaces on a plane cutting through the workspace. Those points are indeed positions of the end-effector on a singular surface.

Similarly, for the same singular surface at point A_1 characterized by $q_1 = 0, q_3 = -10^\circ$ (which is an external boundary), the normal is $\mathbf{N}_o^T = [-0.985 \quad 0 \quad 0.17]^T$ and the eigenvalues are computed as $\{2.5, 1.8505, 0, 0\}$ and $\sigma = 0.868$; same sign as the eigenvalues. This indicates an admissible motion in that direction (but not the opposite direction).

On singular surface $\mathbf{f}^{(8)}$ at point A_3 where $\mathbf{u}^{(8)} = \{q_1, q_2\}$, and at $\mathbf{p}^{(8)} = \{q_3^U, q_4^U\}$ and $q_1 = 0, q_2 = 305^\circ$, the normal is $\mathbf{N}_o^T = [-0.8032 \quad 0 \quad 0.5957]^T$ and the eigenvalues are computed as $\{-2.8974, -1.5389, 0, 0\}$ and $\sigma = -1.5810$. On singular surface $\mathbf{f}^{(3)}$ at point A_4 with $\mathbf{u}^{(3)} = \{q_1, q_3\}$, at $\mathbf{p}^{(4)} = \{q_2^L, q_4^U\}$ at $q_1 = 0, q_3 = 0^\circ$, and $q_4 = q_4^L$, the normal is $\mathbf{N}_o^T = [-0.9487 \quad 0 \quad 0.3160]^T$ and the eigenvalues are computed as $\{4.5529, -1.4901, 0, 0\}$ which is an indefinite quadratic form. Surface patches that do not admit motion and that enclose a volume, are boundary to the workspace. For this example, there are three sets of boundary surface patches characterized by the following surface regions and illustrated in the cross section of Fig. 7. The workspace with interior and exterior boundaries is shown in Fig. 8.

<p>The external boundary is characterized by</p> <p>$\mathbf{f}^{(19)}(q_1, q_2); 0 \leq q_1 \leq 210^\circ \text{ and } 0 \leq q_2 \leq 330^\circ;$</p> <p>$\mathbf{f}^{(17c)}(q_1, q_3); 0 \leq q_1 \leq 210^\circ \text{ and } 34^\circ \leq q_3 \leq 7^\circ$</p> <p>$\mathbf{f}^{(10)}(q_1, q_4); 0 \leq q_1 \leq 210^\circ \text{ and } -20^\circ \leq q_4 \leq 20^\circ$</p> <p>$\mathbf{f}^{(18)}(q_1, q_3); 0 \leq q_1 \leq 210^\circ \text{ and } 0^\circ \leq q_3 \leq 30^\circ$</p>	<p>The boundary to Void 1</p> <p>$\mathbf{f}^{(15)}(q_1, q_2); 0 \leq q_1 \leq 210^\circ \text{ and } 0^\circ \leq q_2 \leq 330^\circ$</p> <p>$\mathbf{f}^{(9)}(q_1, q_4); 0 \leq q_1 \leq 210^\circ \text{ and } -17^\circ \leq q_4 \leq 0^\circ$</p> <p>$\mathbf{f}^{(17b)}(q_1, q_3); 0 \leq q_1 \leq 210^\circ \text{ and } -\pi \leq q_3 \leq -175^\circ$</p>
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The boundary to the void is characterized by the following surface patches:

$\mathbf{f}^{(5)}(q_1, q_2); 0 \leq q_1 \leq 210^\circ \text{ and } 276^\circ \leq q_2 \leq 283^\circ,$

$\mathbf{f}^{(17a)}(q_1, q_3); 0 \leq q_1 \leq 210^\circ \text{ and } -154^\circ \leq q_3 \leq -70^\circ,$

$\mathbf{f}^{(8)}(q_1, q_2); 0 \leq q_1 \leq 210^\circ \text{ and } 293^\circ \leq q_2 \leq 312^\circ,$ and

$\mathbf{f}^{(11)}(q_1, q_3); 0 \leq q_1 \leq 210^\circ \text{ and } -142^\circ \leq q_3 \leq -30^\circ.$

Figure 7 A cross section of the workspace depicting the boundary

Figure 8 The workspace of the 4R manipulator

7.2 Identifying the boundary of a general 4-DOF

Consider the general serial 4-DOF manipulator shown in Fig. 9 that was presented by Ceccarelli and Vinciguerra (1995). This example is treated here for validation purposes and to demonstrate the ability of the proposed method in determining the external boundary.

Figure 9 The general 4-DOF serial manipulator

Applying the rank deficiency criteria to $G_q(\mathbf{q})$ yields singular sets when substituted into $G(\mathbf{q})$ result in singular surfaces (examples are shown in Fig. 10). The trace of these surfaces on an xy -plane at $z = 1.0$ is shown in Fig. 11a and the exterior boundary is shown in Fig. 11b.

Figure 10 Singular surface

Figure 11 (a) A cross section of the general 4-DOF (b) Identification of the boundary

7.3 A 5-DOF RPPRR Manipulator

Consider the 5-DOF manipulator shown in Fig. 12.

Figure 12 A five-DOF manipulator

The vector function of the position of the end-effector is written as

$$\mathbf{G}(\mathbf{q}) = \begin{bmatrix} -(13 + q_2) \sin q_1 - 6.6 \cos q_4 \sin q_1 - 3.9 \cos q_4 \cos q_5 \sin q_1 + 3.9 \sin q_1 \sin q_4 \sin q_5 \\ (13 + q_2) \cos q_1 + 6.6 \cos q_4 \cos q_1 + 3.9 \cos q_4 \cos q_5 \cos q_1 - 3.9 \cos q_1 \sin q_4 \sin q_5 \\ 10 + q_3 + 6.6 \sin q_4 + 3.9 \cos q_5 \sin q_4 + 3.9 \cos q_4 \sin q_5 \end{bmatrix}$$

and is subjected to the following onstraints: $0 \leq q_1 \leq 240^\circ$, $0 \leq q_2 \leq 2.8$, $0 \leq q_3 \leq 4.5$, $0 \leq q_4 \leq 330^\circ$, and $-153^\circ \leq q_5 \leq 128^\circ$. Using the rank deficiency conditions, singular sets are computed and substituted into $\mathbf{G}(\mathbf{q})$. The trace of these surfaces on a xz -cutting plane is shown in Fig. 13.

Figure 13 A cross section of the workspace

The normal acceleration analyses applied to each surface patch identifies admissible directions of output motion. Boundary surface patches are identified. Figure shows these patches that are boundary and their admissible normal directions. A cross section of the workspace depicting the external boundary and the void is shown in Fig. 14.

Figure 14 Identifying the external boundary and the void in the workspace

Because the formulation yields parametric surface patches and because the analysis yields a quadratic form that delineates boundary surface patches, it is possible to represent the boundary to the void in an exact form (Figure 15). The boundary surface patches to the void are identified by the following and are depicted in Figure 16.

$$\mathbf{f}^{(12)}(q_1, q_4), 322^\circ \leq q_4 \leq 330^\circ \text{ and } 0 \leq q_1 \leq 240^\circ$$

$$\mathbf{f}^{(13)}(q_1, q_4), 248^\circ \leq q_4 \leq 277^\circ \text{ and } 0 \leq q_1 \leq 240^\circ$$

$$\mathbf{f}^{(14)}(q_1, q_4), 142^\circ \leq q_4 \leq 171^\circ \text{ and } 0 \leq q_1 \leq 240^\circ$$

$$\mathbf{f}^{(15)}(q_1, q_2), 0 \leq q_2 \leq 1.3 \text{ and } 0 \leq q_1 \leq 240^\circ$$

$$\mathbf{f}^{(16)}(q_1, q_4), 59^\circ \leq q_4 \leq 97^\circ \text{ and } 0 \leq q_1 \leq 240^\circ$$

$$\mathbf{f}^{(17)}(q_1, q_3), 3.18 \leq q_3 \leq 3.54 \text{ and } 0 \leq q_1 \leq 240^\circ$$

Figure 15 A cross section of the workspace depicting the boundary and void surface patches

Figure 16 Boundary surfaces (traces of surface patches) to a void

8. CONCLUSIONS

A broadly applicable formulation for the determination of voids in the workspace of serial mechanical manipulators has been introduced. Based on acceleration analyses of an end-effector point on a singular surface, the difference in normal acceleration components yields a quadratic form.

It was shown that definiteness properties of the quadratic form are used to delineate surface regions that admit motion output normal in the direction of the normal vector at this point. It is emphasized that this work is possible only because of the parametric nature of the singular surfaces delineated using the rank deficiency criteria introduced elsewhere.

While some surface patches in the workspace admit no motion in the normal direction, only enclosures that are bound by such surface patches are identified as a boundary. It was shown that those surface patches that do not admit motion but are not part of the boundary are indeed impediments to motion and may cause the manipulator to exhibit control difficulties. It was also shown that the interior boundary (boundary to voids) and the exterior boundary of the workspace can be determined using this method. The presented formulation is capable of delineating the external boundary of a general n-DOF manipulator.

Compared with the work by Cecarelli and Vinciguerra (1995), holes and voids are equally treated since the differentiation between an apple and a ring void has no significance using the proposed mathematical formulation. In the work by the two authors, a hole in the workspace is identified as a region of unreachable points containing at least a straight line that passes through it without making contact with the workspace.

The formulation presented herein and its accompanying experimental code demonstrate an effective method for understanding the kinematics inside the workspace of serial manipulators.

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