Path Trajectory Verification for Robot Manipulators in a Manufacturing Environment

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In a manufacturing environment where a robotic arm is programmed to follow a specified trajectory such as in welding, painting, and soldering, it is often the case that the arm reaches a singular configuration, where the program is stopped, the arm is switched to a new configuration, and the motion is continued. This difficulty has been a long standing problem. This paper presents a mathematical formulation for verifying whether a trajectory can be completed, uninterrupted, avoiding halting of a planned path. In some cases, this formulation also allows the selection of an initial configuration to ensure a smooth path trajectory. The paper presents an analytical formulation for determining barriers to motion inside the workspace of manipulator arms. Crossability analysis of the end-effector on a barrier are addressed. A criterion for selecting an initial configuration that would result in an un-interrupted motion is introduced. The mathematical theory is validated through numerical examples of planar and spatial manipulator arms.

Keywords: *Trajectory planning, robot singularities, trajectory verification, robot path, and manipulator kinematics*

NOTATION

$\mathbf{x}$ Position vector
 Φ Constraint vector function
 $\mathbf{q}$ Vector of generalized coordinates
 ${}^i\mathbf{T}_j$ (4×4) transformation matrix relating body i to j
 ${}^i\mathbf{R}_j$ (3×3) rotation matrix between frames i and j
 a_i midpoint of an interval

b_i half range of an interval
 $\Phi_{\mathbf{q}}$ Jacobian of Φ with respect to $\mathbf{q}$
 S A set
 $\mathcal{R}^n$ Space of n -generalized coordinates
 $\mathbf{u}$ A vector of generalized coordinates
 $\mathbf{p}_i$ A singular set of constant generalized coordinates
 ζ_o A normal vector
 Ψ A subsurface parametrized same as its corresponding Φ but bounded

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λ_i	A slack variable (generalized coordinate)	ρ_o	Radius of curvature
s	Vector of slack variables	v_t	Tangential velocity
$\mathbf{n}_o$	Normal vector including slack variables	$\mathbf{E}$	A matrix to defined to delete rows
$\mathbf{H}^*$	Matrix of the quadratic form	$\mathbf{B}$	generalized inverse
a_n	Component of normal acceleration	$\mathbf{Q}^*$	Matrix of the quadratic form including joint limits
Λ	Indicator for crossability of an entity	σ	Indicator for crossability (suppl.)
		$\bar{K}$	Indicator for crossability (suppl.)

1 INTRODUCTION

Many tasks performed by a manipulator arm in a manufacturing environment such as arc welding, spray painting, continuous soldering, and deburring, require that the end-effector follows a prescribed trajectory as depicted in Fig. 1. During this process of motion from point A to B, singular behavior of the manipulator may occur. This type of problem has been well documented (1). Many researchers have referred to this problem as determining the admissibility of paths in the task space (2), in joint space (3), or trajectory verification in the workspace (4). Although this type of problem has received considerable attention in recent years, a general formulation for determining whether the specified trajectory can be followed without interruption remains an open problem. Two search methods to determine the path that satisfies all constraints were presented by Milton and Jennings (5). Other works that have dealt with the subject are Althofer et al. (6), Seereeram and Wen (7), Agrawal and Xu (8), and Nenchev (9).

Studies that have addressed difficulties in the control of manipulators due to the appearance of interior curves and surfaces were reported by Waldron (10) and Nielsen et al. (11). In the latter work, the controllability of a mechanical arm is discussed from a differential geometry point of view. The fundamental concept of crossable and non-crossable surfaces inside a manipulator's workspace was addressed by Oblak and Kohli (12). Their report introduced the concept of a crossable surface but did not establish generalized criteria.

Recently, Haug et al. (13) presented a *numerical* algorithm for identifying and analyzing barriers to output control of manipulators using first and second-order Taylor approximations of the output in selected directions. More recently, Abdel-Malek and Yeh (14) presented an analytical method by which all singular surfaces and curves can be analytically determined for a serial manipulator with joint limits. The work was extended and compared with numerical techniques in (15). The work is extended in this paper to the verification of a trajectory subject to the existence of singular geometric entities inside the workspace.

In many situations, however, there are many choices for selecting an initial configuration for the manipulator at point A (Fig. 1). The choice of initial configuration is critical to the completion of the trajectory without interruption. During the motion from point A to B,

the manipulator may reach a configuration at which one or more joints are at their limits. At this configuration, the motion halts, and a different configuration is specified. To illustrate, consider the manipulator arm shown in Fig. 2, where it is required that a point on the end-effector follow the trajectory from A to B. As the end-effector is moving from point A to point B (the orientation is not considered) the manipulator reaches a configuration at point Q where it is no longer possible to continue the motion because joints 2 and 3 reached their limits. At this point, another configuration is specified (as shown in Fig. 3) and the motion continues. Therefore, although the motion can be completed, an interruption has occurred that may cause damage to the manufacturing process.

A choice of a different initial configuration at point A could have resulted in a smooth motion of the trajectory without interruption, where the end-effector is able to move from point A to point B without encountering singular behavior.

It is evident that the formulation for selecting an appropriate trajectory should take into consideration the joint limits of the manipulator. In this paper, a mathematical formulation for determining curves and surfaces inside the workspace upon which the manipulator may exhibit singular behavior will be introduced. For a general multi degree-of-freedom serial manipulator, it will be shown that there are surfaces inside the workspace for which the end-effector cannot cross, or may cross in one direction only. Those surfaces represent certain configurations of the manipulator for which joint constraints are active or for which the manipulator has assumed a singular configuration such as the case that may occur when two consecutive joints are parallel. The direction of movement on those surfaces will also be discussed. Verification of the trajectory is then performed by intersecting the path curve with internal barriers. Selecting an initial configuration at point A will be detrimental to the successful completion of the planned trajectory without interruption. The method will be presented through a simple example of the planar three degree-of-freedom manipulator (Fig. 1). A general formulation will then be deduced and a complete numerical example of a spatial manipulator arm will be presented.

2 SINGULAR GEOMETRIC ENTITIES INSIDE THE WORKSPACE

Inside the workspace of a mechanical manipulator, there exist curves and surfaces that represent impediments to motion that may cause output control difficulties (13, 15). The objective of this section is to determine these barriers.

The position vector of a point on the end-effector of a serial manipulator can be written in terms of joint coordinates as

$$\mathbf{x} = \Phi(\mathbf{q}) \quad (1)$$

where $\mathbf{q} \in \mathfrak{R}^n$ is the vector of n -generalized coordinates, and $\Phi(\mathbf{q})$ can be obtained from the multiplication of the homogeneous transformation matrices defined by the D-H representation method (16, 17) as

$${}^0\mathbf{T}_n = {}^0\mathbf{T}_1 {}^1\mathbf{T}_2 \dots {}^{n-1}\mathbf{T}_n = \begin{bmatrix} {}^i\mathbf{R}_j(\mathbf{q}) & \Phi(\mathbf{q}) \\ \mathbf{0} & 1 \end{bmatrix} \quad (2)$$

where ${}^i\mathbf{R}_j$ is the rotation matrix and ${}^i\mathbf{T}_j$ is the homogeneous transformation matrix relating bodies i to j . In order to extend the formulation to include joint limits in the form of inequality constraints such as $q_i^{\min} \leq q_i \leq q_i^{\max}$, a transformation is introduced as

$$q_i = a_i + b_i \sin \lambda_i \quad (3)$$

where $a_i = (q_i^{\max} + q_i^{\min})/2$ and $b_i = (q_i^{\max} - q_i^{\min})/2$ are the mid point and half range of the inequality constraint and λ is a slack variable. The position constraint function is then written in terms of the extended vector $\mathbf{s} = [\lambda_1 \lambda_2 \dots \lambda_n]^T$ such that

$$\mathbf{x} = \Phi^*(\mathbf{s}) = [\Phi(\mathbf{q}(\lambda_i))] \quad (4)$$

End-effector velocities can be determined by deriving equation (4) with respect to time and using the chain rule as

$$\dot{\mathbf{x}} = \Phi_{\mathbf{q}} \mathbf{q}_s \dot{\mathbf{s}} \quad (5)$$

where $\mathbf{q}_s = \partial \mathbf{q} / \partial \lambda_j$ and $\dot{\mathbf{s}} = d\mathbf{s}/dt$ and $\Phi_{\mathbf{q}}$ is the Jacobian defined by

$$\Phi_{\mathbf{q}} = [\partial \Phi_i(\mathbf{q}) / \partial q_j] \quad (6)$$

Define the subvector of $\mathbf{q}$ as a set of constant generalized coordinates $\mathbf{p}_i \in \mathfrak{R}^m$ where $m \leq n-1$, and $\mathbf{q} = \mathbf{u} \cup \mathbf{p}_i$. Singular sets $\mathbf{p}_i$ of the manipulator can be obtained from studying the dimension of the null space of $\Phi_{\mathbf{q}}^T(\mathbf{q})$, defined by Abdel-Malek and Yeh (14) as the set

$$\mathbf{S} = \{ \mathbf{p}_i \in \mathfrak{R}^m; \dim[Null(\Phi_{\mathbf{q}}^T(\mathbf{q}))] \geq 1, \mathbf{q} = [\mathbf{u} \quad \mathbf{p}_i] \text{ for some constant } \mathbf{p}_i \} \quad (7)$$

where $\mathbf{u} \in \mathfrak{R}^{n-m}$ is the vector of generalized coordinates that are not in $\mathbf{p}_i$. On a singular surface, the term $\Phi_{\mathbf{q}} \mathbf{q}_s|_{\mathbf{q}_o, \mathbf{s}_o}$ is rank-deficient. Therefore, the rows of $\Phi_{\mathbf{q}} \mathbf{q}_s$ are dependent and there exists a set of constants $\mathbf{n}_o = [n_1 \quad n_2 \quad n_3]^T$, that satisfy

$$[\Phi_{\mathbf{q}} \mathbf{q}_s] \mathbf{n}_o^T = \mathbf{0} \quad (8)$$

where $\mathbf{n}_o$ is the basis to the null space of $[\Phi_{\mathbf{q}} \mathbf{q}_s]$.

It was shown that substituting $\mathbf{p}_i$ into $\Phi(\mathbf{q})$ yields singular surfaces and curves parametrized by $\Psi(\mathbf{u})$ where $\mathbf{u}$ is the new vector of generalized coordinates such that

$$\Phi(\mathbf{u}, \mathbf{p}_i) = \Psi(\mathbf{u}) \quad (9)$$

and $\mathbf{n}_o$ is indeed the normal vector to these geometric entities defined by equation (9).

To demonstrate the above formulation, consider the planar 3DOF manipulator shown in Fig. 1. The position vector of a point on the end-effector of the planar three degree-of-freedom manipulator is defined by

$$\Phi(\mathbf{q}) = \begin{bmatrix} 3 \cos q_1 + 2 \cos(q_1 + q_2) + \cos(q_1 + q_2 + q_3) \\ 3 \sin q_1 + 2 \sin(q_1 + q_2) + \sin(q_1 + q_2 + q_3) \end{bmatrix} \quad (10)$$

where $\mathbf{q} = [q_1 \quad q_2 \quad q_3]^T$ is the vector of generalized coordinates and the joints are subject to the following inequality constraints $-\frac{\pi}{3} \leq q_i \leq \frac{\pi}{3}$. In order to determine the

manipulator's singular characteristics, it is necessary to study the manipulator's (2×3) Jacobian $\Phi_q = [\partial \Phi_i / \partial q_j]$ where

$$\Phi_q = \begin{bmatrix} -3\sin q_1 - 2\sin(q_1 + q_2) - \sin(q_1 + q_2 + q_3) & -2\sin(q_1 + q_2) - \sin(q_1 + q_2 + q_3) & -\sin(q_1 + q_2 + q_3) \\ 3\cos q_1 + 2\cos(q_1 + q_2) + \cos(q_1 + q_2 + q_3) & 2\cos(q_1 + q_2) + \cos(q_1 + q_2 + q_3) & \cos(q_1 + q_2 + q_3) \end{bmatrix} \quad (11)$$

The dimension of the nullspace of the Jacobian is studied. In this case, only one singular set results from the $\dim[Null(\Phi_q^T(\mathbf{q}))] \geq 1$ defined by $S_1: \mathbf{p}_1 = \{q_2 = 0, q_3 = 0\}$.

Singularities resulting from the reduced order manipulator are $S_2 = \{\mathbf{p}_i, i = 2 \dots 7\}$ where

$\mathbf{p}_2 = \{q_1 = -\pi/3, q_3 = 0\}$, $\mathbf{p}_3 = \{q_1 = \pi/3, q_3 = 0\}$, $\mathbf{p}_4 = \{q_1 = -\pi/3, q_3 = \tan^{-1}(\sqrt{3}/2)\}$, $\mathbf{p}_5 = \{q_1 = \pi/3, q_3 = -\tan^{-1}(\sqrt{3}/2)\}$, $\mathbf{p}_6 = \{q_1 = -\pi/3, q_2 = \tan^{-1}(\sqrt{3}/5)\}$, and $\mathbf{p}_7 = \{q_1 = \pi/3, q_2 = -\tan^{-1}(\sqrt{3}/5)\}$. Singularity sets resulting from a combination of joint limits are $\{\mathbf{p}_i = \{q_i^o, q_j^o\}, i \neq j, i, j = 8 \dots 19\}$ where q_i^o and q_j^o are the limits of the inequality constraints. Substituting each singularity set $\mathbf{p}_i$, $\{i=1 \dots 19\}$ into equation (10) yields singular curves Ψ_i part of which are shown in Fig. (4).

3 IMPEDIMENTS TO MOTION

The determination of the crossability of a surface/curve inside the workspace can be established from the definiteness properties of a quadratic form representing the normal acceleration of the system. The acceleration of the end-effector can be written in matrix form as

$$\ddot{\mathbf{x}} = \dot{\mathbf{s}} \left\{ \mathbf{q}_s^T \Phi_{qq} \mathbf{q}_s + \sum_{i=1}^n \frac{d\Phi}{dq_i} \cdot [q_i]_{ss} \right\} \dot{\mathbf{s}} + [\Phi_q \mathbf{q}_s] \ddot{\mathbf{s}} \quad (12)$$

where $\Phi_{qq} = [\partial^2 \Phi / \partial q_i \partial q_j]$ and $[q_i]_{ss} = [\partial^2 q_i / \partial s_i \partial s_j]$. To obtain the component of normal acceleration, $\ddot{\mathbf{x}}$ is projected onto the normal $\mathbf{n}_o$. Multiplying both sides of equation (12) by the vector $\mathbf{n}_o^T$ eliminates the last term of the right hand side (definition of the normal as the basis of the null space in equation 8). The component of the normal acceleration is then

$$a_n = \mathbf{n}_o^T \ddot{\mathbf{x}} = \dot{\mathbf{s}}^T \mathbf{H}^* \dot{\mathbf{s}} \quad (13)$$

where

$$\mathbf{H}^*(\mathbf{q}_o, \mathbf{s}_o) = \mathbf{q}_s^T [\mathbf{n}_o^T \Phi]_{qq} \mathbf{q}_s + \sum_{i=1}^n \frac{d(\mathbf{n}_o^T \Phi)}{dq_i} \cdot [q_i]_{ss} \quad (14)$$

In order to determine whether the manipulator's end-effector can cross a singular entity, it is necessary to study acceleration components that would cause motion normal to the surface or curve. However, to generalize the discussion, the normal curvature of a singular surface is considered. The criteria for the end-effector to cross a singular surface is specified by the normal acceleration indicator Λ given by

$$\Lambda = a_n - \frac{|v_t^2|}{\rho_o} \quad (15)$$

where $v_t = \dot{\mathbf{x}}$ is the tangential velocity, $1/\rho_o$ is the normal curvature of the singular surface *with respect to the tangent direction* of v_t (ρ_o is the radius of curvature), and the component $|v_t|^2/\rho_o$ defined in terms of the parametrized entity equation is

$$|v_t|^2/\rho_o = \dot{\mathbf{u}}^T [\mathbf{n}^T \Psi]_{uu} \dot{\mathbf{u}} \quad (16)$$

If a_n and v_t are on the same side of the surface, then the end-effector admits motion in the normal direction of this side of the surface (Fig. 5a). In specific, a positive value of Λ indicates that the end-effector will move in the normal direction away from the center, a zero value indicates the manipulator moving on the surface, and a negative value indicates the manipulator admitting motion towards the center of curvature as illustrated in Fig. 5b.

At a singular configuration, the tangent velocity in terms of the entity Ψ or in terms of the constant constraint function Φ , can be written as

$$\Psi_u \dot{\mathbf{u}} = \Phi_q \dot{\mathbf{q}} \quad (17)$$

In order to obtain an expression for $\dot{\mathbf{u}}$, an inverse for Ψ_u is needed. Because Ψ_u is a nonsquare (3×2) matrix, define a matrix $\mathbf{E}$ such that

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{if the first and second rows of } \Psi_u \text{ are independent} \quad (18)$$

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{if the first and third rows of } \Psi_u \text{ are independent} \quad (19)$$

$$\mathbf{E} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{if the second and third rows of } \Psi_u \text{ are independent} \quad (20)$$

Multiply both sides of equation (17) from the left by $\mathbf{E}$ yields

$$\mathbf{E} \Psi_u \dot{\mathbf{u}} = \mathbf{E} \Phi_q \dot{\mathbf{q}} \quad (21)$$

Multiplying both sides of equation (21) by the inverse of $[\mathbf{E} \Psi_u]$ yields

$$\dot{\mathbf{u}} = [\mathbf{E} \Psi_u]^{-1} \mathbf{E} \Phi_q \mathbf{q}_s \dot{\mathbf{s}} \quad (22)$$

where the chain rule $\dot{\mathbf{q}} = \mathbf{q}_s \dot{\mathbf{s}}$ is used. Equation (22) can be written as

$$\dot{\mathbf{u}} = \mathbf{B} \Phi_q \mathbf{q}_s \dot{\mathbf{s}} \quad (23)$$

where the matrix $\mathbf{B}$ is a generalized inverse of Ψ_u defined by

$$\mathbf{B} \equiv [\mathbf{E} \Psi_u]^{-1} \mathbf{E} \quad (24)$$

In this extended formulation (considering joint limits), the quadratic form of the normal acceleration indicator Λ is written in terms of the $\mathbf{Q}^*$ matrix as

$$\Lambda = a_n - \frac{|v_t|^2}{\rho_o} = \dot{\mathbf{s}}^T \mathbf{Q}^* \dot{\mathbf{s}} \quad (25)$$

where

$$\mathbf{Q}^* = \mathbf{H}^* - \mathbf{q}_s^T \Phi_q^T \mathbf{B}^T [\mathbf{n}^T \Psi]_{uu} \mathbf{B} \Phi_q \mathbf{q}_s \quad (26)$$

Definiteness properties of the quadratic form of equation (26) define the crossability of a singular surface/curve. If $\mathbf{Q}^*$ is indefinite (has both positive and negative eigenvalues), the singular surface/curve is **crossable**. Because of the transformation used to include joint limits in this formulation, the derivative of equation (3) at a joint limit yields a zero velocity. The nullification of joint velocities gives rise to an apparent loss of information from the matrix of the quadratic form (equation 26). Therefore, these supplemental criteria are introduced. If $\mathbf{Q}^*$ is either positive semi-definite or negative semi-definite, the following additional criteria must be evaluated.

- (a) If $\mathbf{n}_o$ is not perpendicular to $\mathbf{x}_{q_i}$, then define σ as a projection of a variational motion $\mathbf{x}_{q_i} \delta q_i$ onto the normal, evaluated by

$$\sigma = \mathbf{n}^T \mathbf{x}_{q_i} \delta q_i \quad (27)$$

where δq_i is given a magnitude of ± 1 as follows

$$\delta q_i = \begin{cases} +1 & \text{if } q_i \text{ is at lower bound} \\ -1 & \text{if } q_i \text{ is at upper bound} \end{cases} \quad (28)$$

- (b) If $\mathbf{n}_o$ is perpendicular to $\mathbf{x}_{q_i}$, then the difference in the normal curvature between the two surfaces as

$$\bar{K} = \mathbf{x}_{q_i}^T \mathbf{B}^T [\mathbf{n}^T \mathbf{x}]_{\mathbf{u}\mathbf{u}} \mathbf{B} \mathbf{x}_{q_i} - [\mathbf{n}^T \mathbf{x}]_{q_i q_i} \quad (29)$$

where equation (29) is the difference in curvature between the surfaces.

- (i) If any of σ or $\bar{K}$ has a different sign than the nonzero eigenvalues of $\mathbf{Q}^*$, the singular surface/curve is crossable.
- (ii) If σ and $\bar{K}$ have the same sign as the nonzero eigenvalues of $\mathbf{Q}^*$, the singular surface/curve is non-crossable.

The direction along which the end-effector can move, either in the positive or negative direction of $\mathbf{n}$, will be in the same sense of signs of σ and $\bar{K}$.

The above criteria establish whether a manipulator at a singular surface may exhibit output control difficulties, i.e., a manipulator following a trajectory, crossing one of those barriers will have to halt.

4 VERIFYING A TRAJECTORY

For a trajectory defined as a curve in space parametrized in terms of one variable $\mathbf{x}''(t)$, it is necessary to determine all intersections of this curve with non-crossable singular entities $\Psi(\mathbf{u})$ inside the workspace. Therefore, at each intersection point satisfying

$$\mathbf{x}''(t) - \Psi(\mathbf{u}) = \mathbf{0} \quad (30)$$

it is necessary to study the crossability criteria characterized by the definiteness properties of the matrix $\mathbf{Q}^*$.

For the planar three-bar example, it is first necessary to evaluate the criteria above for each independent segment, i.e., curves that cross other curves are segmented. For

example, consider a point $\mathbf{p}_1$ on curve Ψ_{19} at $q_1 = \pi/4$, $q_2 = \pi/3$, and $q_3 = \pi/3$. Evaluating the normal $\mathbf{n}$ from the basis of the null space of

$$[\Phi_q \mathbf{q}_\lambda]^T = \begin{bmatrix} -3.4765 & 0 & 0 \\ 0.9315 & 0 & 0 \end{bmatrix}$$

at $\mathbf{p}_1$ yields the normal $\mathbf{n} = [0.2588 \quad 0.9659]^T$. Evaluating the matrix $\mathbf{H}^*$ of equation (14) yields

$$\mathbf{H}^* = \begin{bmatrix} -2.4930 & 0 & 0 \\ 0 & 2.0944 & 0 \\ 0 & 0 & 1.0472 \end{bmatrix}$$

Since this is a planar example, the singular curve is parametrized by only one joint variable (i.e., $\Psi_{19}(q_1)$), therefore, $\mathbf{x}_{q_1}$ is a (2×1) matrix. The matrix $\mathbf{E}$ is therefore a $\mathbf{E} = [1 \quad 0]$ vector and $\mathbf{B} = [\mathbf{E} \Psi_u]^{-1} \mathbf{E} = [-0.1992 \quad 0]$. The matrix $\mathbf{Q}^*$ of the quadratic form is evaluated as

$$\mathbf{Q}^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2.094 & 0 \\ 0 & 0 & 1.047 \end{bmatrix}$$

The eigenvalues of $\mathbf{Q}^*$ are evaluated $\{0, 1.047, \text{ and } 2.094\}$, which indicates a positive semi-definite quadratic form. Since the constraints of joints 2 and 3 are active (upper bounds) and $\mathbf{x}_{q_2}$ and $\mathbf{x}_{q_3}$ are not parallel to $\mathbf{n}$, the additional value of σ is evaluated. For q_2 at its upper limit, $\delta q_2 = -1$, and $\sigma = \mathbf{n}^T \mathbf{x}_{q_2} \delta q_2 = 2$ which is positive. For q_3 , choose $\delta q_3 = -1$ and evaluate $\sigma = \mathbf{n}^T \mathbf{x}_{q_3} \delta q_3 = 1$ which is also positive. Both values for σ are positive (same signs as the eigenvalues of $\mathbf{Q}^*$). Therefore, surface Ψ_{19} only admits movement into the positive direction of the normal $\mathbf{n}$ (i.e., only towards the internal of the workspace) as expected.

To further demonstrate, consider point $\mathbf{p}_2$ on Ψ_6 at $q_1 = \pi/8$, $q_2 = \tan^{-1}(\sqrt{3}/5)$, and $q_3 = -\pi/3$. The matrix $\mathbf{Q}^*$ of the quadratic form is indefinite (both negative and positive eigenvalues) indicating a crossable curve at $\mathbf{p}_2$. Similar analysis is performed for all curve segments. For example, the curve Ψ_8 is crossable for the segment of $0.7137 \leq q_3 \leq \pi/3$; non-crossable for the segment $0 \leq q_3 \leq 0.7137$; and crossable for $-\pi/3 \leq q_3 \leq 0$. The curve Ψ_9 is non-crossable for the segment of $0 \leq q_3 \leq \pi/3$; crossable for the segment $-0.7137 \leq q_3 \leq 0$; and non-crossable for $-\pi/3 \leq q_3 \leq -0.7137$. All barriers are numbered and shown in Fig. (6a). Only crossable curves are shown in Fig. (6b) while non-crossable lines are removed. The direction of admissible movement is also shown by arrows on each curve segment in (6b).

Consider the trajectory shown in Fig. 1 and reproduced in Fig. 7. It is required to study this trajectory such that the end-effector can travel from point A to B without interruption.

The curve is intersected with only those singular entities that are non-crossable (barriers) since they may be impediments to motion. Performing the intersection defined by equation (30) yields point Q on singular curve Ψ_{18} where $\mathbf{p}_{18} = \{q_2 = \pi/3, q_3 = -\pi/3\}$.

In order to plan a trajectory that would not be interrupted requires knowledge of all inverse kinematic solutions along the path. The trajectory planner generates a number of solutions, one of which is selected. The initial configuration is chosen such that when the end-effector reaches point Q, it is not in a singular configuration.

To illustrate, Fig. 8 depicts the trajectory planned for the three-bar manipulator in the workspace. An alternative configuration is specified at point Q as shown in Fig. 8. Let the manipulator start with the initial configuration specified by posture III, such that at point Q, the manipulator is in posture II, therefore, sustaining a smooth motion and avoiding the configuration defined by posture I.

5 A SPATIAL EXAMPLE

In general, manipulators used in manufacturing are spatial with three or more degrees-of-freedom. The singular entities generated inside the workspace of such manipulators are surfaces parametrized in terms of two or three parameters. Consider, for example, the five degree-of-freedom manipulator shown in Fig. (9). The joints are constrained as $0 \leq q_1 \leq 2\pi$, $-\pi/4 \leq q_2 \leq 5\pi/4$, $0 \leq q_3 \leq 5$, $0 \leq q_4 \leq 2\pi$, and $-\pi/2 \leq q_5 \leq \pi/2$.

The position vector of point $\mathbf{p}$ in Fig. 9 is determined as

$$\Phi(\mathbf{q}) = \begin{bmatrix} -5\sin q_5 \cos q_1 \sin q_2 \sin q_4 - 5\sin q_5 \sin q_1 \cos q_4 + 5\cos q_1 \cos q_2 \cos q_5 + 15\cos q_1 \cos q_2 + q_3 \cos q_1 \cos q_2 \\ -5\sin q_5 \sin q_1 \sin q_2 \sin q_4 + 5\sin q_5 \cos q_1 \cos q_4 + 5\sin q_1 \cos q_2 \cos q_5 + 15\sin q_1 \cos q_2 + q_3 \sin q_1 \cos q_2 \\ 5\cos q_2 \sin q_4 \sin q_5 + 5\sin q_2 \cos q_5 + 15\sin q_2 + q_3 \sin q_2 + 25 \end{bmatrix} \quad (31)$$

Singularity analysis yields the following singular sets.

$\Psi_1: q_2 = \pi/2, q_4 = 0, \text{ and } q_5 = \pi/2$	$\Psi_7: q_3 = 5, q_4 = 0, \text{ and } q_5 = 0$
$\Psi_2: q_2 = -\pi/4, q_4 = 0, \text{ and } q_5 = 0$	$\Psi_8: q_2 = \pi/2, q_3 = 5, \text{ and } q_4 = 0$
$\Psi_3: q_2 = -\pi/4, q_4 = \pi/2, \text{ and } q_5 = \pi/2$	$\Psi_9: q_2 = -\pi/4, q_3 = 0, \text{ and } q_4 = \pi/2$
$\Psi_4: q_2 = -\pi/4, q_4 = 3\pi/2, \text{ and } q_5 = \pi/2$	$\Psi_{10}: q_2 = -\pi/4, q_3 = 5, \text{ and } q_4 = \pi/2$
$\Psi_5: q_3 = 0, q_4 = 0, \text{ and } q_5 = 0$	$\Psi_{11}: q_3 = 0 \text{ and } q_5 = \pi/2$
$\Psi_6: q_2 = \pi/2, q_3 = 0, \text{ and } q_4 = 0$	$\Psi_{12}: q_3 = 5 \text{ and } q_5 = \pi/2$

Substituting each singularity set into equation (31) yields a parametric equation of a singular surface. Note that singular surfaces Ψ_{11} and Ψ_{12} are two irregular parametric surfaces since they are parametrized in three variables (q_1 , q_2 , and q_4). A cross section of the workspace of the 5-DOF manipulator depicting all singular surfaces is shown in Fig. (10). Because it is possible to obtain parametric equations of all singular entities, a mathematics symbolic manipulator is used to depict the entities.

Crossability of singular surfaces can now be determined. For example, consider a point located on surface Ψ_{11} at $q_1 = \pi/2$, $q_2 = \pi/4$, $q_3 = 0$, $q_4 = \pi/2$, and $q_5 = \pi/2$. The normal vector is evaluated as $\mathbf{n} = [0 \quad 1/\sqrt{5} \quad 2/\sqrt{5}]^T$ and the Eigenvalues of the $\mathbf{Q}^*$ matrix are $\{0, 0, 0, 2.3717, 7.4509\}$. Since the quadratic form of equation (25) is positive semi-definite (q_3 and q_5 have active constraints) then equation (27) is used to evaluate σ . Both values of σ for q_3 and q_5 are positive (0.9487 and 4.7434). Therefore, Ψ_{11} is non-crossable (only in the positive direction of $\mathbf{n}$). Similarly for a point on surface Ψ_{11} located at ($q_1 = \pi/2$, $q_2 = \pi/4$, $q_3 = 5$, $q_4 = \pi/2$, and $q_5 = \pi/2$), the normal vector is evaluated as $\mathbf{n} = [0 \quad 3/\sqrt{34} \quad 5/\sqrt{34}]^T$. The Eigenvalues of $\mathbf{Q}^*$ are $\{0, 0, 0, -2.425, 7.6195\}$ indicating an indefinite form. Therefore, the surface is crossable. A cross-section of the workspace is shown in Fig. (11a) where each curve is numbered. Non-crossable surfaces are shown in Fig. (11b) with their respective direction of admissible movement.

A trajectory planned in the workspace, e.g., the trajectory from A to B on Fig. 11a, depicts intersections with only the barrier due to surface Ψ_2 . This singular surface characterizes a configuration for which $q_2 = -\pi/4$ and $q_4 = q_5 = 0$ as shown in Fig. 12a. For this configuration, the singular surface is shown in Fig. 12b.

6 CONCLUSIONS

The problem of verifying a path trajectory in the workspace of a robot manipulator subject to joint constraints has been addressed. The manipulator may reach a singular configuration where the motion is interrupted. A general formulation for determining barriers to motion inside the workspace is presented whereby the singular behavior of the underlying mechanism is characterized in the form of parametric geometric entities. It was shown that at the intersection of a path trajectory with a barrier, the manipulator's end-effector can be analytically investigated subject to definiteness properties of a quadratic form. Admissible normal movements of the end-effector across the singular surface are determined from this quadratic form. The formulation is applicable to manipulators that can be represented using the Denavit-Hartenberg method.

It was also shown that all barriers to motion that may cause interruption to the motion are analytically determined. The intersection of a path trajectory with those surfaces can readily be determined. Therefore, although it is always possible to verify whether a

trajectory admits an uninterrupted motion (smooth motion), it is not always possible to compute an initial configuration that does admit a smooth motion.

Although the formulation is presented using planar three-degree-of-freedom and spatial five degree-of-freedom manipulators, it is valid for the general class of serial manipulators. While the experimental computer code used to implement the numerical algorithm is not refined, it nevertheless has demonstrated a generalized method for trajectory verification of serial manipulators.

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Fig. 1 A manipulator following a trajectory connecting points A and B

Fig. 2 A manipulator at a singular configuration interrupting the motion

Fig. 3 A configuration that allows continuation of the motion

Fig. 4 Singular curves of the planar 3-bar linkage

Fig. 5. (a) End-effector can move to side 1 (b) end-effector's movement into side 2 depends on a_n and v_t

Fig. 6 (a) Singular curves

Fig. 6(b) Admissible normal movement directions

Fig. 7 The manipulator arm at a singular configuration with all non-crossable surfaces depicted

Fig. 8 The manipulator at a singular configuration (I), at an initial configuration (II), and at a configuration (III) that would allow the continuation of the motion.

Fig. 9 A five degree-of-freedom manipulator

Fig. 10 A cross-section of the workspace of the 5-DOF manipulator

Fig. 11 (a) Crossable surfaces and (b) Admissible directions of normal movement

Fig. 12 (a) The manipulator in the singular configuration corresponding to Ψ_2

Fig. 12 (b) the singular surface Ψ_2 shown in the workspace