1. Time Domain Analysis

Consider the following graphic representing the input/output characteristics of the system.

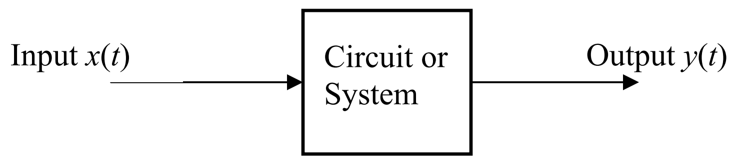

Figure 1-1 Schematic of input/output characteristics of a system.

Analysis and design are made easier if the engineer takes advantage of linear system theory. To wit, a linear time-invariant lumped system with input $x(t)$ and output $y(t)$ is described by the following equation:

$$a_n \frac{d^n y}{dt^n} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + \cdots + b_0 x \tag{1.1}$$

The Laplace Transform of Equation (1.1) is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} \tag{1.2}$$

The function $H(s)$ is the transfer function. The inverse Laplace Transform of $H(s)$ is the impulse response $h(t)$. The output $y(t)$ is found by convolution given by

$$y(t) = \int_0^t h(\tau) x(t-\tau) d\tau \tag{1.3}$$

Time-response of First-Order Systems

Consider the simple first-order system given by

$$H(s) = \frac{b_0}{a_1 s + a_0} = \frac{b_0/a_1}{s + a_0/a_1} = \frac{Ka}{s+a} = \frac{K}{\frac{1}{a}s + 1} = \frac{K}{\tau s + 1} \tag{1.4}$$

where K is the static gain, and $-a$ is the system pole (or eigenvalue). If the input is a unit step, then the output is given by

$$y(t) = K\left(1 - e^{-t/\tau}\right) \tag{1.5}$$

where $\tau = 1/a$ (sec) is known as the time constant. The step response (for K=1) is plotted in Figure 1-2.

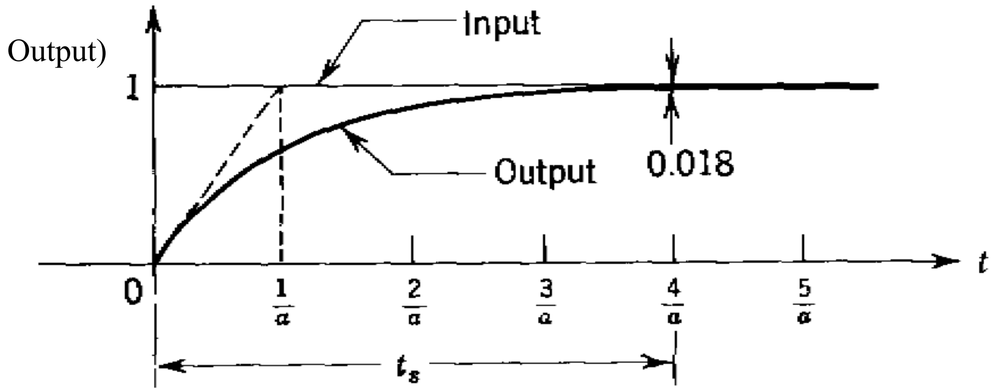

Figure 1-2 Step response of unity gain LTI first-order system.

The settling time is defined as the time required by a system, given a disturbance, to settle to within some fixed percentage of its final value. For our purposes, the settling time is defined as the time to settle within 1.8% of the final value. For a first-order system, this time is equal to four time constants, i.e., $t_s = 4\tau$.

Time-response of Second-Order Systems

The standard form for a second-order transfer function is

$$H(s) = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{1.6}$$

In practice, we will usually have

$$0 \le \xi \le 1, \text{ and } \omega_n > 0$$

Zeta (ξ) is the damping factor. The natural frequency is ω_n. The transfer function has poles at

$$s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} \tag{1.7}$$

When $\xi = 1$, the system is said to be critically damped. When $\xi = 0$, the system is said to be undamped. When $0 < \xi < 1$ the system is said to be underdamped. Most real systems are underdamped with $0.3 < \xi < 0.8$.

If the input is a unit step function, then it can be easily shown that the step response (for *K*=1) is given by

$$y(t) = 1 - e^{-\xi\omega_n t}\,\frac{\sin\left(\omega_n\sqrt{1-\xi^2}\,t + \cos^{-1}\xi\right)}{\sqrt{1-\xi^2}} \tag{1.8}$$

The step responses for various values of the damping coefficient are plotted in Figure 1-3.

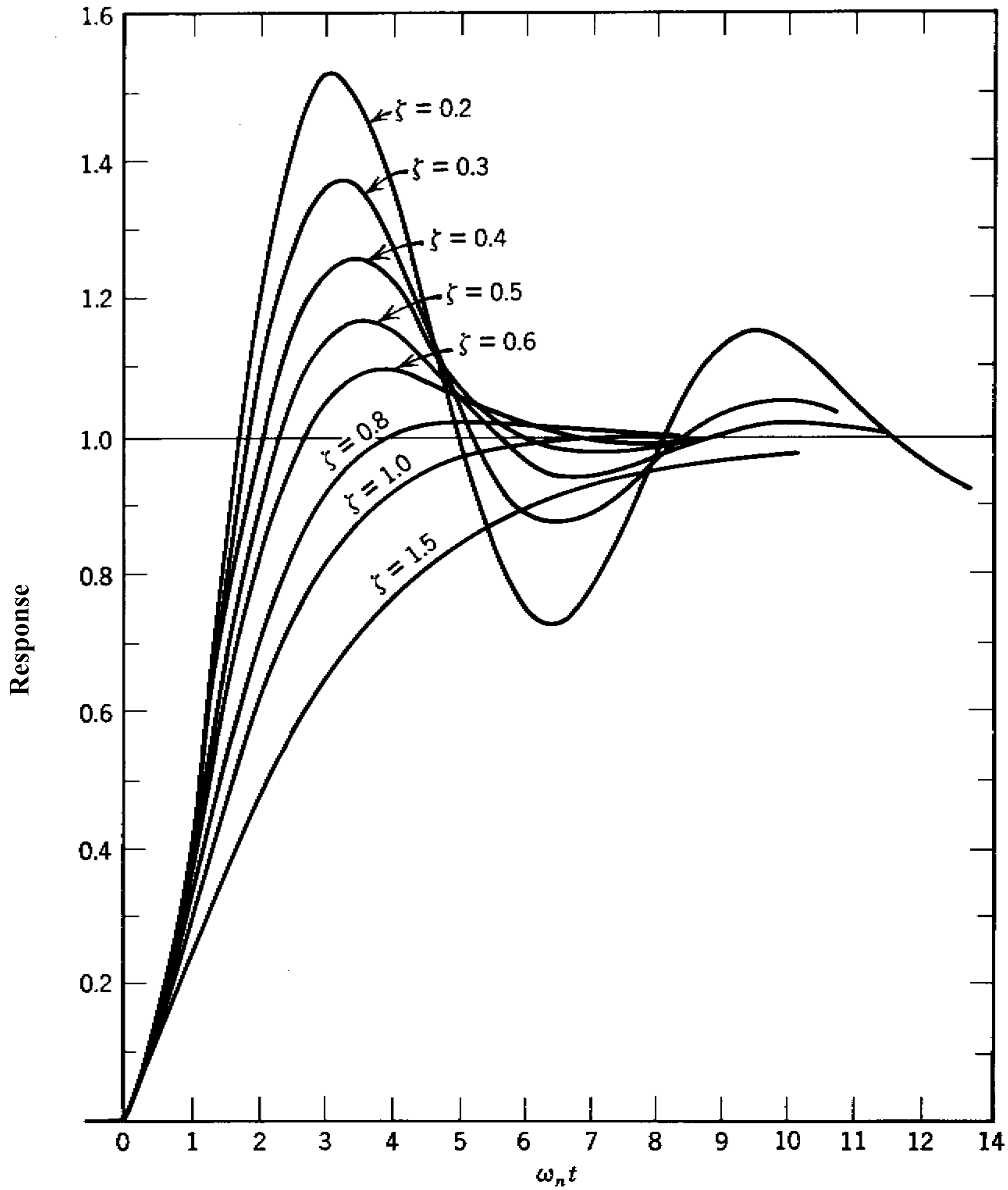

Figure 1-3 Step response of unity gain LTI second-order system.

The step response oscillates at a damped natural frequency ω_d given by

$$\omega_d = \omega_n \sqrt{1-\xi^2} \tag{1.9}$$

We see from the above figure that the magnitude of the oscillations decreases as $e^{-\xi\omega_n t}$. We now can define the time constant for second-order systems as

$$\tau = \frac{1}{\xi\omega_n} \tag{1.10}$$

As in first-order systems, the settling time is defined as four times the time constant.

If one calculates the derivative of the step response given in Equation (1.8) and sets this derivative to zero, then the times to the peaks of the step response are available. These are given by

$$t_k = \frac{k\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{k\pi}{\omega_d}, \quad k = 0,1,2,\cdots \tag{1.11}$$

The overshoot, which is the difference between the peak value and the final value, is thus given by

$$\text{Overshoot} = \exp\left[-\left(\frac{\xi\pi}{\sqrt{1-\xi^2}}\right)\right] \tag{1.12}$$

2. Frequency Domain Analysis

The frequency response is defined as the transfer function evaluated with *s* replaced by $j\omega$. Thus, the frequency response is

$$H(j\omega) = H(s)\,|_{s\to j\omega} \tag{2.1}$$

The magnitude portion of the frequency response is usually plotted in deciBels (dB) and is calculated as

$$20\log_{10}\left|H(j\omega)\right| \text{ versus } \log_{10}\omega$$

with the companion plot

$$\angle H(j\omega) \text{ versus } \log_{10}\omega$$

Such a plot is known as a Bode plot. Thus the two parts of a Bode plot are magnitude and phase plots, respectively.

The reason that the frequency response is so useful is suggested by the following well-known property of LTI systems. Suppose that the linear time-invariant system given in Equation (1.1) and with the transfer function given in Equation (1.2) has a sinusoidal input given by $x(t) = M\sin(\omega t)$. Then it is easy to show that in the steady-state (i.e., once the transients have died out) the output is given by

$$y(t) = M\left|H(jw)\right|\sin\left(\omega t + \angle H(jw)\right) \tag{2.2}$$

This is a fundamental property of all linear time-invariant systems. This property will become more useful after applying Fourier analysis, which tells us that ALL "buildable" signals can be approximated to any arbitrary degree of accuracy as a sum of sinusoids (there will be more later on this topic).

Frequency-response of First-Order Systems

Consider the frequency response of a unity-gain first-order system. The corresponding frequency response will consist of terms of the form

$$\frac{1}{1+jw\tau}$$

As $\omega \to 0$, then $20\log_{10}\limits_{\omega\to 0}\left|\frac{1}{(1+jw\tau)}\right| = -20\log_{10}\limits_{\omega\to 0}\left|(1+jw\tau)\right| \to 0$. Thus, at low frequencies, the gain goes to 0 dB, or unity.

On the other hand, as ω increases to where $\omega\tau \gg 1$, then we get $20\log_{10}\limits_{\omega\to\text{large}}\left|\frac{1}{(1+jw\tau)}\right| \approx -20\log_{10}\limits_{\omega\to\text{large}} w - 20\log_{10}\limits_{\omega\to\text{large}} \tau$. Thus a plot of this as a function of $\log_{10}\omega$ is a straight line of slope –20dB/decade, or –6 dB/octave. When $\omega = 1/\tau$, then the magnitude is 0.707 times the static gain. This frequency is the 3-dB point, half-power point, or corner frequency (these terms are synonymous).

The following figure demonstrates the phase and angle portions of the Bode plot for a first-order system.

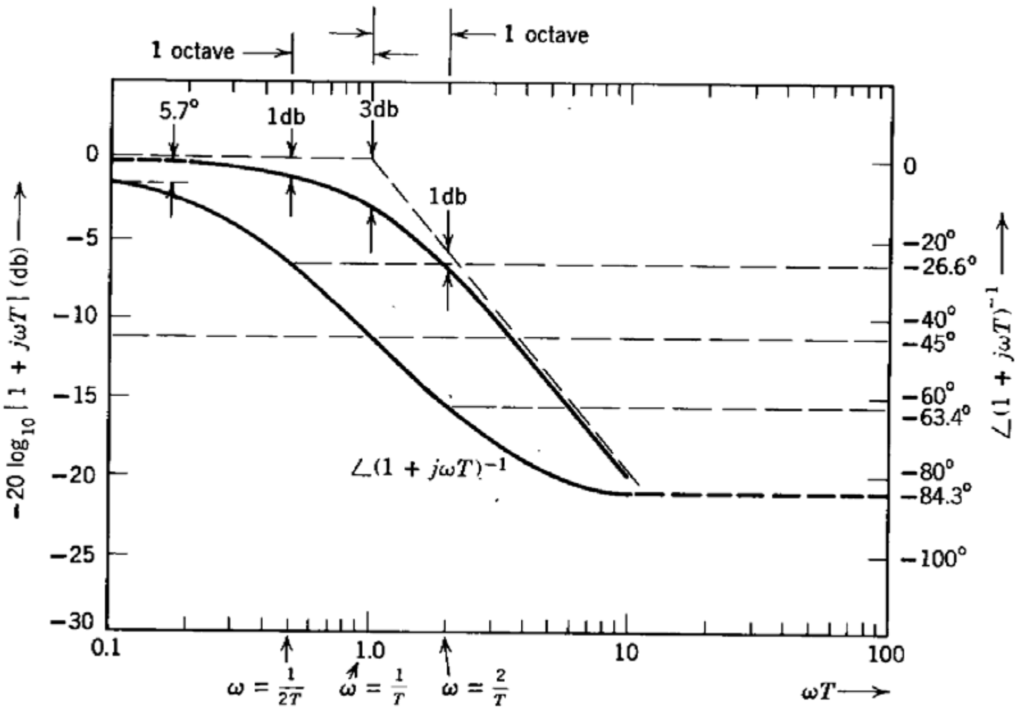

Figure 2-1 Frequency response of unity gain LTI first-order system. The variable *T* in this figure is the time constant (variable τ in our treatment).

Frequency-response of Second-Order Systems

Consider the frequency response of a unity-gain second-order system given by Equation (1.6) (with *K*=1). Replacing $s \to j\omega$, we get

$$\frac{1}{1 + j\left(2\xi / \omega_n\right)\omega - \left(\omega^2 / \omega_n^2\right)} \tag{2.3}$$

The shape Bode plot of the standard second-order system depends on two parameters: ω_n and ξ. Let's first consider the magnitude part of the plot, i.e., the plot of

$$-20\log_{10}\left|1 + 2\xi\frac{j\omega}{\omega_n} - \frac{\omega^2}{\omega_n^2}\right| \tag{2.4}$$

As the frequency decreases, the magnitude portion of the Bode plot goes to zero. Thus, as the frequency goes to zero, the magnitude portion of the Bode plot approaches a horizontal line through 0 dB.

As the frequency increases, then Equation (2.4) becomes

$$-20\log_{10}\left|1+2\xi\frac{j\omega}{\omega_n}-\frac{\omega^2}{\omega_n^2}\right| \to -40\log_{10}\frac{\omega}{\omega_n} \qquad (2.5)$$
$$\to -40\log_{10}\omega + 40\log_{10}\omega_n$$

Thus as the frequency increases, the magnitude portion of the Bode plot approaches a straight line with a slope of –40 dB/decade. The corner frequency in this case is $\omega = \omega_n$.

From Equation (2.5) we see that neither the low nor the high frequency asymptotes depend on ξ. However, the behavior near the corner frequency strongly depends on ξ. This is shown in the following diagram.

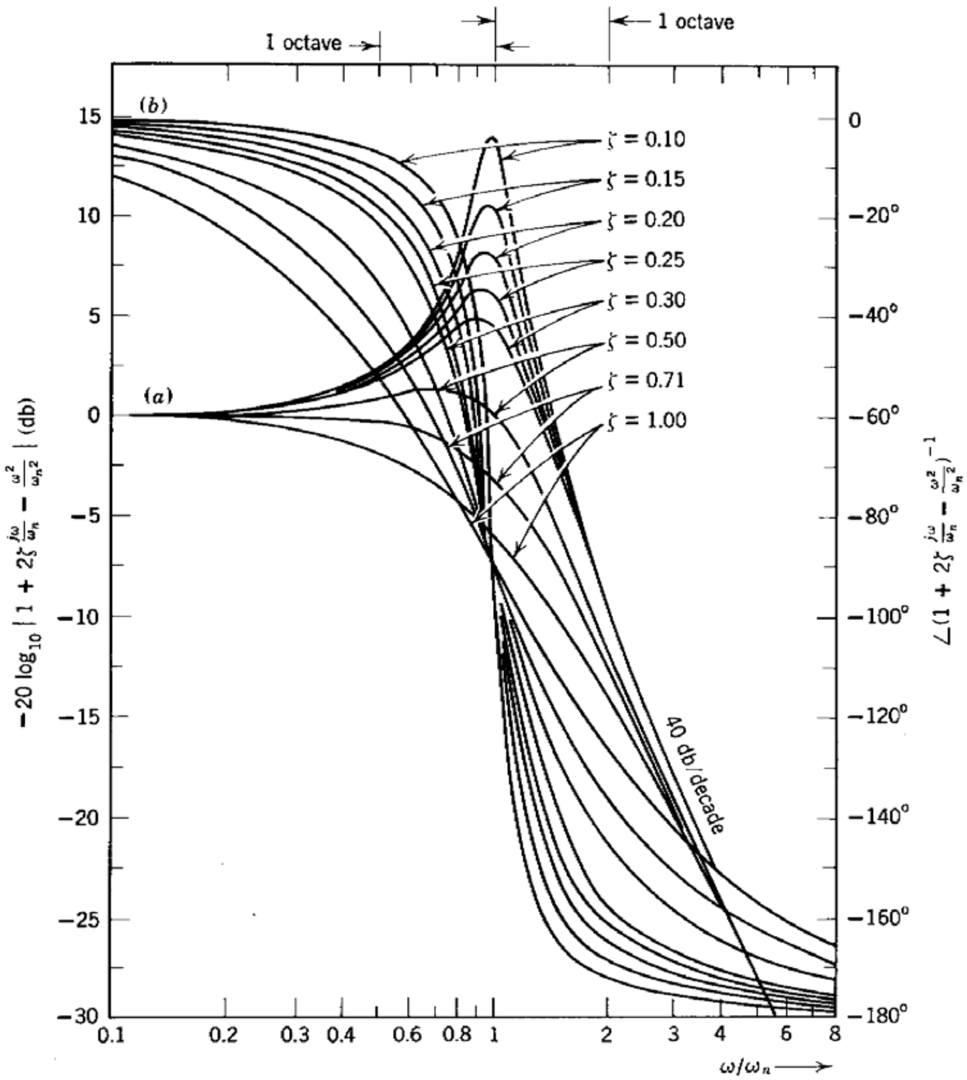

Figure 2-2 Frequency response of unity gain LTI second-order system.

A typical frequency response of a second-order system is shown in the following illustration.

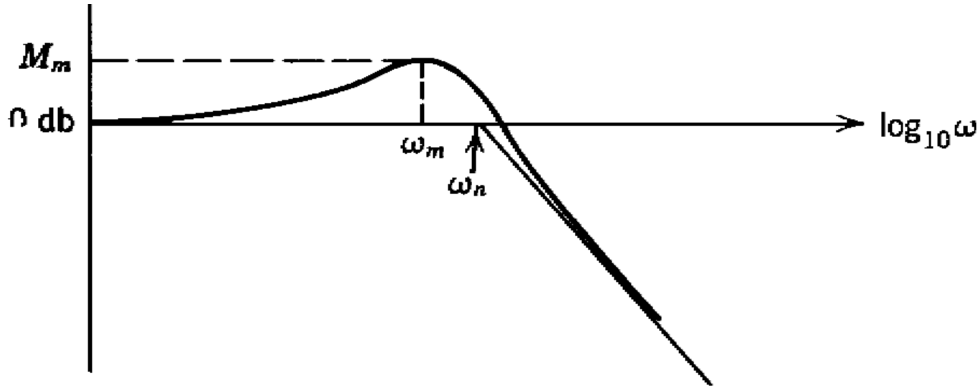

Figure 2-3 Frequency response of a typical unity gain LTI second-order system with damping coefficient less than 0.707.

The peak gain will occur when $\left|1+2\xi\frac{j\omega}{\omega_n}-\frac{\omega^2}{\omega_n^2}\right|$ is a minimum. After simple differentiation, we find that the minimum ω_m is given by

$$\omega_m = \omega_n\sqrt{1-2\xi^2} \tag{2.6}$$

The corresponding peak value M_m is given by

$$M_m = -20\log_{10}\left(2\xi\sqrt{1-\xi^2}\right) \tag{2.7}$$

Note that as ξ goes to zero, the peak value goes off to infinity, and the peak frequency approaches ω_n. This is exactly what one should expect for an undamped system with resonance.

For $\xi = 0.707$, $\omega_m = 0$. Thus for this value of damping, the magnitude portion of the Bode plot does not have a peak. In engineering parlance, this value of ξ gives a maximally plat frequency response. This value of damping yields the widest possible bandwidth.

Now let's look at the angle portion of the Bode plot of a second-order system. The angle contribution is derived from

$$-\angle\left(1+2\xi\frac{j\omega}{\omega_n}-\frac{\omega^2}{\omega_n^2}\right) = -\tan^{-1}\left(\frac{2\xi\omega\omega_n}{\omega_n^2-\omega^2}\right) \tag{2.8}$$

The curves for this for various values of ξ are shown in Figure 2-2.

Asymptotic Behavior of Frequency-response of Systems

As a brief overview, the following table contains the asymptotic behavior of systems up to second-order.

Term	Corner Frequency	Magnitude Slope	Angle
Constant K	None	0 dB/dec	0˚
$1/j\omega$	0	-20 dB/dec	-90˚
$\dfrac{1}{1+jw\tau}$	$1/\tau$	-20 dB/dec	-90˚
$\dfrac{1}{1+j\left(2\xi/\omega_n\right)\omega-\left(\omega^2/\omega_n^2\right)}$	$\omega_n;\xi$	-40 dB/dec	-180˚