Dr. Rodgers Homework # 2 52:217

1. If $\underline{v} = (z-y)\underline{i} + (x-z)\underline{j} + (y-x)\underline{k}$ compute $\nabla \cdot \underline{v}$

$$\nabla \cdot \underline{v} = \sum \frac{\partial v_i}{\partial x_i} = \frac{\partial}{\partial x}(z-y) + \frac{\partial}{\partial y}(x-z) + \frac{\partial}{\partial z}(y-x) = 0$$

$$\nabla \times \underline{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (z-y) & (x-z) & (y-x) \end{vmatrix} = \left[\frac{\partial}{\partial y}(y-x) - \frac{\partial}{\partial z}(x-z)\right]\underline{i} + \left[\frac{\partial}{\partial z}(z-y) - \frac{\partial}{\partial x}(y-x)\right]\underline{j} + \left[\frac{\partial}{\partial x}(x-z) - \frac{\partial}{\partial y}(z\right.$$

$$= [2]\underline{i} + [2]\underline{j} + [2]\underline{k}$$

2. If s (scalar), $\underline{u}$ & $\underline{v}$ (vectors) are differentiable continuously, verify in Cartesian coordinates that:

a) $\nabla \cdot (\underline{u} + \underline{v}) = \nabla \cdot \underline{u} + \nabla \cdot \underline{v}$

let $\underline{u} = u_x\underline{i} + u_y\underline{j} + u_z\underline{k}$, $\underline{v} = v_x\underline{i} + v_y\underline{j} + v_z\underline{k}$

$\underline{u} + \underline{v} = (u_x + v_x)\underline{i} + (u_y + v_y)\underline{j} + (u_z + v_z)\underline{k}$

$$\nabla \cdot (\underline{u} + \underline{v}) = \sum \frac{\partial (u_i + v_i)}{\partial x_i}$$

$$= \frac{\partial (u_x + v_x)}{\partial x} + \frac{\partial (u_y + v_y)}{\partial y} + \frac{\partial (u_z + v_z)}{\partial z}$$

$$= \frac{\partial u_x}{\partial x} + \frac{\partial v_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial v_y}{\partial y} + \frac{\partial u_z}{\partial z} + \frac{\partial v_z}{\partial z}$$

$$= \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} + \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$= \sum \frac{\partial u_i}{\partial x_i} + \sum \frac{\partial v_i}{\partial x_i}$$

$$= \nabla \cdot \underline{u} + \nabla \cdot \underline{v}$$

2b) $\nabla \cdot (\nabla \times \underline{v}) = 0$

$$\nabla \times \underline{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\underline{i} - \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right)\underline{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\underline{k}$$

$$\nabla \cdot (\nabla \times \underline{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right)$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 v_z}{\partial x \partial y}} - \cancel{\frac{\partial^2 v_y}{\partial x \partial z}} - \cancel{\frac{\partial^2 v_z}{\partial y \partial x}} + \cancel{\frac{\partial^2 v_x}{\partial y \partial z}}$$

$$+ \cancel{\frac{\partial^2 v_y}{\partial z \partial x}} - \cancel{\frac{\partial^2 v_x}{\partial z \partial y}}$$

$= 0$ as functions are continuously differentiable $\left[\frac{\partial^2 v_z}{\partial x \partial y} = \frac{\partial^2 v_z}{\partial y \partial x}\right]$

2c) $\nabla \times (\nabla s) = 0$

$$\nabla s = \frac{\partial s}{\partial x} + \frac{\partial s}{\partial y} + \frac{\partial s}{\partial z}$$

$$\nabla \times (\nabla s) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} & \frac{\partial s}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 s}{\partial y \partial z} - \frac{\partial^2 s}{\partial z \partial y}\right)\underline{i} - \left(\frac{\partial^2 s}{\partial x \partial z} - \frac{\partial^2 s}{\partial z \partial x}\right)\underline{j} + \left(\frac{\partial^2 s}{\partial x \partial y} - \frac{\partial^2 s}{\partial y \partial x}\right)\underline{k}$$

$= 0$ as each term $= 0$

d) $\nabla \cdot (s\,\underline{\underline{D}}) = s(\nabla \cdot \underline{\underline{D}}) + (\nabla s) \cdot \underline{\underline{D}}$

LHS

$$\left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right) \cdot \Big(s\,D_{ii}\,\underline{i}\,\underline{i} + s\,D_{ij}\,\underline{i}\,\underline{j} + s\,D_{ik}\,\underline{i}\,\underline{k} + s\,D_{ji}\,\underline{j}\,\underline{i}$$

$$+ s\,D_{jj}\,\underline{j}\,\underline{j} + s\,D_{jk}\,\underline{j}\,\underline{k} + s\,D_{ki}\,\underline{k}\,\underline{i} + s\,D_{kj}\,\underline{k}\,\underline{j} + s\,D_{kk}\,\underline{k}\,\underline{j}\Big)$$

$$= \frac{\partial}{\partial x} s\,D_{ii}\,\underline{i} + \frac{\partial s\,D_{ij}}{\partial x}\,\underline{j} + \frac{\partial}{\partial z} s\,D_{ik}\,\underline{k}$$

$$+ \frac{\partial}{\partial y} s\,D_{ji}\,\underline{i} + \frac{\partial}{\partial y} s\,D_{jj}\,\underline{j} + \frac{\partial}{\partial z} s\,D_{jk}\,\underline{k}$$

$$+ \frac{\partial}{\partial z} s\,D_{ki}\,\underline{i} + \frac{\partial}{\partial z} s\,D_{kj}\,\underline{j} + \frac{\partial}{\partial z} s\,D_{kk}\,\underline{k}$$

$$= \left(\frac{\partial}{\partial x} s\,D_{ii} + \frac{\partial}{\partial y} s\,D_{ji} + \frac{\partial}{\partial z} s\,D_{ki}\right)\underline{i}$$

$$+ \left(\frac{\partial}{\partial x} s\,D_{ij} + \frac{\partial}{\partial y} s\,D_{jj} + \frac{\partial}{\partial z} s\,D_{kj}\right)\underline{j}$$

$$+ \left(\frac{\partial}{\partial x} s\,D_{ik} + \frac{\partial}{\partial y} s\,D_{jk} + \frac{\partial}{\partial z} s\,D_{kk}\right)\underline{k}$$

$$= \left(D_{ii}\frac{\partial s}{\partial x} + D_{ji}\frac{\partial s}{\partial y} + D_{ki}\frac{\partial s}{\partial z}\right)\underline{i} + s\left(\frac{\partial D_{ii}}{\partial x} + \frac{\partial D_{ji}}{\partial y} + \frac{\partial D_{ki}}{\partial z}\right)$$

$$+ \left(D_{ij}\frac{\partial s}{\partial x} + D_{jj}\frac{\partial s}{\partial y} + D_{kj}\frac{\partial s}{\partial z}\right)\underline{j} + s\left(\frac{\partial D_{ij}}{\partial x} + \frac{\partial D_{jj}}{\partial y} + \frac{\partial D_{kj}}{\partial z}\right)$$

$$+ \left(D_{ik}\frac{\partial s}{\partial x} + D_{jk}\frac{\partial s}{\partial y} + D_{kk}\frac{\partial s}{\partial z}\right)\underline{k} + s\left(\frac{\partial D_{ik}}{\partial x} + \frac{\partial D_{jk}}{\partial y} + \frac{\partial D_{kk}}{\partial z}\right)$$

$$= (\nabla s) \cdot \underline{\underline{D}} + s(\nabla \cdot \underline{\underline{D}})$$

2e) $\frac{1}{2}\nabla(\underline{V}\,\underline{V}) = \underline{V}\cdot\nabla\underline{V} + \underline{V}\times(\nabla\times\underline{V})$

$(V_x \underline{i} + V_y \underline{j} + V_z \underline{k})\cdot(V_x\underline{i} + V_y\underline{j} + V_z\underline{k}) = (V_x^2 + V_y^2 + V_z^2)$

$\nabla(V_x^2+V_y^2+V_z^2) = \frac{\partial}{\partial x}(V_x^2+V_y^2+V_z^2)\underline{i} + \frac{\partial}{\partial y}(V_x^2+V_y^2+V_z^2)\underline{j}$

$+ \frac{\partial}{\partial z}(V_x^2+V_y^2+V_z^2)\underline{k}$

$$= \left(2V_x\frac{\partial V_x}{\partial x} + 2V_y\frac{\partial V_y}{\partial x} + 2V_z\frac{\partial V_z}{\partial x}\right)\underline{i}$$
$$+ \left(2V_x\frac{\partial V_x}{\partial y} + 2V_y\frac{\partial V_y}{\partial y} + 2V_z\frac{\partial V_z}{\partial y}\right)\underline{j}$$
$$+ \left(2V_x\frac{\partial V_x}{\partial z} + 2V_y\frac{\partial V_y}{\partial z} + 2V_z\frac{\partial V_z}{\partial z}\right)\underline{k}$$

so $\frac{1}{2}\nabla(\underline{V}\cdot\underline{V}) = \left(V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_y}{\partial x} + V_z\frac{\partial V_z}{\partial x}\right)\underline{i}$
$$+ \left(V_x\frac{\partial V_x}{\partial y} + V_y\frac{\partial V_y}{\partial y} + V_z\frac{\partial V_z}{\partial y}\right)\underline{j}$$
$$+ \left(V_x\frac{\partial V_x}{\partial z} + V_y\frac{\partial V_y}{\partial z} + V_z\frac{\partial V_z}{\partial z}\right)\underline{k}$$

RHS $\underline{V}\cdot\nabla\underline{V} = (V_x\underline{i} + V_y\underline{j} + V_z\underline{k})\cdot\left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right)(V_x\underline{i} + V_y\underline{j} + V_z\underline{k})$

$$= (V_x\underline{i} + V_y\underline{j} + V_z\underline{k})\cdot\left(\frac{\partial}{\partial x}V_x\underline{i}\,\underline{i} + \frac{\partial}{\partial x}V_y\underline{i}\,\underline{j} + \frac{\partial}{\partial x}V_z\underline{i}\,\underline{k}\right.$$
$$+ \frac{\partial}{\partial y}V_x\underline{j}\,\underline{i} + \frac{\partial}{\partial y}V_y\underline{j}\,\underline{j} + \frac{\partial}{\partial y}V_z\underline{j}\,\underline{k}$$
$$\left. + \frac{\partial}{\partial z}V_x\underline{k}\,\underline{i} + \frac{\partial}{\partial z}V_y\underline{k}\,\underline{j} + \frac{\partial}{\partial z}V_z\underline{k}\,\underline{k}\right)$$

$$\underline{v}\cdot\nabla\underline{v} =$$

$$v_x \frac{\partial}{\partial x} v_x \underline{i} + v_x \frac{\partial}{\partial x} v_y \underline{j} + v_x \frac{\partial}{\partial x} v_z \underline{k}$$

$$+ v_y \frac{\partial}{\partial y} v_x \underline{i} + v_y \frac{\partial}{\partial y} v_y \underline{j} + v_y \frac{\partial}{\partial y} v_z \underline{k}$$

$$+ v_z \frac{\partial}{\partial z} v_x \underline{i} + v_z \frac{\partial}{\partial z} v_y \underline{j} + v_z \frac{\partial}{\partial z} v_z \underline{k}$$

Now $$\nabla\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial}{\partial y} v_z - \frac{\partial v_y}{\partial z}\right)\underline{i} - \left(\frac{\partial}{\partial x} v_z - \frac{\partial v_x}{\partial z}\right)\underline{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial}{\partial y} v_x\right)\underline{k}$$

$$\underline{v}\times(\nabla\times\underline{v}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ v_x & v_y & v_z \\ \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) & \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) & \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \end{vmatrix}$$

$$= \left[v_y\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial y}\right) - v_z\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\right]\underline{i}$$

$$+ \left[-v_x\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) + v_z\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\right]\underline{j}$$

$$+ \left[v_x\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) - v_y\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\right]\underline{k}$$

RHS $\underline{v}\ \nabla\underline{v} + \underline{v}\times(\nabla\times\underline{v}) =$

$$\left[v_x\frac{\partial}{\partial x}v_x + \cancel{v_y\frac{\partial}{\partial y}v_x} + \cancel{v_z\frac{\partial}{\partial z}v_x} + v_y\frac{\partial v_y}{\partial x} - \cancel{v_y\frac{\partial}{\partial y}v_x} - \cancel{v_z\frac{\partial}{\partial z}v_x} + v_z\frac{\partial}{\partial x}v_z\right]$$

$$\left[v_x\cancel{\frac{\partial}{\partial x}v_y} + v_y\frac{\partial}{\partial y}v_y + v_z\cancel{\frac{\partial v_y}{\partial z}} + v_x\frac{\partial v_x}{\partial y} - v_x\cancel{\frac{\partial v_y}{\partial x}} + v_z\frac{\partial v_z}{\partial y} - v_z\cancel{\frac{\partial v_y}{\partial z}}\right]\underline{j}$$

$$\left[v_x\cancel{\frac{\partial v_z}{\partial x}} + v_y\cancel{\frac{\partial v_z}{\partial y}} + v_z\frac{\partial v_z}{\partial z} + v_x\frac{\partial v_x}{\partial z} - v_x\cancel{\frac{\partial v_z}{\partial x}} - v_y\cancel{\frac{\partial v_z}{\partial y}} + v_y\frac{\partial v_y}{\partial z}\right]\underline{k}$$

$$= \left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_y}{\partial x} + v_z\frac{\partial v_z}{\partial x}\right)\underline{i}$$

$$+ \left(v_x\frac{\partial v_x}{\partial y} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_z}{\partial y}\right)\underline{j}$$

$$+ \left(v_x\frac{\partial v_x}{\partial z} + v_y\frac{\partial v_y}{\partial z} + v_z\frac{\partial v_z}{\partial z}\right)\underline{k} \quad //$$

2f) $\nabla\cdot(\underline{u}\,\underline{v}) = (\nabla\cdot\underline{u})\underline{v} + \underline{u}\cdot\nabla\underline{v}$

$$\underline{u}\,\underline{v} = (u_x\underline{i} + u_y\underline{j} + u_z\underline{k})(v_x\underline{i} + v_y\underline{j} + v_z\underline{k})$$

$$= u_xv_x\underline{i}\,\underline{i} + u_xv_y\underline{i}\,\underline{j} + u_xv_z\underline{i}\,\underline{k} + u_yv_x\underline{j}\,\underline{i} + u_yv_y\underline{j}\,\underline{j} + u_yv_z\underline{j}\,\underline{k}$$

$$+ u_zv_x\underline{k}\,\underline{i} + u_zv_y\underline{k}\,\underline{j} + u_zv_z\underline{k}\,\underline{k}$$

$$\nabla\cdot(\underline{u}\,\underline{v}) = \left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right)\cdot\underline{u}\,\underline{v}$$

$$=$$

$$= \left(\frac{\partial}{\partial x}u_xv_x\underline{i} + \frac{\partial}{\partial x}u_xv_y\underline{j}\ \frac{\partial u_xv_z\underline{k}}{\partial x} + \frac{\partial}{\partial y}u_yv_x\underline{i} + \frac{\partial}{\partial y}u_yv_y\underline{j} + \frac{\partial}{\partial y}u_yv_z\underline{k}\right.$$

$$+ \frac{\partial}{\partial z}u_zv_x\underline{i} + \frac{\partial}{\partial z}u_zv_y\underline{j} + \frac{\partial}{\partial z}u_zv_z\underline{k}$$

$$= u_x \frac{\partial v_x}{\partial x}\,\underline{i} + v_x \frac{\partial u_x}{\partial x}\,\underline{i} + u_x \frac{\partial v_y}{\partial x}\,j + v_y \frac{\partial u_x}{\partial x}\,j + u_x \frac{\partial v_z}{\partial x}\,\underline{k} + v_z \frac{\partial u_x}{\partial x}$$

$$+ u_y \frac{\partial v_x}{\partial y}\,\underline{i} + v_x \frac{\partial u_y}{\partial y}\,\underline{i} + u_y \frac{\partial v_y}{\partial y}\,j + v_y \frac{\partial u_y}{\partial y}\,j + u_y \frac{\partial v_z}{\partial y}\,\underline{k} + v_z \frac{\partial u_y}{\partial y}$$

$$+ u_z \frac{\partial v_x}{\partial z}\,\underline{i} + v_x \frac{\partial u_z}{\partial z}\,\underline{i} + u_z \frac{\partial v_y}{\partial z}\,j + v_y \frac{\partial u_z}{\partial z}\,j + u_z \frac{\partial v_z}{\partial z}\,\underline{k} + v_z \frac{\partial u_z}{\partial z}$$

HS

$$= u \cdot \nabla \underline{v} + (\nabla \cdot u)\underline{v}$$

$$\underline{\nabla}\underline{v} = \left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}\underline{k}\right)\left(v_x \underline{i} + v_y j + v_z \underline{k}\right)$$

$$= \Big(\frac{\partial v_x}{\partial x}\underline{i}\,\underline{i} + \frac{\partial v_y}{\partial x}\underline{i}j + \frac{\partial}{\partial x}v_z\,\underline{i}\,\underline{k} + \frac{\partial v_x}{\partial y}j\underline{i} + \frac{\partial v_y}{\partial y}jj + \frac{\partial v_z}{\partial y}j\underline{k}$$

$$+ \frac{\partial}{\partial z}v_x\,\underline{k}\,\underline{i} + \frac{\partial v_y}{\partial z}\underline{k}j + \frac{\partial v_z}{\partial z}\underline{k}\,\underline{k} \Big)$$

$$\underline{u} \cdot \nabla \underline{v} = u_x \underline{i} + u_y j + u_z \underline{k} \cdot \nabla \underline{v}$$

$$= u_x \frac{\partial v_x}{\partial x}\underline{i} + u_x \frac{\partial v_y}{\partial x}j + u_x \frac{\partial v_z}{\partial x}\underline{k}$$

$$+ u_y \frac{\partial v_x}{\partial y}\underline{i} + u_y \frac{\partial v_y}{\partial y}j + u_y \frac{\partial v_z}{\partial y}\underline{k}$$

$$+ u_z \frac{\partial v_x}{\partial z}\underline{i} + u_z \frac{\partial v_y}{\partial z}j + u_z \frac{\partial v_z}{\partial z}\underline{k}$$

$$\nabla \cdot \underline{u} = \frac{\partial}{\partial x}u_x + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$(\nabla \cdot u)\underline{v} = v_x \frac{\partial u_x}{\partial x}\underline{i} + v_y \frac{\partial u_x}{\partial x}j + v_z \frac{\partial u_x}{\partial x}\underline{k}$$

Compare terms //

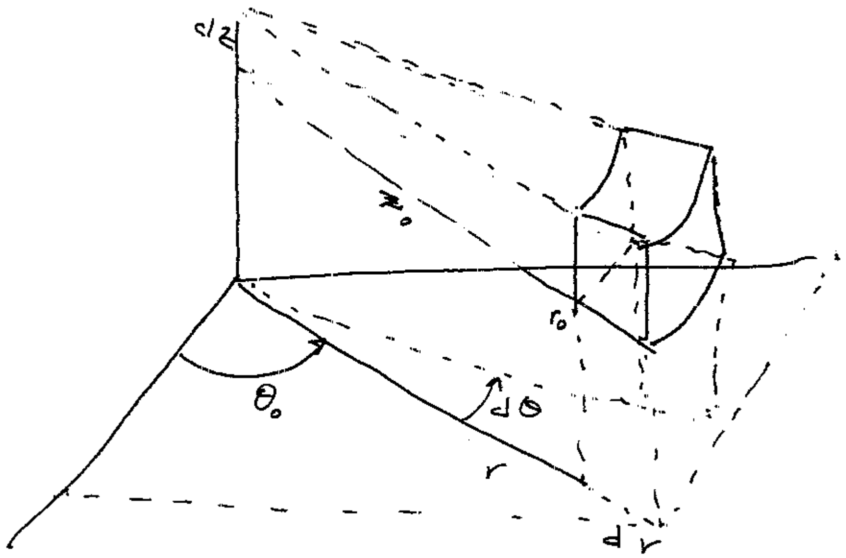
$$+ v_x \frac{\partial u_y}{\partial y}\underline{i} + v_y \frac{\partial u_y}{\partial y}j + v_z \frac{\partial u_y}{\partial y}\underline{k}$$

$$+ v_x \frac{\partial u_z}{\partial z}\underline{i} + v_y \frac{\partial u_z}{\partial z}j + v_z \frac{\partial u_z}{\partial z}\underline{k}$$

$$(P^{+} : \nabla \underline{v}) = P_{ii}\frac{\partial v_x}{\partial x} + P_{ij}\frac{\partial v_y}{\partial x} + P_{ik}\frac{\partial v_z}{\partial x} + P_{ji}\frac{\partial v_x}{\partial y} + P_{jj}\frac{\partial v_y}{\partial y}$$
$$+ P_{jk}\frac{\partial v_z}{\partial y} + P_{ki}\frac{\partial v_x}{\partial z} + P_{kj}\frac{\partial v_y}{\partial z} + P_{kk}\frac{\partial v_z}{\partial z}$$

Compare LHS with RHS

3. Obtain expression for $\nabla \underline{v}$ in cylindrical coord.



$$\nabla \underline{v} = \lim_{V \to 0} \frac{1}{V}\int_A \underline{n}\,\underline{v}\, da$$

Six faces. look at constant r_0 face.

$$\int_{A_1} \underline{n}\,\underline{v}\, da \qquad \underline{n} = -\underline{e}_r \qquad \underline{v} = (r_0, \theta, z) \qquad A_1 = r_0\, d\theta\, dz$$

$$\int_{A_1} \underline{n}\,\underline{v}\, da = \int -\underline{e}_r\,(v_r\underline{e}_r + v_\theta\underline{e}_\theta + v_z\underline{e}_z)\, r_0\, d\theta\, dz$$

for A_2 $\quad n = \underline{e}_r \qquad v = (r_0 + dr, \theta, z)$

$$\int_{A_2} \underline{n}\,\underline{v}\, da = \int \underline{e}_r\,(v_r\underline{e}_r + v_\theta e_\theta + v_z e_z)(r_0 + dr)\, d\theta\, dz$$

$$\int_{A_1} \underline{n}\,\underline{v}\,da = \int -V_r(r_0,\theta,z)(r_0)\,\underline{e}_r\underline{e}_r\,d\theta\,dz - \int V_\theta(r_0,\theta,z)(r_0)\,\underline{e}_r\underline{e}_\theta\,d\theta\,dz$$

$$\int - V_z(r_0,\theta,z)(r_0)\,\underline{e}_r\underline{e}_z\,d\theta\,dz$$

$$\int_{A_2} \underline{n}\,\underline{v}\,da = \int V_r(r_0+dr,\theta,z)(r_0+dr)\,\underline{e}_r\underline{e}_r\,d\theta\,dz + \int V_\theta(r_0+dr,\theta,z)(r_0+dr)\,\underline{e}_r\underline{e}_\theta\,d\theta\,dz$$

$$+ \int V_z(r_0+dr,\theta,z)(r_0+dr)\,\underline{e}_r\underline{e}_z\,d\theta\,dz$$

Mean Value Theorem

$$\int_{A_1} \underline{n}\,\underline{v}\,da = -V_r(r_0,\bar{\theta},\bar{z})\,r_0\,\underline{e}_r\underline{e}_r\int d\theta\,dz - V_\theta(r_0,\bar{\theta},\bar{z})\,r_0\,\underline{e}_r\underline{e}_\theta\int d\theta\,dz$$

$$- V_z(r_0+dr,\bar{\theta},\bar{z})\,r_0\,\underline{e}_r\underline{e}_z\int d\theta\,dz$$

$$\int d\theta\,dz = \Delta\theta\,\Delta z$$

Likewise for A_2

$$\int_{A_2} \underline{n}\,\underline{v}\,da = V_r(r_0+dr,\bar{\bar{\theta}},\bar{\bar{z}})(r_0+dr)\,\underline{e}_r\underline{e}_r\,\Delta\theta\Delta z + V_\theta(r_0+dr,\bar{\bar{\theta}},\bar{\bar{z}})(r_0+dr)\,\underline{e}_r\underline{e}_\theta\,\Delta\theta\Delta z$$

$$+ V_z(r_0+dr,\bar{\bar{\theta}},\bar{\bar{z}})(r_0+dr)\,\underline{e}_r\underline{e}_z\,\Delta\theta\Delta z$$

$$\int_{A_1+A_2} = \left[V_r(r_0+dr,\bar{\bar{\theta}},\bar{\bar{z}})(r_0+dr) - V_r(r_0,\bar{\theta},\bar{z})(r_0)\right]\underline{e}_r\underline{e}_r\,\Delta\theta\Delta z$$

$$+ \left[V_\theta(r_0+dr,\bar{\bar{\theta}},\bar{\bar{z}})(r_0+dr) - V_\theta(r_0,\bar{\theta},\bar{z})(r_0)\right]\underline{e}_r\underline{e}_\theta\,\Delta\theta\Delta z$$

$$+ \left[V_z(r_0+dr,\bar{\bar{\theta}},\bar{\bar{z}})(r_0+dr) - V_z(r_0,\bar{\theta},\bar{z})(r_0)\right]\underline{e}_r\underline{e}_z\,\Delta\theta\,\Delta z$$

$$V = r\,\Delta\theta\,\Delta z\,\Delta r$$

$$\text{So } \frac{1}{V}\int_{A_1+A_2} \underline{n}\,\underline{v}\,da = \frac{\left[V_r(r_0+dr,\bar{\bar{\theta}},\bar{\bar{z}})(r_0+dr) - V_r(r_0,\bar{\theta},\bar{z})(r_0)\right]}{r\,\Delta r}\underline{e}_r\underline{e}_r$$

$$+ \frac{\left[V_\theta(r_0+dr,\bar{\bar{\theta}},\bar{\bar{z}})(r_0+dr) - V_\theta(r_0,\bar{\theta},\bar{z})(r_0)\right]}{r\,\Delta r}\underline{e}_r\underline{e}_\theta$$

$$+ \frac{\left[V_z(r_0+dr,\bar{\bar{\theta}},\bar{\bar{z}})(r_0+dr) - V_\theta(r_0,\bar{\theta},\bar{z})(r_0)\right]}{r\,\Delta r}\underline{e}_r\underline{e}_z$$

$$\lim_{\Delta r \Delta\theta \Delta z \to 0} \frac{1}{V}\int_{A_1+A_2} \underline{n}\,\underline{v}\,da = \frac{[V_r(r_o+dr,\theta_o,z_o)(r_o+dr) - V_r(r_o,\theta_o,z_o)(r_o)]\,\underline{e}_r\underline{e}_r}{r\Delta r}$$

$$+ \frac{[V_\theta(r_o+dr,\theta_o,z_o)(r_o+dr) - V_\theta(r_o,\theta_o,z_o)(r_o)]\,\underline{e}_r\underline{e}_\theta}{r\Delta r}$$

$$+ \frac{[V_z(r_o+dr,\theta_o,z_o)(r_o+dr) - V_z(r_o,\theta_o,z_o)(r_o)]\,\underline{e}_r\underline{e}_z}{r\Delta r}$$

$$\lim_{\to 0} \frac{1}{V}\int_{A_1+A_2} \underline{n}\,\underline{v}\,da = \frac{d(V_r r)}{r\,dr}\underline{e}_r\underline{e}_r + \frac{1}{r}\frac{d(V_\theta r)}{dr}\underline{e}_r\underline{e}_\theta + \frac{1}{r}\frac{d(V_z r)}{dr}\underline{e}_r\underline{e}_z$$

$$= \underline{e}_r \frac{d(\underline{V} r)}{r\,dr} = \underline{e}_r\left(\frac{d\underline{V}}{dr} + \frac{\underline{V}}{r}\right) //$$

For constant θ faces

$$\frac{1}{V}\int_{A_3} \underline{n}\,\underline{v}\,da = \frac{1}{V}\int -\underline{e}_\theta(\theta)\,\underline{V}(r,\theta_o,z)\,dr\,dz = -\frac{1}{V}\underline{e}_\theta(\theta)\underline{V}(\bar{r},\theta,\bar{z})\int dr\,dz$$

$$A_4 \quad \frac{1}{V}\int_{A_4} \underline{n}\,\underline{v}\,da = \frac{1}{V}\int \underline{e}_\theta(\theta)\,\underline{V}(r,\theta_o+d\theta,z)\,dr\,dz = \frac{1}{V}\underline{e}_\theta(\theta)\,V(\bar{r},\theta_o+d\theta,\bar{z})\int dr\,dz$$

$$\underline{n}\,\underline{v}\,da = \left(\underline{e}_\theta(\theta)\,\underline{V}(\bar{r},\theta_o+d\theta,\bar{z}) - \underline{e}_\theta(\theta)\,\underline{V}(\bar{r},\theta_o,\bar{z})\right)\Delta r\Delta z$$

$3+A_4$

$$\lim_{V\to 0} \frac{1}{V}\int_{A_3+A_4} \underline{n}\,\underline{v}\,da = \frac{d(\underline{e}_\theta \underline{V})}{r\,d\theta} = \frac{\underline{e}_\theta}{r}\frac{d\underline{V}}{d\theta} + \frac{1}{r}\left(\frac{d\underline{e}_\theta}{d\theta}\right)\underline{V} = \frac{\underline{e}_\theta}{r}\frac{d\underline{V}}{d\theta} - \frac{1}{r}\underline{e}_r\underline{V} \;/$$

For const z face Similarly

$$\lim_{V\to 0} \frac{1}{V}\int_{A_5+A_6} \underline{n}\,\underline{v}\,da = \underline{e}_z\frac{d\underline{V}}{dz} \quad //$$

Therefore $$\nabla\underline{V} = \underline{e}_r\frac{d\underline{v}}{dr} + \cancel{\frac{\underline{e}_r\underline{V}}{r}} + \frac{\underline{e}_\theta}{r}\frac{d\underline{V}}{d\theta} - \cancel{\frac{\underline{e}_r\underline{V}}{r}} + \underline{e}_z\frac{d\underline{V}}{dz}$$

$$\boxed{\nabla\underline{V} = \underline{e}_r\frac{d\underline{v}}{dr} + \underline{e}_\theta\frac{d\underline{v}}{r\,d\theta} + \underline{e}_z\frac{d\underline{v}}{dz}}$$

4) $\nabla \cdot (\nabla v)^{+} = \nabla(\nabla \cdot \underline{v})$

$$v = v_x \underline{i} + v_y \underline{j} + v_z \underline{k}$$

$$\nabla = \left(\frac{\partial}{\partial x}\right)\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$$

$$\nabla \underline{v} = \left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right)\left(v_x \underline{i} + v_y \underline{j} + v_z \underline{k}\right)$$

$$= \frac{\partial}{\partial x}v_x \underline{i}\underline{i} + \frac{\partial}{\partial x}v_y \underline{i}\underline{j} + \frac{\partial}{\partial x}v_z \underline{i}\underline{k}$$

$$+ \frac{\partial}{\partial y}v_x \underline{j}\underline{i} + \frac{\partial}{\partial y}v_y \underline{j}\underline{j} + \frac{\partial}{\partial y}v_z \underline{j}\underline{k}$$

$$+ \frac{\partial}{\partial z}v_x \underline{k}\underline{i} + \frac{\partial}{\partial z}v_y \underline{k}\underline{j} + \frac{\partial}{\partial z}v_z \underline{k}\underline{k}$$

$$\nabla v^{+} = \frac{\partial}{\partial x}v_x \underline{i}\underline{i} + \frac{\partial}{\partial x}v_y \underline{j}\underline{i} + \frac{\partial}{\partial x}v_z \underline{k}\underline{i}$$

$$+ \frac{\partial}{\partial y}v_x \underline{i}\underline{j} + \frac{\partial}{\partial y}v_y \underline{j}\underline{j} + \frac{\partial}{\partial y}v_z \underline{k}\underline{j}$$

$$+ \frac{\partial}{\partial z}v_x \underline{i}\underline{k} + \frac{\partial}{\partial z}v_y \underline{j}\underline{k} + \frac{\partial}{\partial z}v_z \underline{k}\underline{k}$$

$$\nabla \cdot (\nabla v^{+}) = \frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x}\right)\underline{i} + \frac{\partial}{\partial \cancel{x}}\left(\frac{\partial v_x}{\partial y}\right)\underline{j} + \frac{\partial}{\partial \cancel{x}}\left(\frac{\partial v_x}{\partial z}\right)\underline{k}$$

$$+ \frac{\partial}{\partial \cancel{y}}\left(\frac{\partial v_y}{\partial x}\right)\underline{i} + \frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial y}\right)\underline{j} + \frac{\partial}{\partial \cancel{y}}\left(\frac{\partial v_y}{\partial z}\right)\underline{k}$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial v_z}{\partial x}\right)\underline{i} + \frac{\partial}{\partial z}\left(\frac{\partial v_z}{\partial y}\right)\underline{j} + \frac{\partial}{\partial z}\left(\frac{\partial v_z}{\partial z}\right)\underline{k}$$

RHS $\quad \nabla \cdot \underline{v} = \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z}$

$$\nabla(\nabla \cdot \underline{v}) = \left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x}\right)\underline{i} + \frac{\partial}{\partial x}\left(\frac{\partial v_y}{\partial y}\right)\underline{i} + \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial z}\right)\underline{i}$$

$$+ \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial x}\right)\underline{j} + \frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial y}\right)\underline{j} + \frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial z}\right)\underline{j}$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial x}\right)\underline{k} + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial y}\right)\underline{k} + \frac{\partial}{\partial z}\left(\frac{\partial v_z}{\partial z}\right)\underline{k}$$

Compare terms noting that $\dfrac{\partial^2 v}{\partial x_i x_j} = \dfrac{\partial^2 v}{\partial x_j \partial x_i}$ //

6 Show that the following vector field is irrotational and determine a scalar potential $a(x,y,z)$ such that $\underline{v} = \nabla a$

$$\underline{v} = 2x^2\,\underline{i} - 2yz\,\underline{j} - (y^2+3)\,\underline{k}$$

if irrotational then $\nabla \times \underline{v} = 0$

$$\nabla \times \underline{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 & -2yz & -(y^2+3) \end{vmatrix}$$

$$= \left(-\frac{\partial (y^2+3)}{\partial y} + \frac{\partial\, 2yz}{\partial z} \right)\underline{i} - \left(\frac{-\partial (y^2+3)}{\partial x} - \frac{\partial\, 2x^2}{\partial z} \right) j$$

$$+ \left(-\frac{\partial\, 2yz}{\partial x} + \frac{\partial\, 2x^2}{\partial y} \right)\underline{k}$$

$$= (-2y + 2y)\,\underline{i} - (0 - 0)\,j$$

$$+ (-0 + 0)\,\underline{k} = \underline{\underline{0}}$$

Since $\nabla \times \underline{v} = 0$ there exist a scalar field a such that $\underline{v} = \nabla a$

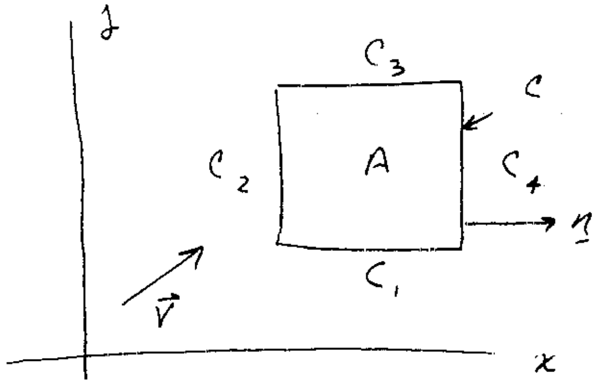
$$\nabla a = \underline{v} \Rightarrow \frac{\partial a}{\partial x} = v_x \quad \frac{\partial a}{\partial y} = v_y \,, \quad \frac{\partial a}{\partial z} = v_z$$

$$\frac{\partial a}{\partial x} = 2x^2 \Rightarrow a = \frac{2x^3}{3} + c_1(y,z)$$

$$\frac{\partial a}{\partial y} = -2yz \Rightarrow a = -2\frac{y^2}{2}z + c_2(x,z)$$

$$\frac{\partial a}{\partial z} = -(y^2+3) \Rightarrow a = -y^2z - 3z + c_3(x,y)$$

$$a = \frac{2x^3}{3} - y^2z - 3z + c_4 \quad //$$



Assume that any line parallel to the coordinate axis intersects the surface C in no more than two points

Choose $\vec{V} = P\vec{i} + Q\vec{j}$ $\quad P = P(x, y)$

$\qquad\qquad\qquad\qquad\quad Q = Q(x, y)$

Then $\int_C \vec{n}\cdot V\,dC = \int_C P\vec{i}\cdot\vec{n}_1\,dy + \int_C Q\vec{j}\cdot\vec{n}_2\,dx$

$$= \int P\vec{i}\cdot\vec{n}_1\,dy + \int \bar{Q}_j\cdot\vec{n}_2\,dx$$

Evaluate 1st integral

$$= \int \{P(x_2, y) - P(x_1, y)\}\,dy + \int (Q(x_1, y_2) - Q(x_1, y_1))\,dx$$

$dA = dx\,dy$

$$\iint_A \nabla\cdot\vec{V}\,dA = \iint_A \nabla\cdot(P\vec{i} + Q\vec{j})\,dx\,dy$$

$$= \iint \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j\right)\cdot\left(Pi + Qj\right)dx\,dy$$

$$\iint_R \nabla \cdot \vec{r}\, dA = \iint \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\, dy$$

$$= \iint \left(\frac{\partial P}{\partial x} \right) dx\, dy + \iint \left(\frac{\partial Q}{\partial y} \right) dx\, dy$$

$$= \int \left[P(x_2, y) - P(x_1, y) \right] dy + \int \left[Q(x, y_2) - Q(x, y_1) \right] dx$$

QED //