Summary of Experimental Uncertainty Assessment Methodology with Example

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Introduction

- Experiments are an essential and integral tool for engineering and science
- Uncertainty estimates are imperative for risk assessments in design both when using data directly or in calibrating and/or validating simulations methods
- True values are seldom known and experiments have errors due to instruments, data acquisition, data reduction, and environmental effects
- Determination of truth requires estimates for experimental errors, i.e., uncertainties

Introduction

- Uncertainty analysis (UA): rigorous methodology for uncertainty assessment using statistical and engineering concepts
- ASME and AIAA standards (e.g., ASME, 1998; AIAA, 1995) are the most recent updates of UA methodologies, which are internationally recognized
- Presentation purpose: to provide summary of EFD UA methodology accessible and suitable for student and faculty use both in classroom and research laboratories

Terminology

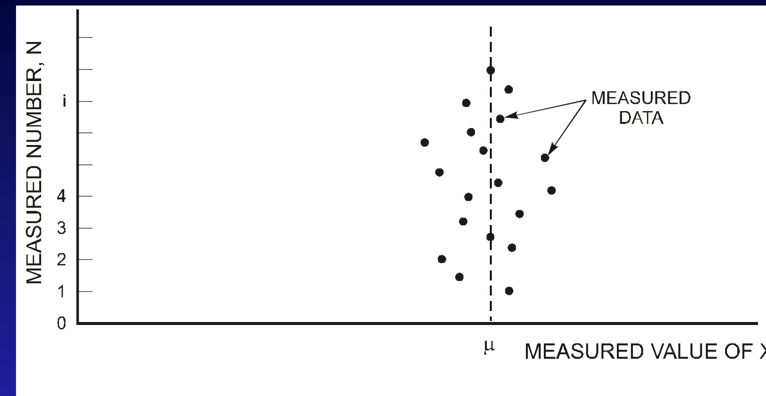
- **Accuracy**: closeness of agreement between measured and true value
- **Error**: difference between measured and true value
- **Uncertainties** (U): estimate of errors in measurements of individual variables X_i (U_{xi}) or results (U_r)
- Estimates of U made at 95% confidence level, on large data samples (at least 10/measurement)

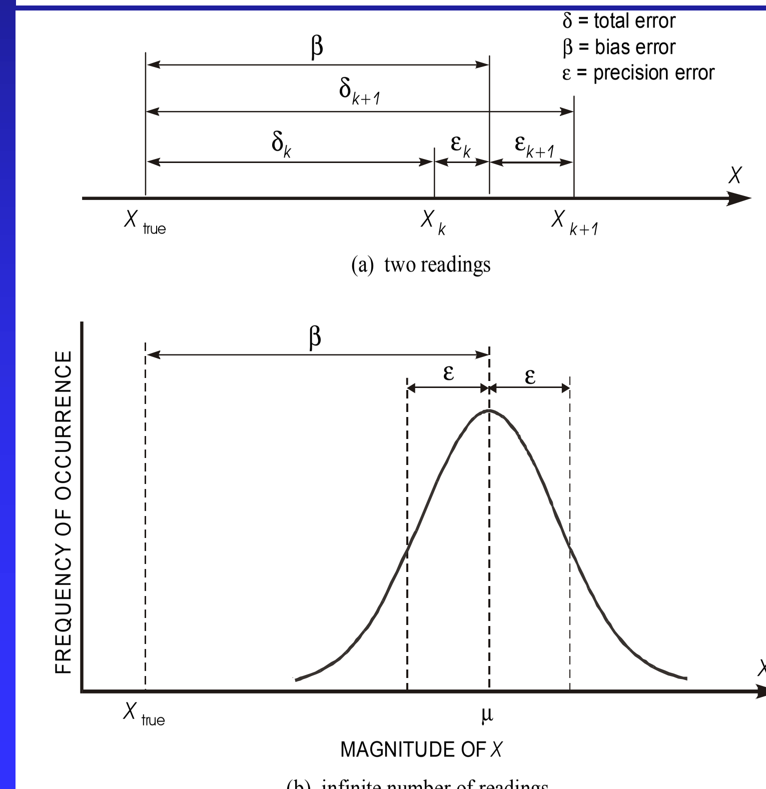
Terminology

- **Bias error** (β): fixed, systematic
- **Bias limit** (B): estimate of β

- **Precision error** (ε): random
- **Precision limit** (P): estimate of ε

- **Total error**: $\delta = \beta + \varepsilon$





(a) two readings

(b) infinite number of readings

Terminology

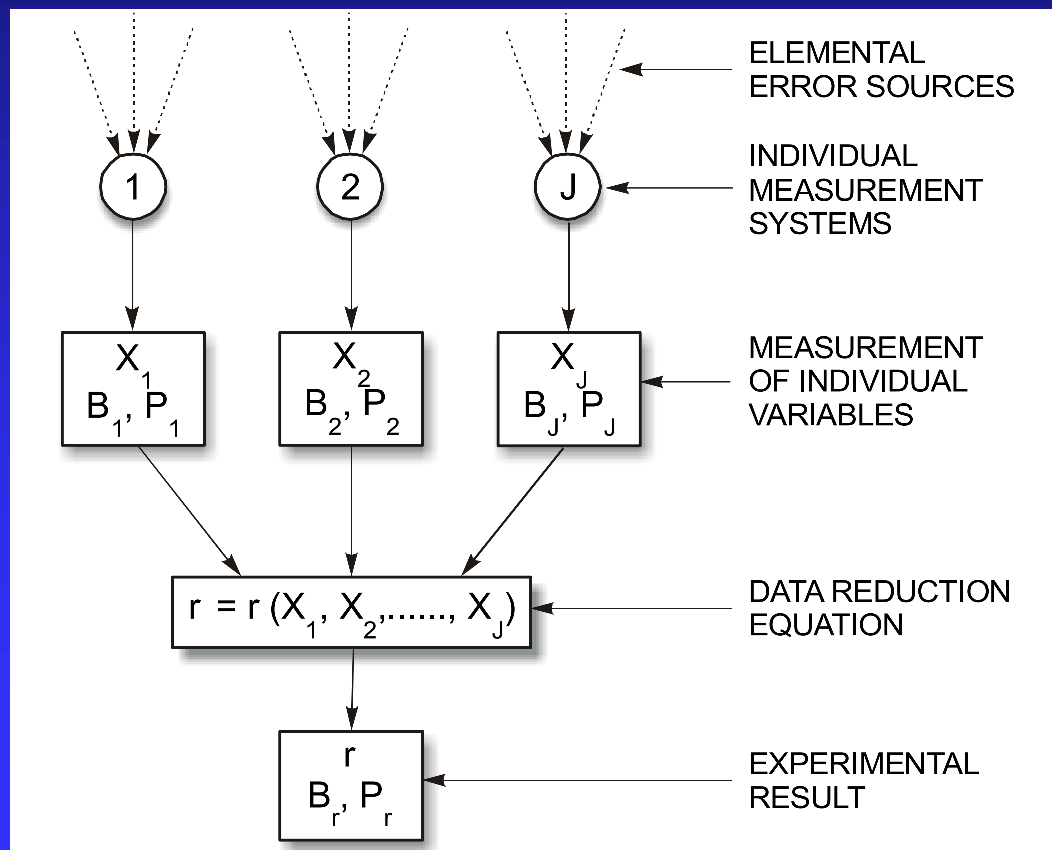
- **Measurement systems** for individual variables X_i: instrumentation, data acquisition and reduction procedures, and operational environment (laboratory, large-scale facility, in situ)
- Results expressed through **data-reduction equations (DRE)**

$$r = r(X_1, X_2, X_3, \ldots, X_j)$$

- Estimates of errors are meaningful only when considered in the **context of the process** leading to the value of the quantity under consideration
- Identification and quantification of **error sources** require considerations of:
  - steps used in the process to obtain the measurement of the quantity
  - the environment in which the steps were accomplished

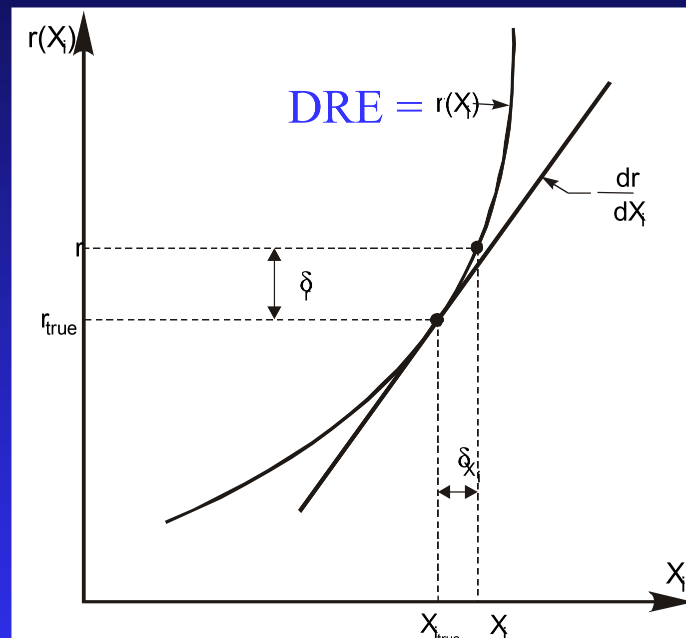
Terminology

- **Block diagram:** elemental error sources, individual measurement systems, measurement of individual variables, data reduction equations, and experimental results



Uncertainty propagation equation

- One variable, one measurement



$$\delta_r = r(X_i) - r_{true}(X_i) = \delta_{X_i} \frac{dr}{dX_i}$$

Uncertainty propagation equation

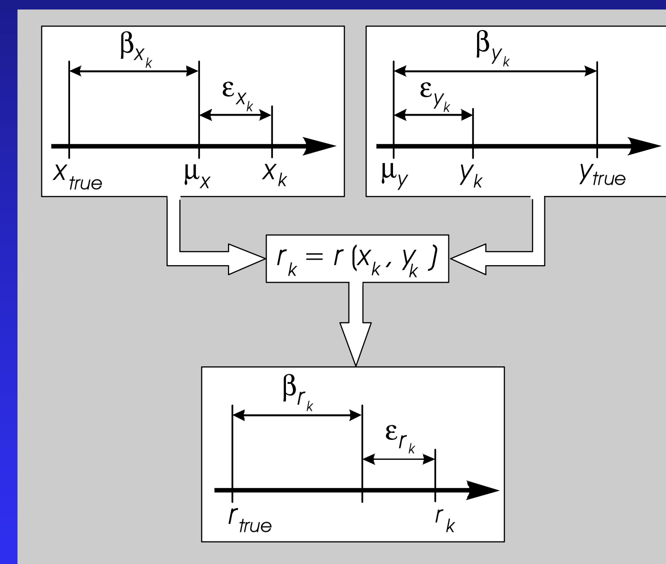
- Two variables, the *k*th set of measurements (x_k, y_k)

$$r = r(x, y)$$

$$x_k = x_{true} + \beta_{x_k} + \varepsilon_{x_k}$$

$$y_k = y_{true} + \beta_{y_k} + \varepsilon_{y_k}$$

$$r_k - r_{true} = \frac{\partial r}{\partial x}(x_k - x_{true}) + \frac{\partial r}{\partial y}(y_k - y_{true}) + R_2$$



The total error in the *k*th determination of *r*

$$\delta_{r_k} = r_k - r_{true} = \theta_x\left(\beta_{x_k} + \varepsilon_{x_k}\right) + \theta_y\left(\beta_{y_k} + \varepsilon_{y_k}\right) \quad (1)$$

Uncertainty propagation equation

- A measure of δ_r is $$\sigma_{\delta_r}^2 = \lim_{N\to\infty}\left[\frac{1}{N}\sum_{k=1}^{N}\left(\delta_{r_k}\right)^2\right] \qquad (2)$$

Substituting (2) in (1), and assuming that bias/precision errors are correlated

$$\sigma_{\delta_r}^2 = \theta_x^2\sigma_{\beta_x}^2 + \theta_y^2\sigma_{\beta_y}^2 + 2\theta_x\theta_y\sigma_{\beta_x\beta_y} + \theta_x^2\sigma_{\varepsilon_x}^2 + \theta_y^2\sigma_{\varepsilon_x}^2 + 2\theta_x\theta_y\sigma_{\varepsilon_x\varepsilon_y} \qquad (3)$$

σ's are not known; use estimates for the variances and covariances of the distributions of the total, bias, and precision errors

$$u_c^2 = \theta_x^2 b_x^2 + \theta_y^2 b_y^2 + 2\theta_x\theta_y b_{xy} + \theta_x^2 S_x^2 + \theta_y^2 S_y^2 + 2\theta_x\theta_y S_{xy}$$

The total uncertainty of the results at a specified level of confidence is

$$U_r = Ku_c$$ ($K = 2$ for 95% confidence level)

Uncertainty propagation equation

- Generalizing (3) for J variables

$$U_r^2 = \underbrace{\sum_{i=1}^{J}\theta_i^2 B_i^2 + 2\sum_{i=1}^{J-1}\sum_{k=i+1}^{J}\theta_i\theta_k B_{ik}}_{B_r^2} + \underbrace{\sum_{i=1}^{J}\theta_i^2 P_i^2 + 2\sum_{i=1}^{J-1}\sum_{k=i+1}^{J}\theta_i\theta_k P_{ik}}_{P_r^2}$$

$$\theta_i = \frac{\partial r}{\partial X_i}$$ sensitivity coefficients

Example:

$$C_D = \frac{D}{1/2\,\rho U^2 A} = C_D\left(D,\rho,U,A\right)$$

$$U_{C_D}^2 = \sum_{i=1}^{J}\theta_i^2 B_i^2 + \sum_{i=1}^{J}\theta_i^2 P_i^2$$

$$= \left(\frac{\partial C_D}{\partial D}\right)^2\left(B_D^2 + P_D^2\right) + \left(\frac{\partial C_D}{\partial \rho}\right)^2\left(B_\rho^2 + P_\rho^2\right) + \left(\frac{\partial C_D}{\partial U}\right)^2\left(B_U^2 + P_U^2\right) + \left(\frac{\partial C_D}{\partial A}\right)^2\left(B_A^2 + P_A^2\right)$$

Single and multiple tests

- Single test: one set of measurements $(X_1, X_2, \ldots, X_j)$ for r
- Multiple tests: many sets of measurements $(X_1, X_2, \ldots, X_j)$ for r

- The total uncertainty of the result (single and multiple)

$$U_r^2 = B_r^2 + P_r^2 \qquad (4)$$

- B_r: determined in the same manner for single and multiple tests
- P_r: determined differently for single and multiple tests

Bias limits (single and multiple tests)

- B_r given by:

$$B_r^2 = \sum_{i=1}^{J} \theta_i^2 B_i^2 + 2\sum_{i=1}^{J-1} \sum_{k=i+1}^{J} \theta_i \theta_k B_{ik}$$

- Sensitivity coefficients

$$\theta_i = \frac{\partial r}{\partial X_i}$$

- B_i: estimate of calibration, data acquisition, data reduction, and conceptual bias errors for X_i

- B_{ik}: estimate of correlated bias limits for X_i and X_k

$$B_{ik} = \sum_{\alpha=1}^{L} (B_i)_\alpha (B_k)_\alpha$$

Precision limits (multiple tests)

- Precision limit of the result (end to end):

$$P_{\bar{r}} = \frac{tS_{\bar{r}}}{\sqrt{M}}$$

t: coverage factor ($t = 2$ for $N > 10$)

$S_{\bar{r}}$: standard deviation for M readings of the result

$$S_{\bar{r}} = \left[\sum_{k=1}^{M} \frac{(r_k - \bar{r})^2}{M - 1} \right]^{1/2}$$

- The average result: $\bar{r} = \frac{1}{M} \sum_{k=1}^{M} r_k$

Precision limits (single test)

- Precision limit of the result (end to end):

$$P_r = tS_r$$

t: coverage factor ($t = 2$ for $N > 10$)

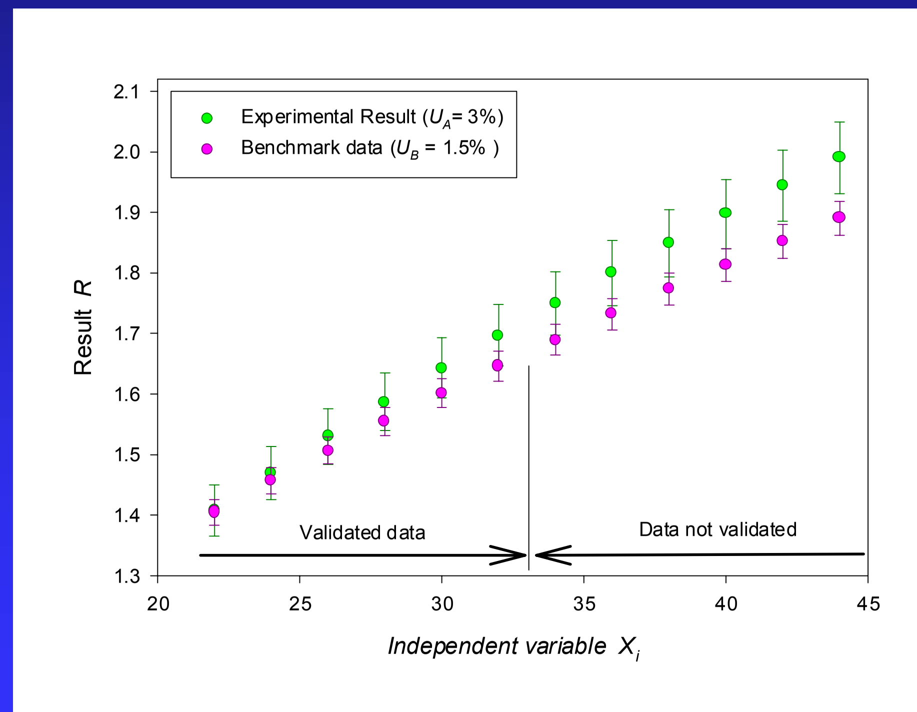
S_r: the standard deviation for the N readings of the result. It is not available for single test. Use of "best available information" (literature, inter-laboratory comparison, etc.) needed.

EFD Validation

- Conduct uncertainty analysis for the results:
  - EFD result: $A \pm U_A$
  - Benchmark or EFD data: $B \pm U_B$

$$E = B - A$$

$$U_E^2 = U_A^2 + U_B^2$$



- Validation:

$$|E| < U_E$$

Recommendations for implementation

- Determine data reduction equation: $r = r(X_1, X_2, \ldots, X_j)$
- Construct the block diagram
- Identify and estimate sources of errors
- Establish relative significance of the bias limits for the individual variables
- Estimate precision limits (end-to-end procedure recommended)
- Calculate total uncertainty using equation (4)
- Report total error, bias and precision limits for the final result

Recommendations for implementation

- Recognition of the uncertainty analysis (UA) importance
- Full integration of UA into all phases of the testing process
- Simplified UA:
  - dominant error sources only
  - use of previous data
  - end-to-end calibration and estimation of errors
- Full documentation:
  - Test design, measurement systems, data-stream in block diagrams
  - Equipment and procedure
  - Error sources considered
  - Estimates for bias and precision limits and estimating procedures
  - Detailed UA methodology and actual data uncertainty estimates

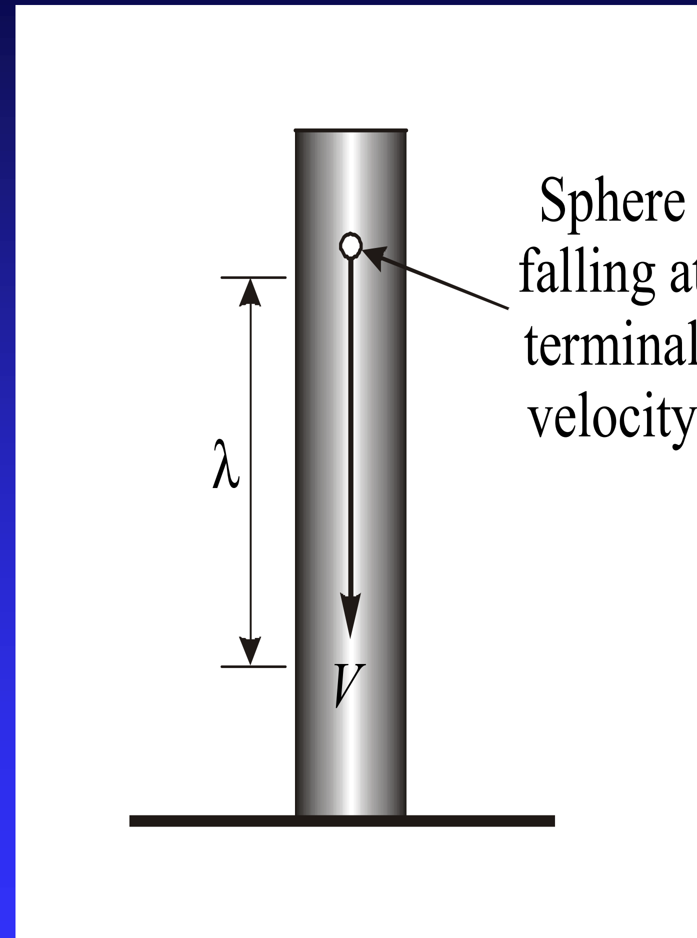
Experimental Uncertainty Assessment Methodology: Example for Measurement of Density and Kinematic Viscosity

Test Design

A sphere of diameter D falls a distance λ at terminal velocity V (fall time t) through a cylinder filled with 99.7% aqueous glycerin solution of density ρ, viscosity μ, and kinematic viscosity ν (= μ/ρ).

Flow situations:

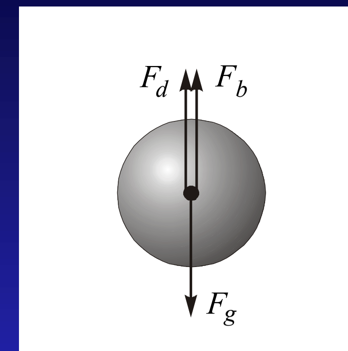
- $Re = VD/\nu << 1$ (Stokes law)
- $Re > 1$ (asymmetric wake)
- $Re > 20$ (flow separates)



Test Design

- Assumption: $Re = VD/\nu << 1$

- Forces acting on the sphere:



$$W_a = F_g - F_b = F_d$$

  - Apparent weight

$$W_a = \gamma \nabla (S - 1)$$

$$\gamma = \rho g; \qquad \nabla = \pi D^3 / 6; \qquad S = \rho_{sphere} / \rho$$

  - Drag force (Stokes law)

$$F_d = 3\pi \mu V D$$

Test design

- Terminal velocity: $V = \frac{gD^2}{18\nu}(S-1); \qquad V = \frac{\lambda}{t}$

- Solving for ν and substituting λ/t for V

$$\nu = \nu(D, t, \lambda, \rho) = \frac{gD^2 t}{18\lambda}(S-1) \qquad (5)$$

- Evaluating ν for two different spheres (e.g., teflon and steel) and solving for ρ

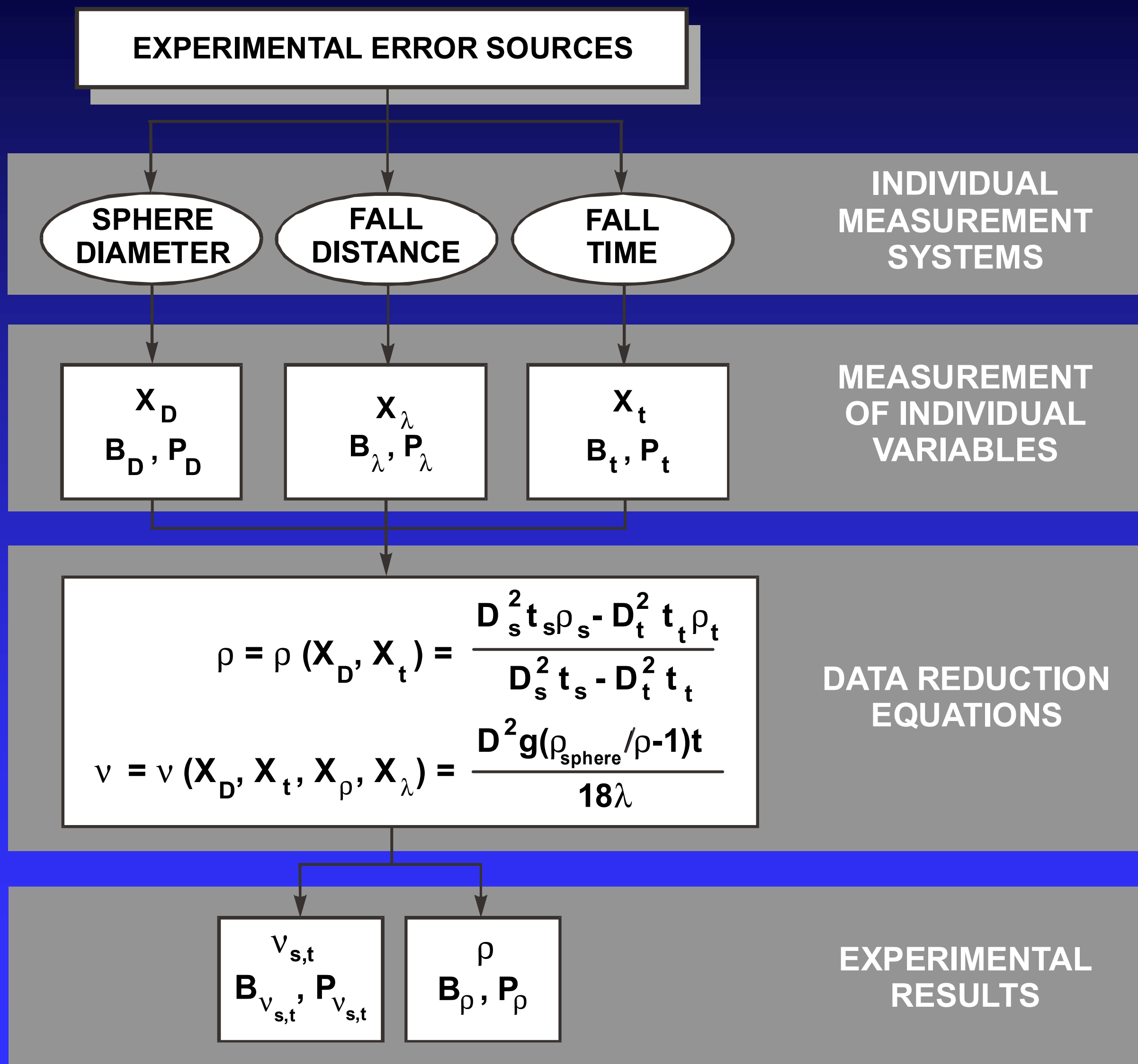
$$\rho = \rho(D_t, t_t, D_s, t_s) = \frac{D_t^2\, t_t\, \rho_t - D_s^2\, t_s\, \rho_s}{D_t^2\, t_t - D_s^2\, t_s} \qquad (6)$$

- Equations (5) and (6): data reduction equations for ν and ρ in terms of measurements of the individual variables: D_t, D_s, t_t, t_s, λ

Measurement Systems and Procedures

- Individual measurement systems:
  - D_t and D_s – micrometer; resolution 0.01mm
  - λ – scale; resolution 1/16 inch
  - t_t and t_s - stopwatch; last significant digit 0.01 sec.
  - T (temperature) – digital thermometer; last significant digit 0.1° F
- Data acquisition procedure:
  1. measure T and λ
  2. measure diameters D_t, and fall times t_t for 10 teflon spheres
  3. measure diameters D_s and fall times t_s for 10 steel spheres
- Data reduction is done at steps (5) and (6) by substituting the measurements for each test into the data reduction equation (6) for evaluation of ρ and then along with this result into the data reduction equation (5) for evaluation of ν

Block-diagram



Test results

Table 1. Gravity and sphere density constants

Definitions	Symbol	Value
Gravitational acceleration	g	9.81 m/s^2
Density of steel	ρ_s	7991 kg/m^3
Density of teflon	ρ_t	2148 kg/m^3

Table 2. Typical test results

Trial	TEFLON		STEEL		RESULTS	
T= 26.4 °C	D_t	t_t	D_s	t_s	ρ	ν
λ = 0.61 m	(m)	(sec)	(m)	(sec)	(kg/m^3)	(m^2/s)
1	0.00661	31.08	0.00359	12.210	1382.14	0.000672
2	0.00646	31.06	0.00358	12.140	1350.94	0.000683
3	0.00634	30.71	0.00359	12.070	1305.50	0.000712
4	0.00632	30.75	0.00359	12.020	1304.66	0.000709
5	0.00634	30.89	0.00359	12.180	1302.38	0.000720
6	0.00633	30.82	0.00359	12.060	1306.70	0.000710
7	0.00637	30.89	0.00359	12.110	1317.75	0.000710
8	0.00634	30.71	0.00359	12.120	1301.50	0.000717
9	0.00633	31.2	0.00359	12.030	1320.75	0.000700
10	0.00634	31.11	0.00359	12.200	1307.64	0.000718
Average	0.00637	30.91	0.00358	12.114	1318.80	0.000706
Std.Dev. (S_i)	$9.17 \cdot 10^{-5}$	0.18	$3.16 \cdot 10^{-6}$	0.0687	26.74	$1.597 \cdot 10^{-5}$

Uncertainty assessment (multiple tests)

- Density ρ (DRE: $\rho = \rho(D_t, t_t, D_s, t_s) = \frac{D_t^2 t_t \rho_t - D_s^2 t_s \rho_s}{D_t^2 t_t - D_s^2 t_s}$)

  - Bias limit $B_\rho^2 = \theta_{D_t}^2 B_{D_t}^2 + \theta_{t_t}^2 B_{t_t}^2 + \theta_{D_s}^2 B_{D_s}^2 + \theta_{t_s}^2 B_{t_s}^2 + 2\theta_{D_t}\theta_{D_s} B_{D_t} B_{D_s} + 2\theta_{t_t}\theta_{t_s} B_{t_t} B_{t_s}$

Bias Limit	Magnitude	Percentage Values	Estimation
$B_D = B_{D_t} = B_{D_s}$	0.000005 m	0.078 % D_t 0.14 % D_s	½ instrument resolution
$B_t = B_{t_t} = B_{t_t}$	0.01 s	0.032% t_t 0.083% t_s	Last significant digit

Sensitivity coefficients: e.g., $\theta_{D_t} = \frac{\partial \rho}{\partial D_t} = \frac{2 D_s^2 t_t t_s D_t (\rho_s - \rho_t)}{\left[D_t^2 t_t - D_s^2 t_s\right]^2} = 296{,}808 \frac{kg}{m^4}$

  - Precision limit $P_{\bar{\rho}} = \frac{2 \cdot S_{\bar{\rho}}}{\sqrt{M}}$

  - Total uncertainty $U_{\bar{\rho}} = \pm\sqrt{B_\rho^2 + P_{\bar{\rho}}^2}$

Uncertainty assessment (multiple tests)

- Density ρ

Term	Without correlated bias errors		With correlated bias errors	
	Magnitude	% Values	Magnitude	% Values
$\theta_{D_t} B_D$	1.48 kg/m^3	22.30% B_ρ^2	1.48 kg/m^3	147.16% B_ρ^2
$\theta_{t_t} B_t$	0.31 kg/m^3	0.95% B_ρ^2	0.31 kg/m^3	4.09% B_ρ^2
$\theta_{D_s} B_D$	-2.63 kg/m^3	70.60% B_ρ^2	-2.63 kg/m^3	464.72% B_ρ^2
$\theta_{t_s} B_t$	-0.78 kg/m^3	6.15% B_ρ^2	-0.78 kg/m^3	38.89% B_ρ^2
$2\theta_{D_t}\theta_{D_s} B_D^2$	-	-	-2.79 kg/m^3	-522.98% B_ρ^2
$2\theta_{t_t}\theta_{t_s} B_t^2$	-	-	-0.69 kg/m^3	-31.88% B_ρ^2
B_ρ	3.13 kg/m^3	0.24% $\overline{\rho}$ 3.3% $U_{\overline{\rho}}^2$	1.22 kg/m^3	0.09% $\overline{\rho}$ 0.47% $U_{\overline{\rho}}^2$
$P_{\overline{\rho}}$	16.91 kg/m^3	1.28% $\overline{\rho}$ 96.70% $U_{\overline{\rho}}^2$	16.91 kg/m^3	1.29% $\overline{\rho}$ 99.53% $U_{\overline{\rho}}^2$
$U_{\overline{\rho}}$	17.20kg/m^3	1.30% $\overline{\rho}$	16.95kg/m^3	1.28% $\overline{\rho}$

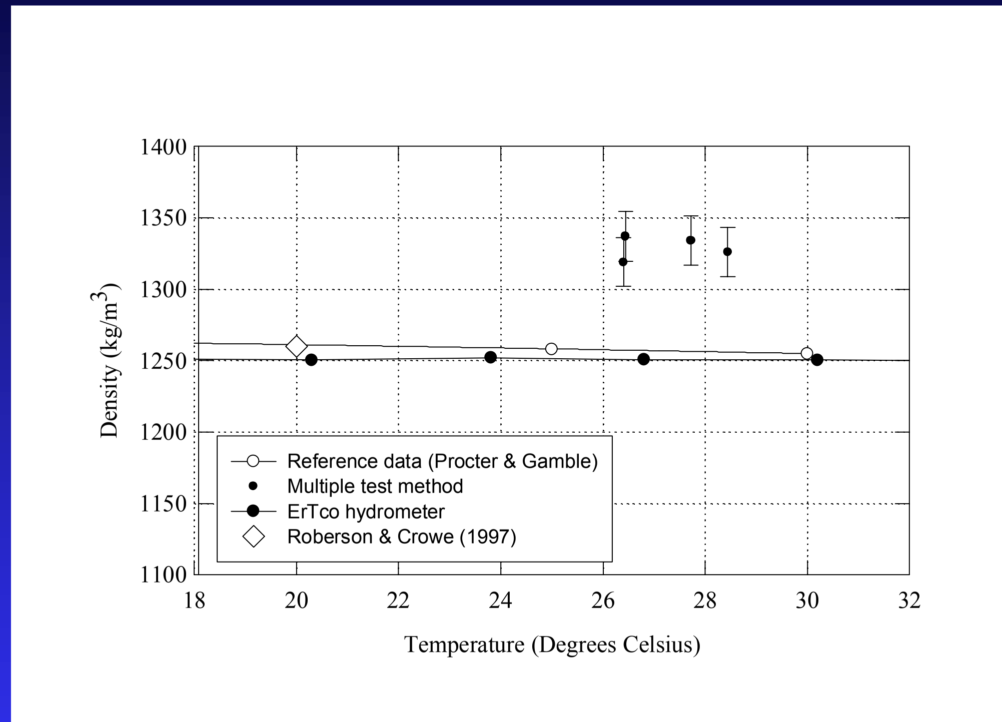
Uncertainty assessment (multiple tests)

- Viscosity ν (DRE : $\nu = \nu(D,t,\lambda,\rho) = \frac{gD^2 t}{18\lambda}(S-1)$)
  - Calculations for teflon sphere
- Bias limit $$B_{\nu_t}^2 = \theta_{D_t}^2 B_D^2 + \theta_\rho^2 B_\rho^2 + \theta_{t_t}^2 B_t^2 + \theta_\lambda^2 B_\lambda^2$$
- Precision limit $$P_{\bar{\nu}_t} = 2 \cdot S_{\bar{\nu}_t} / \sqrt{M}$$
- Total uncertainty $$U_{\bar{\nu}_t}^2 = B_{\nu_t}^2 + P_{\bar{\nu}_t}^2$$

Term	Magnitude	Percentage Values
B_λ	7.9×10^{-4} m	0.13% λ
$\theta_{D_t} B_D$	1.1×10^{-6} m²/s	$5.97\% B_{\nu_t}^2$
$\theta_\rho B_\rho$	4.27×10^{-6} m²/s	$90.03\% B_{\nu_t}^2$
$\theta_{t_t} B_t$	2.29×10^{-7} m²/s	$0.26\% B_{\nu_t}^2$
$\theta_\lambda B_\lambda$	-0.92×10^{-6} m²/s	$3.74\% B_{\nu_t}^2$
B_{ν_t}	4.5×10^{-6} m²/s	$0.64\% \overline{\nu_t}$ $16.43\% U_{\bar{\nu}_t}^2$
$P_{\bar{\nu}_t}$	1.01×10^{-5} m²/s	$1.43\% \overline{\nu_t}$ $83.57\% U_{\bar{\nu}_t}^2$
$U_{\bar{\nu}_t}$	1.11×10^{-5} m²/s	$1.57\% \overline{\nu_t}$

Comparison with benchmark data

- Density ρ



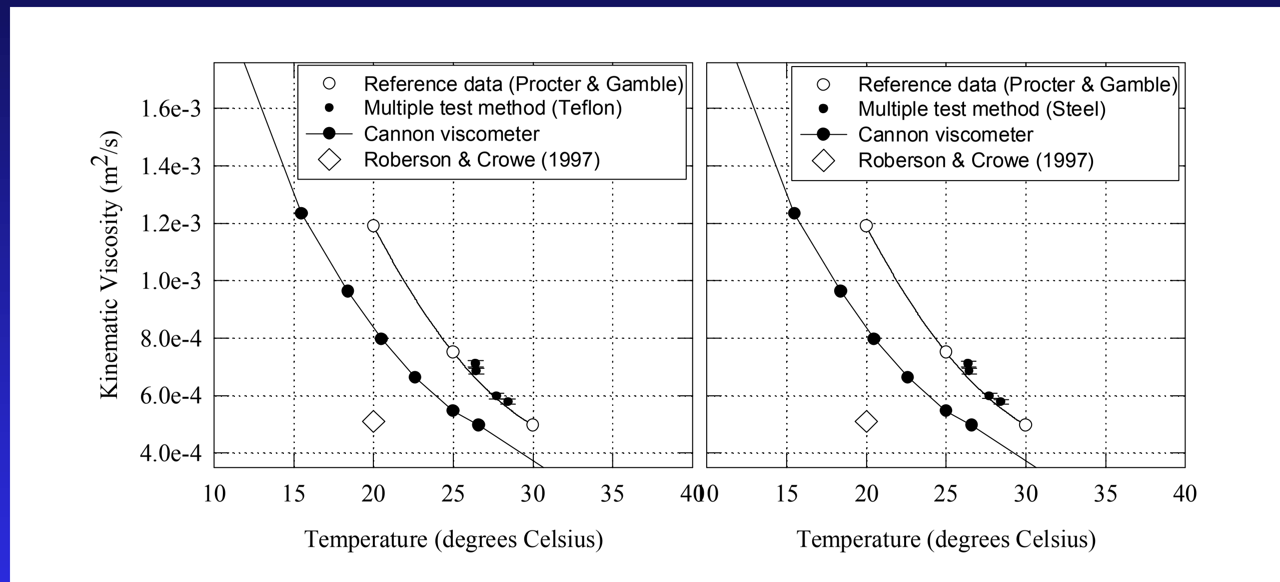
$E = 4.9\%$ (reference data) and $E = 5.4\%$ (ErTco hydrometer)

Neglecting correlated bias errors: $U_E \approx U_D = 1.30\%$

Data not validated: $|E| \geq U_E$

Comparison with benchmark data

- Viscosity ν



$E = 3.95\%$ (reference data) and $E = 40.6\%$ (Cannon capillary viscometer)

Neglecting correlated bias errors:

$$U_E \approx U_D = 1.57\% (teflon)$$

$$U_E \approx U_D = 1.49\% (steel)$$

Data not validated (unaccounted bias error): $|E| \geq U_E$

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