56:171

Operations Research -- Sample Homework Assignments
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Homework #1

(1.) ***Linear Programming Model Formulation.*** SunCo processes oil into aviation fuel and heating oil. It costs $40 to purchase each barrel of oil, which is then distilled and yields 0.5 barrel of aviation fuel and 0.5 barrel of heating oil. Output from the distillation may be sold directly or processed in the catalytic cracker. If sold after distillation without further processing, aviation fuel sells for $60/barrel. If sold after distillation without further processing, heating oil sells for $40/barrel. It takes 1 hour to process 1000 barrels of aviation fuel in the catalytic cracker, and these can be sold for $130/barrel. It takes 45 minutes to process 1000 barrels of heating oil in the cracker, and these can be sold for $90/barrel. Each day, at most 20,000 barrels of oil can be purchased, and 8 hours of catalytic cracker time are available. Formulate an LP to maximize SunCo's profits. (This is #4 of Review Problems, page 113, of text by Winston. Note that I have modified the purchase costs and selling prices, since it seems extremely unrealistic that oil might be purchased for $40/1000 barrels, i.e., $0.04/barrel.) Use LINDO (or other LP code) to find the optimal solution.

(2.) Simplex Algorithm. Use the Simplex algorithm to solve the following problem:

$$\begin{array}{llllll} \text{Max } z = & 2x_1 & - x_2 & + x_3 & \\ \text{subject to} & 3x_1 & + x_2 & + x_3 & \leq 60 \\ & x_1 & - x_2 & + 2x_3 & \leq 10 \\ & x_1 & + x_2 & - x_3 & \leq 20 \\ & x_1,\ x_2,\ x3 \geq 0 \end{array}$$

(This is #3 of §4.3, page 145 of text by Winston.)

Homework #1 Solution

(1.) ***Linear Programming Model Formulation.*** SunCo processes oil into aviation fuel and heating oil. It costs $40 to purchase each barrel of oil, which is then distilled and yields 0.5 barrel of aviation fuel and 0.5 barrel of heating oil. Output from the distillation may be sold directly or processed in the catalytic cracker. If sold after distillation without further processing, aviation fuel sells for $60/barrel. If sold after distillation without further processing, heating oil sells for $40/barrel. It takes 1 hour to process 1000 barrels of aviation fuel in the catalytic cracker, and these can be sold for $130/barrel. It takes 45 minutes to process 1000 barrels of heating oil in the cracker, and these can be sold for $90/barrel. Each day, at most 20,000 barrels of oil can be purchased, and 8 hours of catalytic cracker time are available. Formulate an LP to maximize SunCo's profits.

Solution: Define the decision variables

OIL = # of barrels of oil purchased
HSOLD = # of barrels of heating oil sold
HCRACK = # of barrels of heating oil processed in catalytic cracker
ASOLD = # of barrels of aviation fuel sold
ACRACK = # of barrels of aviation fuel processed in catalytic cracker

The LP model to maximize profit is

Maximize 40 HSOLD + 90 HCRACK + 60 ASOLD + 130 ACRACK - 40 OIL
subject to

OIL ≤ 20000	(available supply)
0.5 OIL = ASOLD + ACRACK	(aviation fuel & heating oil
0.5 OIL = HSOLD + HCRACK	each constitute 50% of product of distilling)
0.001ACRACK + 0.00075 HCRACK ≤ 8	(avail. time for cracker)
OIL ≥ 0, ASOLD≥0, ACRACK≥0, HSOLD≥0, HCRACK≥0	

The fourth constraint (5th row) imposes the available time limitation of the catalytic cracker:

$$\left(1 \text{ hr.}/_{1000 \text{ bbls.}}\right)\text{ACRACK} + \left(0.75 \text{ hr.}/_{1000 \text{ bbls.}}\right)\text{HCRACK} \le 8 \text{ hrs.}$$

Note: 1. Because the coefficients of this constraint are very small relative to those in other rows, LINDO will display a warning, and advise you to scale your problem, defining your variables to all have units of thousands of barrels, for example, so that your constraints would appear as:

OIL ≤ 20	(available supply)
0.5 OIL = ASOLD + ACRACK	(aviation fuel & heating oil
0.5 OIL = HSOLD + HCRACK	each constitute 50% of product of distilling)
ACRACK + 0.75HCRACK ≤ 8.	

and your objective function would have units of $1000's. This will avoid inaccuracies in the computation which might be caused by round-off errors. Because this problem is so small, this should not be a difficulty here, so in obtaining the LINDO output below, I have not scaled my input this way.

2. Other formulations are OK. For example, one could eliminate the variable OIL above by substituting (2ASOLD+2ACRACK) everywhere it appears.
3. When entering the model into LINDO, all variables must be on the left side of a constraint. Furthermore, it is not necessary to define a row for each non-negativity constraint (nonnegativity of every variable is assumed by LINDO).

The output of LINDO follows:

```
MAX      40 HSOLD + 90 HCRACK + 60 ASOLD + 130 ACRACK - 40 OIL
SUBJECT TO
        2)   OIL <=   20000
        3) - ASOLD - ACRACK + 0.5 OIL =    0
```

```
       4) - HSOLD - HCRACK + 0.5 OIL =    0
       5)   0.00075 HCRACK + 0.001 ACRACK <=   8
 END

 LP OPTIMUM FOUND

       OBJECTIVE FUNCTION VALUE

       1)    760000.000

  VARIABLE        VALUE          REDUCED COST
     HSOLD     10000.000000           .000000
    HCRACK          .000000          2.500000
     ASOLD      2000.000300           .000000
    ACRACK      8000.000000           .000000
       OIL     20000.000000           .000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)          .000000        10.000000
        3)          .000000       -60.000000
        4)          .000000       -40.000000
        5)          .000000     70000.000000
```

The optimal solution, then, is to purchase all the available oil (20,000 barrels), which will produce 10,000 barrels of heating oil and 10,000 barrels of aviation fuel. All of the heating oil is sold without further processing (HSOLD), while 8000 barrels of the aviation fuel is processed in the catalytic cracker (ACRACK). The remaining 2000 barrels of aviation fuel is sold without further processing (ASOLD). This plan should generate a profit of $760,000.

LINDO will also display the following information, which is useful in sensitivity analysis:

```
 RANGES IN WHICH THE BASIS IS UNCHANGED:

                            OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
    HSOLD       40.000000         INFINITY         2.500000
   HCRACK       90.000000         2.500000         INFINITY
    ASOLD       60.000000         3.333333        20.000000
   ACRACK      130.000000         INFINITY         3.333333
      OIL      -40.000000         INFINITY        10.000000

                            RIGHTHAND SIDE RANGES
      ROW         CURRENT        ALLOWABLE        ALLOWABLE
                    RHS          INCREASE         DECREASE
        2    20000.000000         INFINITY      4000.001000
        3         .000000      2000.000300         INFINITY
        4         .000000     10000.000000         INFINITY
        5        8.000000         2.000000         8.000000

THE TABLEAU
     ROW  (BASIS)      HSOLD   HCRACK  ASOLD  ACRACK  OIL    SLK 2
```

1	ART	.000	2.500	.000	.000	.000	10.000
2	OIL	.000	.000	.000	.000	1.000	1.000
3	ASOLD	.000	-.750	1.000	.000	.000	.500
4	HSOLD	1.000	1.000	.000	.000	.000	.500
5	ACRACK	.000	.750	.000	1.000	.000	.000

ROW	SLK 5	
1	0.70E+05	0.76E+06
2	.000	20000.000
3	-1000.000	2000.000
4	.000	10000.000
5	1000.000	8000.000

(2.) ***Simplex Algorithm***. We will use the Simplex algorithm to solve the following problem:

$$\begin{array}{llllll} \text{Max } z = & 2x_1 & - x_2 & + x_3 & \\ \text{subject to} & 3x_1 & + x_2 & + x_3 & \leq 60 \\ & x_1 & - x_2 & + 2x_3 & \leq 10 \\ & x_1 & + x_2 & - x_3 & \leq 20 \\ & \multicolumn{4}{l}{x_1,\ x_2,\ x3 \geq 0} \end{array}$$

First the constraints (other than the nonnegativity constraints) are converted to equations:

$$\begin{array}{lllllllll} \text{Max} & -z + & 2x_1 & - x_2 & + x_3 & & & & = 0 \\ \text{subject to} & & 3x_1 & + x_2 & + x_3 & + S_1 & & & = 60 \\ & & x_1 & - x_2 & + 2x_3 & & + S_2 & & = 10 \\ & & x_1 & + x_2 & - x_3 & & & + S_3 & = 20 \\ & & \multicolumn{7}{l}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ S_1 \geq 0,\ S_2 \geq 0,\ S_3 \geq 0} \end{array}$$

We then put the coefficients into tableau form:

-Z	X1	X2	X3	S1	S2	S3	rhs
1	2	¯1	1	0	0	0	0
0	3	1	1	1	0	0	60
0	1	¯1	2	0	1	0	10
0	1	1	¯1	0	0	1	20

Notice that, unlike the text by Winston, I have used (-Z) in the first column, rather than Z. An initial basic feasible solution is already apparent in the above tableau, namely the solution with (-Z), S_1, S_2, and S_3 as the basic variables, and $x_1=x_2=x_3=0$ as the nonbasic variables. The profit for this basic solution is zero.

To improve (i.e., increase) the objective function, we may increase either x_1 or x_3, since they both have positive relative profits (+2 and +1, respectively). Let's arbitrarily choose x1. That is, we will pivot in the x1 column. To determine the row in which to pivot, we must perform the "minimum ratio test": minimim { 60/3, 10/1, 20/1} = 10/1. Therefore, the pivot row will be that row which yielded the ratio 10/1, i.e., row 3:

-Z	X1	X2	X3	S1	S2	S3	rhs
1	2	¯1	1	0	0	0	0
0	3	1	1	1	0	0	60
0	1	¯1	2	0	1	0	10
0	1	1	¯1	0	0	1	20

The result of pivoting in row 3, column 2 is:

-Z	X1	X2	X3	S1	S2	S3	rhs
1	0	1	¯3	0	¯2	0	¯20
0	0	4	¯5	1	¯3	0	30
0	1	¯1	2	0	1	0	10
0	0	2	¯3	0	¯1	1	10

The basic solution corresponding to this tableau has (-z) = -20, i.e., z=20, and x_1=10, S_1=30, S_3=10, and x_2=x_3=S_2=0. This basic solution is not optimal, however, because x2 now has a positive relative profit. We therefore will increase x2 by pivoting in its column. The pivot row, determined by the minimum ratio test (i.e., minimum{30/4, -, 10/2} = 10/2) is the bottom row.

-Z	X1	X2	X3	S1	S2	S3	rhs
1	0	1	¯3	0	¯2	0	¯20
0	0	4	¯5	1	¯3	0	30
0	1	¯1	2	0	1	0	10
0	0	2	¯3	0	¯1	1	10

The result of pivoting here is the tableau

-Z	X1	X2	X3	S1	S2	S3	rhs
1	0	0	¯1.5	0	¯1.5	¯0.5	¯25
0	0	0	1	1	¯1	¯2	10
0	1	0	0.5	0	0.5	0.5	15
0	0	1	¯1.5	0	¯0.5	0.5	5

which, since the relative profits are all nonpositive, is optimal. That is, the optimal solution is (-z) = -25 i.e., z = 25, and x_1=15, x_2=5, S_1=10, x_3=S_2=S_3=0.

If we had used (+z) rather than (-z) as a basic variable, then all the signs in the objective row, except for the first column of course, would be reversed, and the rule for choosing a pivot column is to choose one with a negative value in the objective row.

At the first iteration, we had a choice of either x_1 or x_3 as the pivot column. If we had selected x_3, we would have obtained the following sequence of tableaus, which results in the same optimal tableau:

-Z	X1	X2	X3	S1	S2	S3	rhs
1	2	¯1	1	0	0	0	0
0	3	1	1	1	0	0	60
0	1	¯1	2	0	1	0	10
0	1	1	¯1	0	0	1	20

-Z	X1	X2	X3	S1	S2	S3	rhs
1	1.5	¯0.5	0	0	¯0.5	0	¯5
0	2.5	1.5	0	1	¯0.5	0	55
0	0.5	¯0.5	1	0	0.5	0	5
0	1.5	0.5	0	0	0.5	1	25

-Z	X1	X2	X3	S1	S2	S3	rhs
1	0	1	⁻3	0	⁻2	0	⁻20
0	0	4	⁻5	1	⁻3	0	30
0	1	⁻1	2	0	1	0	10
0	0	2	⁻3	0	⁻1	1	10

-Z	X1	X2	X3	S1	S2	S3	rhs
1	0	0	⁻1.5	0	⁻1.5	⁻0.5	⁻25
0	0	0	1	1	⁻1	⁻2	10
0	1	0	0.5	0	0.5	0.5	15
0	0	1	⁻1.5	0	⁻0.5	0.5	5

This, it turns out, required one more pivot than before... however, in practice it is impossible to determine which choice of pivot column will lead to fewer pivots being required.

Homework #2

(1.) ***Linear Programming Model Formulation.*** Shoemakers of America forecasts the following demand for each of the next six months:

Month t	Demand D_t (pairs)
1	5000
2	6000
3	5000
4	9000
5	6000
6	5000

It takes a shoemaker 15 minuts to produce a pair of shoes. Each shoemaker works 150 hours/month, plus up to 40 hours/month overtime. A shoemaker is paid a regular salary of $2000/month, plus $50/hour for overtime. At the beginning of each month, Shoemakers can either hire or fire workers. It costs the company $1500 to hire a worker and $1900 to fire a worker. The monthly holding cost per pair of shoes is 3% of the cost of producing a pair of shoes with regular-time labor. (The raw materials in a pair of shoes cost $20.) Formulate an LP that minimizes the cost of meeting (on time) the demands of the next six months. At the beginning of month 1, Shoemakers has 13 workers. Use LINDO to compute the plan which will minimize the cost of meeting (on time) the demands of the next six months. (Problem #18 of Chapter 4 Review Problems, page 184 of text by Winston.)

Hints: Assume that, even though a shoemaker is paid for 150 hours of work per month, he may be idle part of the time if he is required in a later month, in order that the company avoid firing and hiring costs. You may find it useful to define the following decision variables for each month t, t=1,2,... 6:

W_t = # of shoemakers in the work force during month t
H_t= # of shoemakers hired at the beginning of month t
F_t= # of shoemakers fired at the beginning of month t
R_t= # of pairs of shoes produced during regular time in month t
O_t= # of pairs of shoes produced during overtime in month t
I_t= # of pairs of shoes in inventory at the end of month t

(2.) The following questions refer to the first problem in HW#1.

a. Suppose that additional oil were available for purchase, but at a price higher than the original price of $40/barrel. How much more should SunCo be willing to pay per barrel?
b. How many barrels should they be willing to purchase for the price you answered in part (a)?
c. Suppose that the catalytic cracker requires maintenance during the next day, reducing its available time from 8 hours to 6 hours. How much loss of profit will result?
d. Based upon the substitution rates in the optimal tableau, what will be the optimal solution with the reduced processing time specified in (c)?
e. In the optimal solution, no heating oil is processed in the catalytic cracker. How much would be required for the selling price of heating oil processed in the catalytic cracker in order to justify this processing?

(3.) Gepbab Corporation produces three products at two different plants. The cost of producing a unit at each plant is:

	Product 1	Product 2	Product 3
Plant 1	$5	$6	$8
Plant 2	$8	$7	$10

Each plant can produce a total of 10,000 units. At least 6000 units of Product 1, at least 8000 units of Product 2, and at least 5000 units of Product 3 must be produced. To minimize the cost of meeting these demands, the following LP should be solved:

$$\text{Min} \quad z = 5x_{11} + 6x_{12} + 8x_{13} + 8x_{21} + 7x_{22} + 10x_{23}$$

subject to

$$x_{11} + x_{12} + x_{13} \le 10000$$
$$x_{21} + x_{22} + x_{23} \le 10000$$
$$x_{11} + x_{21} \ge 6000$$
$$x_{12} + x_{22} \ge 8000$$
$$x_{13} + x_{23} \ge 5000$$
$$x_{ij} \ge 0 \text{ for all } i,j$$

Here, x_{ij} = number of units of product j produced at plant i. Use the LINDO output (which appears on page 205 of the textbook, or you can run LINDO yourself) to answer the following questions. (In each case, answer the question based on the output of the original problem, instead of re-solving the problem with revised costs or right-hand-sides.)

a. What would the cost of producing product 2 at plant 1 have to be in order for the firm to want to produce product 2 at plant 1?
b. What would total cost be if plant 1 had 9000 units of capacity?
c. If it costs \$9 to produce a unit of product 3 at plant 1, what would be the new optimal solution?

Homework #2 Solution

(1.) ***Linear Programming Model Formulation.***

Solution: Using the variables defined above, we can state the objective function as

$$\text{Minimize } 2000\sum_{t=1}^{6} W_t + 1500\sum_{t=1}^{6} H_t + 1900\sum_{t=1}^{6} F_t$$
$$+ 20\sum_{t=1}^{6} R_t + 32.5\sum_{t=1}^{6} O_t + 0.70\sum_{t=1}^{6} I_t$$

Note that the cost of regular labor (\$2000/month) is assigned to the variables W_t rather than R_t, since we are assuming that the regular salary is paid even if a worker is idle. The cost of R_t is only the cost of materials. The overtime labor costs (\$50/hr)(1hr/4 pair) = \$12.5/pair, however, are assigned to variable O_t, in addition to the material cost of \$20/pair. The holding cost per pair of shoes requires that we compute the total regular-time production cost per pair of shoes. The labor cost is $\frac{1 \text{ hr.}}{4 \text{ pr.}} \times \frac{\$2000/\text{month}}{150\text{hr/month}}$ = \$3.33/pair, which with a material cost of \$20/pair, gives a production cost of \$23.33/pair. At 3% per month holding cost, this means that the monthly cost of holding shoes in inventory is \$0.70/pair.

The constraints include material balance equations for the shoes:

$$R_t + O_t + I_{t-1} = D_t + I_t$$

That is, the production in month t plus shoes held from month (t-1) must equal the demand satisfied in month t plus the shoes placed into storage at the end of month t.

We have "material balance" equations as well for the workforce:

$$W_t = W_{t-1} + H_t - F_t$$

That is, the workforce in month t is the previous month's workforce, plus the newly hired workers, minus those fired at the beginning of month t. (When entering these constraints, I have used the specified demands for D_t and the initial workforce, 13, for W_0.)

We also need constraints to relate R_t and O_t to W_t:

$$R_t \le \frac{150\text{hrs}}{\text{month}} \times \frac{4 \text{ pairs}}{\text{hr.}} W_t\ ,\ O_t \le \frac{40\text{hrs}}{\text{month}} \times \frac{4 \text{ pairs}}{\text{hr.}} W_t$$

The problem may then be formulated as shown in the LINDO output below:

```
MIN  2000 W1 +2000 W2 +2000 W3 +2000 W4 +2000 W5 +2000 W6
   + 1500 H1 +1500 H2 +1500 H3 +1500 H4 +1500 H5 +1500 H6
   + 1900 F1 +1900 F2 +1900 F3 +1900 F4 +1900 F5 +1900 F6
   + 0.7 I1  +0.7 I2  +0.7 I3  +0.7 I4  +0.7 I5  +0.7 I6
   + 20 R1   +20 R2   +20 R3   +20 R4   +20 R5   +20 R6
   + 32.5 O1 +32.5 O2 +32.5 O3 +32.5 O4 +32.5 O5 +32.5 O6
```

```
 SUBJECT TO
        2) - I1 + R1 + O1 =    5000
        3)   I1 - I2 + R2 + O2 =    6000
        4)   I2 - I3 + R3 + O3 =    5000
        5)   I3 - I4 + R4 + O4 =    9000
        6)   I4 - I5 + R5 + O5 =    6000
        7)   I5 + R6 + O6 =    5000
        8)   W1 - H1 + F1 =    13
        9) - W1 + W2 - H2 + F2 =    0
       10) - W2 + W3 - H3 + F3 =    0
       11) - W3 + W4 - H4 + F4 =    0
       12) - W4 + W5 - H5 + F5 =    0
       13) - W5 + W6 - H6 + F6 =    0
       14) - 600 W1 + R1 <=   0
       15) - 600 W2 + R2 <=   0
       16) - 600 W3 + R3 <=   0
       17) - 600 W4 + R4 <=   0
       18) - 600 W5 + R5 <=   0
       19) - 600 W6 + R6 <=   0
       20) - 160 W1 + O1 <=   0
       21) - 160 W2 + O2 <=   0
       22) - 160 W3 + O3 <=   0
       23) - 160 W4 + O4 <=   0
       24) - 160 W5 + O5 <=   0
       25) - 160 W6 + O6 <=   0
 END
```

The solution found by LINDO is:

```
LP OPTIMUM FOUND AT STEP     29

        OBJECTIVE FUNCTION VALUE

        1)    852716.620

  VARIABLE        VALUE          REDUCED COST
        W1        10.416670           .000000
        W2        10.416670           .000000
        W3        10.416670           .000000
        W4        10.416670           .000000
        W5        10.000000           .000000
        W6         8.333333           .000000
        F1         2.583333           .000000
        F5          .416667           .000000
        F6         1.666667           .000000
        I1      1250.000000           .000000
        I2      1500.000000           .000000
        I3      2750.000000           .000000
        R1      6250.000000           .000000
        R2      6250.000000           .000000
        R3      6250.000000           .000000
        R4      6250.000000           .000000
        R5      6000.000000           .000000
        R6      5000.000000           .000000
```

Note that in the optimal solution of the LP the size of the workforce in months 1, 2,3, and 4 is non-integer. It is possible to specify that these variables are integer-valued (in recent versions of LINDO, but not the version currently on the Apollo workstations of ICAEN). This is done using a comand GIN. The solution found by LINDO after much more computation is then as follows:

```
       OBJECTIVE FUNCTION VALUE

        1)    854080.000

  VARIABLE        VALUE          REDUCED COST
        F1         2.000000      -7579.997000
        F3         1.000000      -3999.998000
        F6         1.000000       -100.000000
        W1        11.000000           .000000
        W2        11.000000           .000000
        W3        10.000000           .000000
        W4        10.000000           .000000
        W5        10.000000           .000000
        W6         9.000000           .000000
        I1      1400.000000           .000000
        I2      2000.000000           .000000
        I3      3000.000000           .000000
        R1      6400.000000           .000000
        R2      6600.000000           .000000
        R3      6000.000000           .000000
        R4      6000.000000           .000000
        R5      6000.000000           .000000
        R6      5000.000000           .000000
```

As a result of adding the integer restriction, the cost is increased by (854080-852716.62) = \$1363.38, an increase of approximately 0.15%.

(2.) The following questions refer to the first problem in HW#1.

a. Suppose that additional oil were available for purchase, but at a price higher than the original price of \$40/barrel. How much more should SunCo be willing to pay per barrel?

Solution: Row #2, which specifies the upper limit on the oil purchased, has a "DUAL PRICE" of 10.

```
     ROW   SLACK OR SURPLUS     DUAL PRICES
       2)           .000000       10.000000
```

That is, if the right-hand-side of row 2 were to increase, the profit would increase at the rate of \$10/barrel. Therefore, if SunCo is able to obtain additional amounts of oil for less than the original price plus \$10/barrel, i.e., less than \$50/barrel, they would increase their profits.

b. How many barrels should they be willing to purchase for the price you answered in part (a)?

Solution: For row #2, the right-hand-side range, i.e., the range within which the current basis and therefore the current dual price remains unchanged, is:

```
     ROW          CURRENT          ALLOWABLE         ALLOWABLE
                    RHS             INCREASE          DECREASE
       2    20000.000000           INFINITY       4000.001000
```

Therefore, if they can purchase <u>any</u> amount for any price less than \$50/barrel, it is in their best interests to do so.

c. Suppose that the catalytic cracker requires maintenance during the next day, reducing its availability from 8 hours to 6 hours. How much loss of profit will result?

Solution: With this change, the right-hand-side of row (5) will be reduced from 8 to 6:

```
 5)   0.00075 HCRACK + 0.001 ACRACK <=   8
```

(The right-hand-side is hrs/day available time for the catalytic cracker.) The right-hand-side range of row (5) given by LINDO is

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
5	8.000000	2.000000	8.000000

The specified decrease of 2 hours is within the ALLOWABLE DECREASE, and therefore the DUAL PRICE of row 5 gives us the loss in profit for each of the 2 hours decrease. This dual price is

ROW	SLACK OR SURPLUS	DUAL PRICES
5)	.000000	70000.000000

That is, there will be a $140,000 reduction in profits.

d. Based upon the substitution rates in the optimal tableau, what will be the optimal solution with the reduced capacity specified in (c)?

Solution: Row (5), converted to an equation, is

```
0.00075 HCRACK + 0.001 ACRACK + SLK5 =  8
```

We wish to force 2 hours of slack in this row, i.e., we wish to increase the variable SLK 5 by 2 units.

The substitution rates for SLK 5, found in the tableau, are

ROW	BASIS	SLK 5	RHS
1	ART	0.70E+05	.76E+06
2	OIL	.000	20000.000
3	ASOLD	-1000.000	2000.000
4	HSOLD	.000	10000.000
5	ACRACK	1000.000	8000.000

Therefore, the basic variable ACRACK will decrease by 2(1000)=2000 barrels/day and the basic variable ASOLD will increase by the same amount.

e. In the optimal solution, no heating oil is processed in the catalytic cracker. How much would be required for the selling price of heating oil processed in the catalytic cracker in order to justify this processing?

Solution: The objective coefficient ranges, given by LINDO, are

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
HCRACK	90.000000	2.500000	INFINITY

That is, any increase less than 2.5 in the objective coefficient for the variable HCRACK will result in no change in the solution, so that an increase in the selling price of $2.50/barrel is required to change the solution (presumably, increasing HCRACK).

(3.) Gepbab Corporation produces three products at two different plants. The cost of producing a unit at each plant is:

	Product 1	Product 2	Product 3
Plant 1	$5	$6	$8
Plant 2	$8	$7	$10

Each plant can produce a total of 10,000 units. At least 6000 units of Product 1, at least 8000 units of Product 2, and at least 5000 units of Product 3 must be produced. To minimize the cost of meeting these demands, the following LP should be solved:

Min $z = 5x_{11} + 6x_{12} + 8x_{13} + 8x_{21} + 7x_{22} + 10x_{23}$

subject to

$x_{11} + x_{12} + x_{13} \leq 10000$

$x_{21} + x_{22} + x_{23} \leq 10000$

$x_{11} + x_{21} \geq 6000$

$$x_{12} + x_{22} \geq 8000$$
$$x_{13} + x_{23} \geq 5000$$
$$x_{ij} \geq 0 \text{ for all } i,j$$

Here, x_{ij} = number of units of product j produced at plant i. Use the LINDO output below to answer the following questions. (In each case, answer the question based on the output of the original problem, instead of re-solving the problem with revised costs or right-hand-sides.)

```
MIN      5 X11 + 6 X12 + 8 X13 + 8 X21 + 7 X22 + 10 X23
 SUBJECT TO
        2)   X11 + X12 + X13 <=   10000
        3)   X21 + X22 + X23 <=   10000
        4)   X11 + X21 >=   6000
        5)   X12 + X22 >=   8000
        6)   X13 + X23 >=   5000
 END

 LP OPTIMUM FOUND AT STEP      5

        OBJECTIVE FUNCTION VALUE

        1)    128000.000

  VARIABLE        VALUE          REDUCED COST
       X11      6000.000000          .000000
       X12          .000000         1.000000
       X13      4000.000000          .000000
       X21          .000000         1.000000
       X22      8000.000000          .000000
       X23      1000.000000          .000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)          .000000         2.000000
        3)      1000.000000          .000000
        4)          .000000        -7.000000
        5)          .000000        -7.000000
        6)          .000000       -10.000000

 NO. ITERATIONS=       5

 RANGES IN WHICH THE BASIS IS UNCHANGED:

                           OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
      X11        5.000000        1.000000         7.000000
      X12        6.000000        INFINITY         1.000000
      X13        8.000000        1.000000         1.000000
      X21        8.000000        INFINITY         1.000000
      X22        7.000000        1.000000         7.000000
      X23       10.000000        1.000000         1.000000

                           RIGHTHAND SIDE RANGES
      ROW         CURRENT        ALLOWABLE        ALLOWABLE
                    RHS          INCREASE         DECREASE
```

```
        2    10000.000000      1000.000000      1000.000000
        3    10000.000000         INFINITY      1000.000000
        4     6000.000000      1000.000000      1000.000000
        5     8000.000000      1000.000000      8000.000000
        6     5000.000000      1000.000000      1000.000000
```

a. What would the cost of producing product 2 at plant 1 have to be in order for the firm to want to produce product 2 at plant 1?

Solution: The variable X12=0 in the optimal solution. Unless the cost were to decrease by more than $1.00 (the ALLOWABLE DECREASE for variable X12), the solution will not change. Therefore, the cost must be less than $5 in order that X12 be positive.

b. What would total cost be if plant 1 had 9000 units of capacity?

Solution: The ALLOWABLE DECREASE for row (2) is 1000.00. The DUAL PRICE of row (2), which is the capacity restriction on plant 1, is 2.00. That is, if the right-hand-side of row (2) were to decrease by 1000 units (from 10000 to 9000), each unit of decrease will increase our cost by $2.00.

c. If it costs $9 to produce a unit of product 3 at plant 1, what would be the new optimal solution?

Solution: The ALLOWABLE INCREASE in the objective coefficient of X13 is 1.00, so that an increase of $1.00 in the cost of X13 will result in no change in the solution.

Homework #3

(1.) ***Linear Programming Sensitivity Analysis***

(Problem #10, page 224) Use LINDO to solve the "Sailco" problem of §3.10 (Example 12, pages 100-103). The LP model there is

Minimize $z=400x_1 + 400x_2 + 400x_3 + 400x_4 + 450y_1 + 450y_2 + 450y_3 + 450y_4 + 20i_1 + 20i_2 + 20i_3 + 20i_4$

subject to $i_1 = 10+x_1+y_1-40$ $\quad i_2 = i_1 + x_2 + y_2 - 60$

$i_3 = i_2 + x_3 + y_3 - 75$ $\quad i_4 = i_3 + x_4 + y_4 - 25$

$x_t \leq 40$ for t=1,2,3,4; $x_t \geq 0$, $y_t \geq 0$, & $i_t \geq 0$, t=1,2,3,4

where

x_t = # of sailboats produced by regular-time labor (at $400/boat) during quarter t

y_t = # of sailboats produced by overtime labor (at $450/boat) during quarter t

The LINDO output is as follows:

```
MIN     400 X1 + 400 X2 + 400 X3 + 400 X4 + 450 Y1 + 450 Y2 + 450 Y3
      + 450 Y4 + 20 I1 + 20 I2 + 20 I3 + 20 I4
 SUBJECT TO
        2)   X1 + Y1 - I1 =    30
        3)   X2 + Y2 + I1 - I2 =    60
        4)   X3 + Y3 + I2 - I3 =    75
        5)   X4 + Y4 + I3 - I4 =    25
 END
 SUB       X1       40.00000
 SUB       X2       40.00000
 SUB       X3       40.00000
 SUB       X4       40.00000

 LP OPTIMUM FOUND AT STEP      7

        OBJECTIVE FUNCTION VALUE
```

```
        1)    78450.0000

 VARIABLE        VALUE          REDUCED COST
       X1        40.000000        -30.000000
       X2        40.000000        -50.000000
       X3        40.000000        -50.000000
       X4        25.000000           .000000
       Y1          .000000         20.000000
       Y2        10.000000           .000000
       Y3        35.000000           .000000
       Y4          .000000         50.000000
       I1        10.000000           .000000
       I2          .000000         20.000000
       I3          .000000         70.000000
       I4          .000000        420.000000


      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)          .000000       -430.000000
       3)          .000000       -450.000000
       4)          .000000       -450.000000
       5)          .000000       -400.000000

 NO. ITERATIONS=       7

 RANGES IN WHICH THE BASIS IS UNCHANGED:

                          OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
      X1       400.000000        30.000000         INFINITY
      X2       400.000000        50.000000         INFINITY
      X3       400.000000        50.000000         INFINITY
      X4       400.000000        50.000000       420.000000
      Y1       450.000000         INFINITY        20.000000
      Y2       450.000000        20.000000        20.000000
      Y3       450.000000        20.000000        50.000000
      Y4       450.000000         INFINITY        50.000000
      I1        20.000000        30.000000        20.000000
      I2        20.000000         INFINITY        20.000000
      I3        20.000000         INFINITY        70.000000
      I4        20.000000         INFINITY       420.000000

                          RIGHTHAND SIDE RANGES
      ROW         CURRENT        ALLOWABLE        ALLOWABLE
                    RHS          INCREASE         DECREASE
        2       30.000000        10.000000        10.000000
        3       60.000000         INFINITY        10.000000
        4       75.000000         INFINITY        35.000000
        5       25.000000        15.000000        25.000000
```

(Notice that, rather than adding 4 additional rows, the constraints $x_t \leq 40$ were entered as "simple upper bounds", using the "sub" command.) Use the LINDO output to answer the following questions:

a. If quarter* #1 demand decreased to 35 sailboats, what would be the total cost of satisfying the demands during the next four quarters?

b. If the cost of producing a sailboat with regular-time labor during quarter[*] #1 were \$420, what would be the new optimal solution to the Sailco problem?

c. Suppose a new customer is willing to pay \$425 for a sailboat. If his demand must be met during quarter 1, should Sailco fill the order? How about if his demand must be met during quarter 4?

---

[*]I assume that the author intended the time unit here to correspond to that in §3.10

(2.) The following questions refer to the Fine-Webb Paper Company LP model in §A of the Class Notes.

a. Which variables are basic in the optimal solution shown in the notes?

b. How can you determine that the optimal solution shown in the notes is not unique?

c. Which nonbasic variable(s) may be increased with no effect on the objective function?

d. Using the "Minimum Ratio Test", how much will a variable which you specified in (c) increase if entered into the basis?

e. Using to the substitution rates, determine a basic optimal solution different from that shown in the notes. (Are the variables integer-valued?)

Homework #3 Solution

a. If quarter 1 demand decreased to 35 sailboats, what would be the total cost of satisfying the demands during the next four quarters?

If quarter #1 demand decreased by 5 (from 40 to 35), then the right-hand-side in row 2 will also decrease by 5. The ALLOWABLE DECREASE is 10, so the cost will decrease by 430 (the DUAL PRICE) for each of the 5 units decrease in RHS. That is, the cost will decrease by 430x5 = \$2150 to 78450-2150 = \$76300.

b. If the cost of producing a sailboat with regular-time labor during quarter [*]1 were \$420, what would be the new optimal solution to the Sailco problem?

The ALLOWABLE INCREASE in the objective coefficient of X1 is 30. That is, the basis is not changed if the cost of X1 is $\leq$400+30, and therefore the optimal solution is unchanged (except for the increase in the total cost to \$78450 + \$20x40 = \$79250.)

c. Suppose a new customer is willing to pay \$425 for a sailboat. If his demand must be met during quarter 1, should Sailco fill the order? How about if his demand must be met during quarter 4?

If the demand in quarter 1 were increased by 1, the cost will increase by \$ 430 (the DUAL PRICE of row (2)). Since this is greater than the price offered, it is not to Sailco's advantage to produce an additional boat to be delivered in quatter #1. On the other hand, the DUAL PRICE in row (5) is \$400, and so Sailco should accept the offer of \$425 for a boat to be delivered at that time, since it will cost them only \$400 to build the additional boat to be delivered.

---

[*]I assume that the author intended the time unit here to correspond to that in §3.10

(2.) The following questions refer to the Fine-Webb Paper Company LP model in §A of the Class Notes.

a. Which variables are basic in the optimal solution shown in the notes?

There will be **3** basic variables (in addition to the objective, -Z). These are: X1, X4, and X9.

b. How can you determine that the optimal solution shown in the notes is not unique?

Notice that five nonbasic variables have a reduced cost equal to zero, namely X2, X3, X5, X7, and X8.

*c. Which **non**basic variable(s) may be increased with no effect on the objective function?*

Increasing any one of the five nonbasic variables mentioned in (b) will have no effect on the objective function.

d. Using the "Minimum Ratio Test", how much will a variable which you specified in (c) increase if entered into the basis?

The minimum ratios for each column are shown below, with the minimum ratio in each column indicated:

X_2	X_3	X_5	X_7	X_8
$20/0.5$	$20/0.75$	$20/0.25$	∞	∞
$50/0.5$	∞	$50/0.25$	$50/0.5$	$50/0.75$
∞	$20/0.5$	$20/0.5$	∞	$20/0.5$

X_2	X_3	X_5	X_7	X_8
$\boxed{40}$	$\boxed{26.67}$	80	∞	∞
100	∞	200	$\boxed{200}$	66.67
∞	100	$\boxed{40}$	∞	$\boxed{40}$

That is, if X_2 is chosen to enter the basis, it's value will become 40, while if X_3 is selected to enter the basis, its value increases to 26.67, etc.

e. Using to the substitution rates, determine a basic optimal solution different from that shown in the notes. (Are the variables integer-valued?)

If you choose X_2, for example, then when X_2 increases by 40, since the substitution rates in the X_2 column of the tableau are 0.5 in both rows in which X_1 and X_4 are basic, the values of X_1 and X_4 will each decrease by 0.5x40 = 20. The new basic solution would then be X_2=40, X_4= 50-20=30, X_9=20 (with other X's nonbasic and equal to zero.)

If you choose X_5, then X_5 increases by 40, while X_1 and X_4 each decrease by 0.25x40=10 and X_9 decreases by 0.5x40=20, giving the new basic solution X_1=10, X_4=40, X_5=40 (with other X's nonbasic and therefore zero.) In both this case and the last, the new solution has integer values for the X's. If, however, you choose to enter X_3 into the basis, the result will be a non-integer solution.

1. Revised Simplex Method: Consider the LP problem:

Minimize $2X_1 + 5X_2 \quad + 7X_4 + 15X_5 + 14X_6$

subject to $X_1 + 2X_2 - X_3 + X_4 + 4X_5 + 5X_6 = 10$

$X_1 + 3X_2 - 2X_3 + 2X_4 + 5X_5 + 7X_6 = 12$

$X_j \geq 0, \ j=1,2,\dots 6$

After several iterations of the "revised simplex method", the current basis is **B** = {3,5}.

(a.) What is the current basis matrix?

(b.) What is the basis inverse matrix?

(c.) What is the current basic solution?

(d.) What are the values of the simplex multipliers?

(e.) Price the second column of the coefficient matrix. Would entering this column into the basis matrix result in an improvement in the solution?

(f.) Assume that column 2 is to be entered into the basis (regardless of whether doing so improves the solution). What is the "updated" column, i.e. the column of substitution rates?

(g.) Which column of the current basis will be replaced by column 2?

2. Linear Programming Model Formulation: (Exercise 30 of Review Problems, page 119 of text by Winston) "A paper recycling plant processes box board, tissue paper, newsprint, and book paper into pulp that can be used to produce three grades of recycled paper (grades 1, 2, and 3). The prices per ton and the pulp contents of the four inputs are:

Input type	Cost $/ton	Pulp content
Box board	5	15%
Tissue paper	6	20%
Newsprint	8	30%
Book paper	10	40%

Two methods, de-inking and asphalt dispersion, can be used to process the four inputs into pulp. It costs $20 to de-ink a ton of any input. The process of de-inking removes 10% of the input's pulp. It costs $15 to apply asphalt dispersion to a ton of material. The asphalt dispersion process removes 20% of the input's pulp. At most 3000 tons of input can be run through the asphalt dispersion process or the de-inking process. Grade 1 paper can only be produced with newsprint or book paper pulp; grade 2 paper, only with book paper, tissue paper, or box board pulp; and grade 3 paper, only with newsprint, tissue paper, or box board pulp. To meet its current demands, the company needs 500 tons of pulp for grade 1 paper, 500 tons of pulp for grade 2 paper, and 600 tons of pulp for grade 3 paper. Formulate <u>and solve</u> (using LINDO) an LP to minimize the cost of meeting the demands for pulp."

Hint:: *you may wish to define the variables*

BOX = tons of purchased boxboard
TISS = tons of purchased tissue
NEWS = tons of purchased newsprint
BOOK = tons of purchased book paper

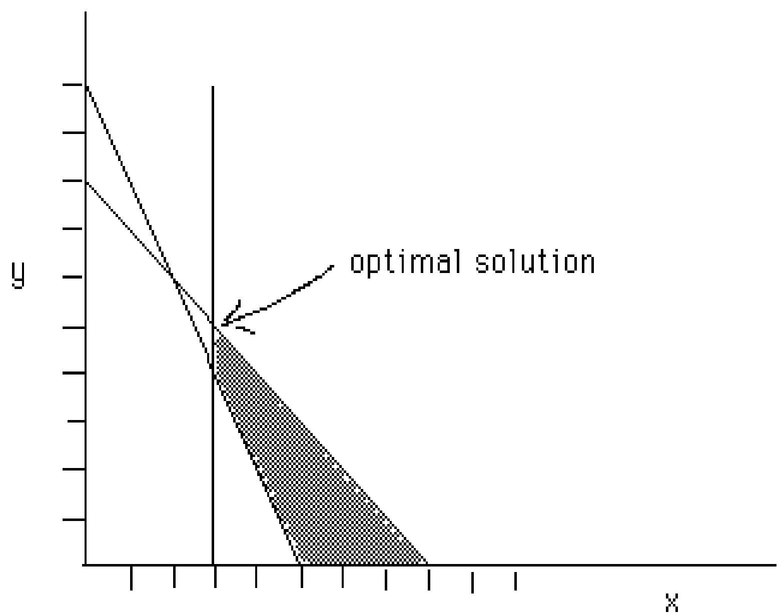
BOX1 = tons of boxboard sent through de-inking
TISS1 = tons of tissue sent through de-inking
NEWS1 = tons of newsprint sent through de-inking
BOOK1 = tons of book paper sent through de-inking
BOX2 = tons of boxboard sent through asphalt dispersion
TISS2 = tons of tissue sent through asphalt dispersion
NEWS2 = tons of newsprint sent through asphalt dispersion
BOOK2 = tons of book paper sent through asphalt dispersion
PBOX = tons of pulp recovered from boxboard
PTISS = tons of pulp recovered from tissue
PNEWS= tons of pulp recovered from newsprint
PBOOK = tons of pulp recovered from book paper
PBOX1 = tons of boxboard pulp used for grade 1 paper,
PBOX2 = tons of boxboard pulp used for grade 2 paper, etc.
…
PBOOK3 = tons of book paper pulp used for grade 3 paper.

3. **Duality:** Consider the following LP and its graphical solution:

Maximize $x + 2y$

subject to

$$\begin{aligned} x + y &\le 8 \\ 2x + y &\ge 10 \\ x &\ge 3 \\ x \ge 0,\ y &\ge 0 \end{aligned}$$



(a.) What is the optimal solution? the optimal objective value? What are the basic variables?

(b.) Write the dual LP for this problem.

(c.) By the complementary slackness theorem,, what must be the value of the optimal dual variable corresponding to the constraint $2x + y \ge 10$?

(d.) By the complementary slackness theorem, which dual constraints must be tight? What is the optimal solution of the dual problem?

1. Revised Simplex Method:

(a.) What is the current basis matrix?

The basis matrix consists of the columns of A corresponding to the two basic variables, i.e.,

$$A^B = \begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix}$$

(b.) What is the basis inverse matrix?

$$\left(A^B\right)^{-1} = \begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5/3 & -4/3 \\ 2/3 & -1/3 \end{bmatrix}$$

(c.) What is the current basic solution?

$$X_B = \begin{bmatrix} X_3 \\ X_5 \end{bmatrix} = \left(A^B\right)^{-1} b = \begin{bmatrix} 5/3 & -4/3 \\ 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} 10 \\ 12 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 8/3 \end{bmatrix}$$

i.e., $X_3 = 2/3$, $X_5 = 8/3$.

(d.) What are the values of the simplex multipliers?

$$\pi = \begin{bmatrix} \pi_1, & \pi_2 \end{bmatrix} = C_B\left(A^B\right)^{-1} = \begin{bmatrix} 0, & 15 \end{bmatrix}\begin{bmatrix} 5/3 & -4/3 \\ 2/3 & -1/3 \end{bmatrix} = \begin{bmatrix} 10, & -5 \end{bmatrix}$$

(e.) Price the second column of the coefficient matrix. Would entering this column into the basis matrix result in an improvement in the solution?

The reduced cost is computed using the simplex multipliers by

$$\overline{C}_j = C_j - \pi A^j$$

for j not in the basis B:

$$\overline{C}_1 = C_1 - \pi A^1 = 2 - \begin{bmatrix} 10, & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 - 5 = -3$$

$$\overline{C}_2 = 5 - \begin{bmatrix} 10, & -5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 5 - 5 = 0$$

$$\overline{C}_4 = 7 - \begin{bmatrix} 10, & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 7 - 0 = 7$$

$$\overline{C}_6 = 14 - \begin{bmatrix} 10, & -5 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = 14 - 15 = -1$$

(f.) Assume that column 2 is to be entered into the basis (regardless of whether doing so improves the solution). What is the "updated" column, i.e. the column of substitution rates?

Since the reduced cost of X2 is zero, entering X2 into the basis (i.e., increasing X2) will have no effect on the objective function value. The updated column (used in pivoting), i.e., the column of substitution rates of X2 for the basic variables, is

$$\hat{A}^2 = \left(A^B\right)^{-1} A^2 = \begin{bmatrix} 5/3 & -4/3 \\ 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \end{bmatrix}$$

i.e., each unit of X_2 which enters the solution will require an additional $2/3$ unit of the first basic variable (X_3) and will replace $1/3$ unit of the second basic variable (X_5).

(g.) Which column of the current basis will be replaced by column 2?

The column to be replaced by X_2 is generally found by the minimum ratio test. Here, however, there is only one positive substitution rate, and therefore only one variable being reduced as X_2 increases, namely X_5.

2. Linear Programming Model Formulation:

Define the variables

BOX = tons of purchased boxboard
TISS = tons of purchased tissue
NEWS = tons of purchased newsprint
BOOK = tons of purchased book paper
BOX1 = tons of boxboard sent through de-inking
TISS1 = tons of tissue sent through de-inking
NEWS1 = tons of newsprint sent through de-inking
BOOK1 = tons of book paper sent through de-inking
BOX2 = tons of boxboard sent through asphalt dispersion
TISS2 = tons of tissue sent through asphalt dispersion
NEWS2 = tons of newsprint sent through asphalt dispersion
BOOK2 = tons of book paper sent through asphalt dispersion
PBOX = tons of pulp recovered from boxboard
PTISS = tons of pulp recovered from tissue
PNEWS= tons of pulp recovered from newsprint
PBOOK = tons of pulp recovered from book paper
PBOX1 = tons of boxboard pulp used for grade 1 paper,
PBOX2 = tons of boxboard pulp used for grade 2 paper, etc.
... etc.
PBOOK3 = tons of book paper pulp used for grade 3 paper.

The LP model using these variables is:

```
MIN  5 BOX +6 TISS +8 NEWS +10 BOOK +20 BOX1 +20 TISS1 +20 NEWS1
           +20 BOOK1 +15 BOX2 +15 TISS2 +15 NEWS2 +15 BOOK2
 SUBJECT TO
        2) - BOX + BOX1 + BOX2 <=   0
        3) - TISS + TISS1 + TISS2 <=   0
        4) - NEWS + NEWS1 + NEWS2 <=   0
        5) - BOOK + BOOK1 + BOOK2 <=   0
        6)   0.135 BOX1 + 0.12 BOX2 - PBOX =    0
        7)   0.18 TISS1 + 0.16 TISS2 - PTISS =    0
        8)   0.27 NEWS1 + 0.24 NEWS2 - PNEWS =    0
        9)   0.36 BOOK1 + 0.32 BOOK2 - PBOOK =    0
       10) - PBOX + PBOX2 + PBOX3 <=   0
       11) - PTISS + PTISS2 + PTISS3 <=   0
       12) - PNEWS + PNEWS1 + PNEWS3 <=   0
       13) - PBOOK + PBOOK1 + PBOOK2 <=   0
       14)   PNEWS1 + PBOOK1 >=   500
       15)   PBOX2 + PTISS2 + PBOOK2 >=   500
       16)   PBOX3 + PTISS3 + PNEWS3 >=   600
       17)   BOX1 + TISS1 + NEWS1 + BOOK1 <=   3000
       18)   BOX2 + TISS2 + NEWS2 + BOOK2 <=   3000
 END
```

- Rows 2-5 state that only the supply of each input material which is purchased can be processed, either by de-inking or asphalt dispersion.

- Row 6 states that the recovered pulp in the boxboard is 90% of that in the boxboard which is processed by de-inking, i.e., (0.90)(0.15)BOX1, since boxboard is 15% pulp, plus 80% of that in the boxboard which is processed by asphalt dispersion, i.e., (0.80)(0.15)BOX2.
- Rows 7-9 are similar to row 6, but for different input materials.
- Rows 10-13 state that no more than the pulp which is recovered from each input may be used in making paper (grades 1, 2, &/or 3). Note that some variables are omitted, e.g., PBOX1, since boxboard cannot be used in Grade 1 paper.
- Rows 14-16 state that the demand for each grade of paper must be satisfied.
- Rows 17-18 state that each process (de-inking & asphalt dispersion) has a maximum thruput of 3000 tons. (The problem statement is unclear here, and might be interpreted as stating the total thruput of both processes cannot exceed 3000 tons, in which case rows 17&18 would be replaced by

```
17)   BOX1 + TISS1 + NEWS1 + BOOK1
    + BOX2 + TISS2 + NEWS2 + BOOK2 <=   3000
```

The solution found by LINDO is as follows:

```
                                        P P P P P P
              T N B   T N B   P P P P P T T N N B B
      T N B B I E O B I E O P T N B B B I I E E O O
    B I E O O S W O O S W O B I E O O O S S W W O O
    O S W O X S S K X S S K O S W O X X S S S S K K
    X S S K 1 1 1 1 2 2 2 2 X S S K 2 3 2 3 1 3 1 2

 1: 5 6 8 A B B B B B B B B '     '     '     '     MIN
 2:-1     ' 1   '   1 '     '     '     '     <
 3: '-1'  '  '1 '  '  1  '  '  '  '  '  '  '  '  '  <
 4:    -1 '     1     ' 1   '     '     '     '     <
 5:      -1     ' 1   '   1 '     '     '     '     <
 6: '  '  ' T'  '  'T '  ' -1  '  '  '  '  '  '  '  =
 7:       '   T '     T     '-1   '     '     '     =
 8:       '     T     ' T   '  -1 '     '     '     =
 9: '  '  '  '  ' T'  '  'T '  ' -1  '  '  '  '  '  =
10:       '     '     '    -1     ' 1 1 '     '     <
11:       '     '     '     '-1   '     1 1   '     <
12: '  '  '  '  '  '  '  '  '  -1 '  '  '  '1 1  '  <
13:       '     '     '     '    -1     '     ' 1 1 <
14:       '     '     '     '     '     '   1 ' 1   > C
15: '  '  '  '  '  '  '  '  '  '  ' 1'  1  '  '  '1 > C
16:       '     '     '     '     '   1 ' 1   1     > C
17:       ' 1 1 1 1   '     '     '     '     '     < D
18: '  '  '  '  '  '1 1 1'1 '  '  '  '  '  '  '  '  < D
```

```
LP OPTIMUM FOUND AT STEP     23

       OBJECTIVE FUNCTION VALUE

       1)    140000.000

 VARIABLE        VALUE          REDUCED COST
     NEWS      2500.000000           .000000
     BOOK      2833.333400           .000000
```

BOOK1	2333.333400	.000000
NEWS2	2500.000000	.000000
BOOK2	499.999930	.000000
PNEWS	600.000000	.000000
PBOOK	1000.000000	.000000
PNEWS3	600.000000	.000000
PBOOK1	500.000000	.000000
PBOOK2	500.000000	.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	.000000	.000000
3)	.000000	.000000
4)	.000000	8.000000
5)	.000000	10.000000
6)	.000000	-102.777800
7)	.000000	-102.777800
8)	.000000	-102.777800
9)	.000000	-83.333340
10)	.000000	102.777800
11)	.000000	102.777800
12)	.000000	102.777800
13)	.000000	83.333340
14)	.000000	-83.333340
15)	.000000	-83.333340
16)	.000000	-102.777800
17)	666.666500	.000000
18)	.000000	1.666666

That is, the optimal solution is to purchase only newsprint and book paper, process 500 tons of the book paper and all of the newsprint by asphalt dispersion, and the remaining book paper by de-inking. This yields 600 tons of pulp from the newsprint and 1000 tons of pulp from the book paper. One-half of the pulp from book paper is used in each of grades 1 & 2 paper, and the newsprint is used in grade 3 paper. This plan will use all of the de-inking capacity and (2333/3000)= 77.8% of the alphalt dispersion capacity.

3. Duality:

(a.) What is the optimal solution? the optimal objective value? What are the basic variables?

The optimal solution, indicated in the graph, is the intersection of the lines x=3 and x+y=8, i.e., the point (3,5).

(b.) Write the dual LP for this problem.

The dual problem will be a minimization problem (since the primal is a maximization problem), with two "≥" constraints, since both primal variables are nonnegative. Because the first primal constraint is of type "≤", the sign restriction on the first dual variable is nonnegativity, but the other two primal constraints are type "≥", so the corresponding dual variables are restricted to be non-positive:

$$\begin{aligned} &\text{Minimize } 8U_1 + 10U_2 + 3U_3 \\ &\text{subject to } U_1 + 2U_2 + U_3 \geq 1 \\ &U_1 + U_2 \geq 2 \\ &U_1 \geq 0,\ U_2 \leq 0,\ U_3 \leq 0 \end{aligned}$$

(If you first transform the 2nd and 3rd primal constraints to get type "≤" constraints, in order to obtain a symmetric primal/dual pair, the dual problem will be

$$\text{Minimize } 8U_1 - 10U_2 - 3U_3$$

$$\text{subject to} \quad U_1 - 2U_2 - U_3 \geq 1$$
$$U_1 - U_2 \geq 2$$
$$U_1 \geq 0,\ U_2 \geq 0,\ U_3 \geq 0$$

which is equivalent to that above if you make the change of variable $U_2' = -U_2$ and $U_3' = -U_3$.

(c.) By the complementary slackness theorem,, what must be the value of the optimal dual variable corresponding to the constraint $2x + y \geq 10$?

By inspection of the graph above, we see that this constraint is not tight, i.e., its surplus variable is positive. According to the complementary slackness theorem, the corresponding dual variable, U_2, should be zero at optimum.

(d.) By the complementary slackness theorem, which dual constraints must be tight?

Both x & y are positive in the primal solution, so both of the dual inequality constraints must be tight (have zero slack or surplus).

What is the optimal solution of the dual problem?

Since we have already determined that U2=0 at the optimum, we need only solve the 2 equations for the remaining 2 variables:

$$U_1 - U_3 = 1$$
$$U_1 = 2$$

so that U1=2 and U3=1 and the optimal dual solution is U=(2,0,1).

mmmmmm Homework #5 mmmmmm

1. **Transportation Model for Production Planning:** (Exercise #3 of Review Problems, page 371 of text by Winston.) A company must meet the following demands for a product:

January	30
February	30
March	20

Demand may be backlogged at a cost of $5/unit/month. Of course, all demand must be met by the end of March. Thus, if 1 unit of January demand is met during March, a backlogging cost of $5(2) = $10 is incurred. Monthly production capacity and unit production cost during each month are:

Month	Production Capacity	Unit Production Cost
January	35	$400
February	30	$420
March	35	$410

A holding cost of $20/unit is assessed on the inventory at the end of each month.

a. Formulate a balanced transportation problem that could be used to determine how to minimize the total cost (including backlogging, holding, and production costs) of meeting demand. (It is sufficient to display a transportation tableau with rows for sources and columns for destinations.)

b. Use *either* Vogel's method *or* the Northwest-corner method to find a basic feasible solution of the transportation problem.

c. Use the transportation simplex method to determine how to meet each month's demand. Make sure to give an interpretation of your optimal solution. (For example, 20 units of month 2 demand is met from month 1 production, etc.)

2. **Assignment Problem:** (Exercise 2 of Review Problems, page 371 of text by Winston) Five workers are available to perform four jobs. The time (in hours) which it takes each worker to perform each job is:

	Job 1	Job 2	Job 3	Job 4
Worker #1	10	15	10	15
Worker #2	12	8	20	16
Worker #3	12	9	12	18
Worker #4	6	12	15	18
Worker #5	16	12	8	12

The goal is to assign workers to jobs so as to minimize the total time required to perform the four jobs. Use the Hungarian method to solve the problem. *(Hint: define a "dummy" job which requires zero time when assigned to any worker.)*

mmmmmm Homework #5 Solution mmmmmm

1. **Transportation Model for Production Planning:** (Exercise #3 of Review Problems, page 371 of text by Winston.)

a. Formulate a balanced transportation problem that could be used to determine how to minimize the total cost (including backlogging, holding, and production costs) of meeting demand.

Demand

Production	JAN	FEB	MAR	dummy	
JAN	400	420	440	0	35
FEB	425	420	440	0	30
MAR	420	415	410	0	35
	30	30	20	20	100

Note that a "dummy" destination is required, so that the total demand will equal the total supply (which is 100). The "shipping" cost actually consists of production and either backlogging or storage costs. For example, the cost of "shipping" from the JAN source to the MAR demand is \$400 (production cost) + (\$20/month)(2 months), the second term being the holding cost for inventory.

b. *Use either Vogel's method or the Northwest-corner method to find a basic feasible solution of the transportation problem.*

If we use the "Northwest-corner" method, we obtain the initial basic feasible solution:

	JAN	FEB	MAR	dummy	
JAN	30 / 400	5 / 420	440	0	35
FEB	425	25 / 420	5 / 440	0	30
MAR	420	415	15 / 410	20 / 0	35
	30	30	20	20	

The cost of this "shipping" plan (actually a production plan) is 30(\$400) +5(\$420) +25(\$425) + 5(\$440) + 15(\$410) + 20(\$0) = \$32950.

If we use Vogel's Approximation Method (VAM), we obtain the initial basic feasible solution below. Note that the feasible solution found by VAM has a lower cost than that found by NW-corner method.

step 1

penalties

	20	5	30	
400	400	420	440	0
→ 420	425	420	440	0
410	420	415	410	0

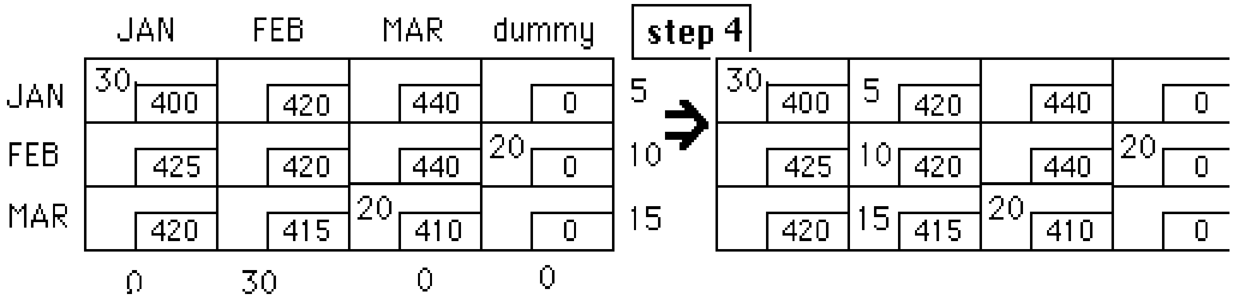
	JAN	FEB	MAR	dummy	
JAN	400	420	440	0	35
FEB	425	420	440	20 / 0	30
MAR	420	415	410	0	35
	30	30	20	20	

step 2

penalties

	20	5	30 ↓
20	400	420	440
5	425	420	440
5	420	415	410

	JAN	FEB	MAR	dummy	
JAN	400	420	440	0	35
FEB	425	420	440	20 / 0	10
MAR	420	415	20 / 410	0	~~35~~ 15
	30	30	~~20~~ 0	0	

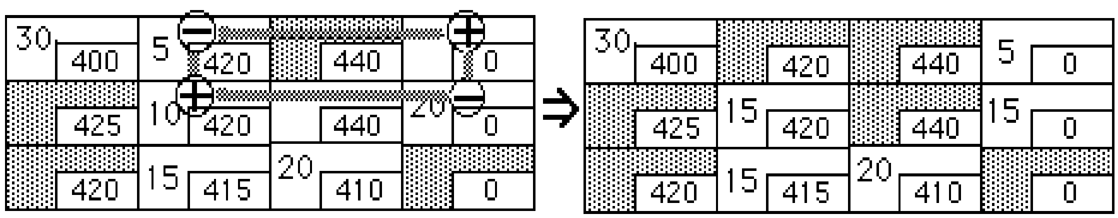


At step four of VAM, we obtain a matrix with nonzero demand in only one column, so the remaining shipments (to FEB demand) are determined. The total cost of this shipping plan is 30($400) +5($420) +10($420) +15($415) +20($410) +20($0) = $32725.

c. *Use the transportation simplex method to determine how to meet each month's demand.*

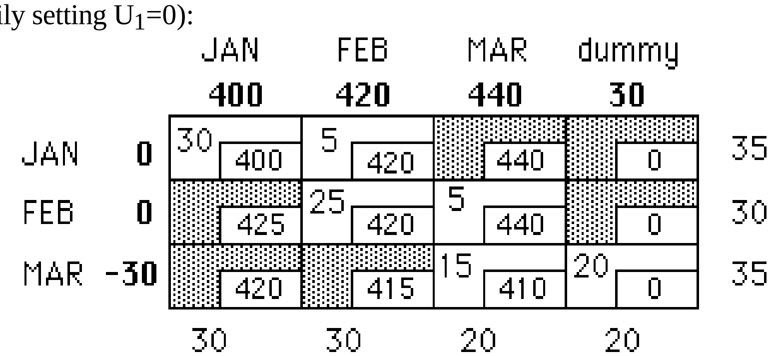
Let's use the starting basic feasible solution found by VAM.

- First, we calculate the dual variables. If we let U_1=0, then we obtain U_1=0, U_2= 0, U_3=-5, V_1= 400, V_2=420, V_3= 415, and V_4= 0.
- We then use these dual variables to compute the reduced costs $\underline{C}_{ij}$ of the nonbasic variables: $\underline{C}_{21}$ = 425-(0+400) = 25; $\underline{C}_{31}$ = 420-(-5+400) = 25; $\underline{C}_{13}$ = 440 - (0+415) = 25; $\underline{C}_{23}$ = 440 - (0+400) = 40; $\underline{C}_{14}$ = 0 - (0+0) = 0; $\underline{C}_{34}$ = 0 - (-5+0) = 5.
- Since $\underline{C}_{ij} \geq 0$ for all i and j, the solution is optimal!
- However, since $\underline{C}_{14}$=0, this is not the only optimal solution. If we enter X_{14} into the basis, we obtain the "shipping" plan on the right below:



Suppose that we use the initial basic feasible solution found by the "Northwest-Corner" method:

In order to price the nonbasic variables, we first compute the dual variables as shown (after arbitrarily setting U_1=0):



- The reduced costs are now: $\underline{C}_{21}$=425-400>0, $\underline{C}_{31}$=420-370>0, $\underline{C}_{32}$=415-390>0, $\underline{C}_{13}$=440-440=0, $\underline{C}_{14}$=0-30<0, and $\underline{C}_{24}$=0-30<0. The solution can be improved by entering either X_{14} or X_{24} into the basis. Suppose that we choose X_{14}. We then identify the "loop" of adjustments required to increase X_{14}:

	JAN	FEB	MAR	dummy	
JAN	30 / 400	5 (−) / 420	440	(+) / 0	35
FEB	425	25 (+) / 420	5 (−) / 440	0	30
MAR	420	415	15 (+) / 410	20 (−) / 0	35
	30	30	20	20	

• For each unit increase in X_{14}, the variables X_{12}, X_{23}, and X_{34} are all decreased by one unit. When X_{14}=5, the basic variables X_{12} and X_{23} have both decreased (simultaneously) to zero. Only one of these should leave the basis, however (since the number of basic variables must remain constant at 6). Suppose we arbitrarily select X12 to leave the basis. Then the next basic tableau is

		JAN **400**	FEB **450**	MAR **410**	dummy **0**	
JAN	**0**	30 / 400	420	440	5 / 0	35
FEB	**-30**	425	30 / 420	0 / 440	0	30
MAR	**0**	420	415	20 / 410	15 / 0	35
		30	30	20	20	

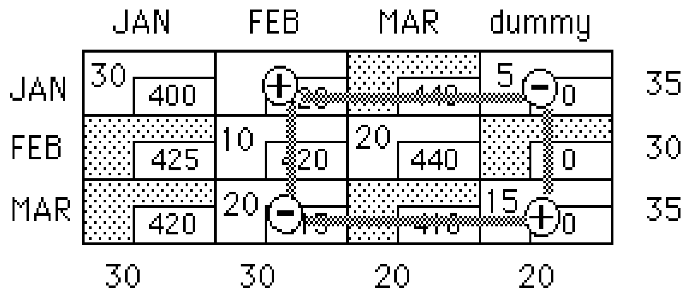
with the new values of the dual variables as show. The reduced costs must now be re-computed: $\underline{C}_{12}$=420-450<**0**, $\underline{C}_{13}$=440-410>0, $\underline{C}_{21}$=425-370>0, $\underline{C}_{24}$=0-(-30)>0, $\underline{C}_{31}$=420-400>0, and $\underline{C}_{32}$=415-450<**0**. Increaseing either X_{12} or X_{32} could improve the solution. Let's arbitrarily choose to increase X_{23}. We identify the loop of adjustments which must be made:

	JAN	FEB	MAR	dummy	
JAN	30 / 400	420	440	5 / 0	35
FEB	425	30 (−) / 420	0 (+) / 440	0	30
MAR	420	(+) / 415	20 (−) / 410	15 / 0	35
	30	30	20	20	

When X_{23} enters the basis, then, X_{33} leaves the basis, X_{22} decreases to 10, and X_{23} increases to 20. The new basic feasible solution and new values of the dual variables are:

		JAN **400**	FEB **415**	MAR **435**	dummy **0**	
JAN	**0**	30 / 400	420	440	5 / 0	35
FEB	**5**	425	10 / 420	20 / 440	0	30
MAR	**0**	420	20 / 415	410	15 / 0	35
		30	30	20	20	

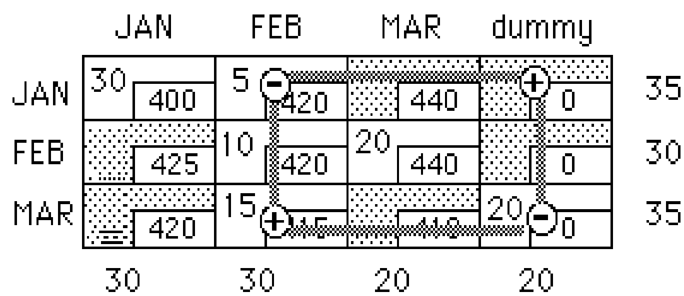
and the reduced costs are: $\underline{C}_{12}$=420-415<**0**, $\underline{C}_{13}$=440-435>0, $\underline{C}_{21}$=425-420>0, $\underline{C}_{24}$=0 - 5 <0, $\underline{C}_{31}$=420-400>0, and $\underline{C}_{33}$=410-435<**0**. So Increasing either X_{12}, X_{24}, or X_{33} will improve the solution. Let's arbitrarily choose X_{12}, and obtain the following loop of adjustments:



We see that X12 will replace X14 in the basis when X12=5. The new basic solution and the new dual variables are:

		JAN **400**	FEB **420**	MAR **440**	dummy **5**	
JAN	**0**	30 400	5 420	440	0	35
FEB	**0**	425	10 420	20 440	0	30
MAR	**-5**	420	15 415	410	20 0	35
		30	30	20	20	

- The new reduced costs are: $\underline{C}_{13}$=440-440=0, $\underline{C}_{14}$=0-5<**0**, $\underline{C}_{21}$=425-400>0, $\underline{C}_{24}$=0-5<**0**, $\underline{C}_{31}$=420-395>0, and $\underline{C}_{33}$=410-435<**0**. Let's enter X_{14} into the basis. The loop of adjustments is:



and we see that X12 leaves the basis, giving us the new basic solution (& new dual variables):

		JAN **400**	FEB **415**	MAR **435**	dummy **0**	
JAN	**0**	30 400	420	440	5 0	35
FEB	**5**	425	10 420	20 440	0	30
MAR	**0**	420	20 415	410	15 0	35
		30	30	20	20	

- The new reduced costs are: $\underline{C}_{12}$=420-415>0, $\underline{C}_{13}$=440-435>0, $\underline{C}_{21}$=425-400>0, $\underline{C}_{24}$=0-5<**0**, $\underline{C}_{31}$=420-400>0, and $\underline{C}_{33}$=410-440<**0**. Let's enter X_{33} into the basis. Tbe loop of adjustments is:

	JAN	FEB	MAR	dummy	
JAN	30 $\boxed{400}$	$\boxed{420}$	$\boxed{440}$	5 $\boxed{0}$	35
FEB	$\boxed{425}$	10 ⊕ $\boxed{420}$	20 ⊖ $\boxed{440}$	$\boxed{0}$	30
MAR	$\boxed{420}$	20 ⊖ $\boxed{415}$	⊕ $\boxed{410}$	15 $\boxed{0}$	35
	30	30	20	20	

Either X_{32} or X_{23} will become nonbasic. Let's arbitrarily keep X_{32} in the basis to obtain the degenerate feasible solution:

		JAN	FEB	MAR	dummy	
		400	**415**	**410**	**0**	
JAN	**0**	30 $\boxed{400}$	$\boxed{420}$	$\boxed{440}$	5 $\boxed{0}$	35
FEB	**5**	$\boxed{425}$	30 $\boxed{420}$	$\boxed{440}$	$\boxed{0}$	30
MAR	**0**	$\boxed{420}$	0 $\boxed{415}$	20 $\boxed{410}$	15 $\boxed{0}$	35
		30	30	20	20	

The new reduced costs are: $\underline{C}_{12}$=420-415>0, $\underline{C}_{13}$=440-410>0, $\underline{C}_{21}$=425-405>0, $\underline{C}_{23}$=440-415>0, $\underline{C}_{24}$=0-5<**0**, and $\underline{C}_{31}$=420-400>0. The solution is still not optimal, and so we enter X_{24} into the basis, obtaining the loop of adjustments:

	JAN	FEB	MAR	dummy	
JAN	30 $\boxed{400}$	$\boxed{420}$	$\boxed{440}$	5 $\boxed{0}$	35
FEB	$\boxed{425}$	30 ⊖ $\boxed{420}$	$\boxed{440}$	⊕ $\boxed{0}$	30
MAR	$\boxed{420}$	0 ⊕ $\boxed{415}$	20 $\boxed{410}$	15 ⊖ $\boxed{0}$	35
	30	30	20	20	

and, when X34 leaves the basis, obtain the solution

		JAN	FEB	MAR	dummy	
		400	**420**	**415**	**0**	
JAN	**0**	30 $\boxed{400}$	$\boxed{420}$	$\boxed{440}$	5 $\boxed{0}$	35
FEB	**0**	$\boxed{425}$	15 $\boxed{420}$	$\boxed{440}$	15 $\boxed{0}$	30
MAR	**-5**	$\boxed{420}$	15 $\boxed{415}$	20 $\boxed{410}$	$\boxed{0}$	35
		30	30	20	20	

- The reduced costs are now: $\underline{C}_{12}$=420-420=0, $\underline{C}_{13}$=440-415>0, $\underline{C}_{21}$=425-400>0, $\underline{C}_{23}$=440-415>0, $\underline{C}_{31}$=420-395>0, and $\underline{C}_{34}$=0-(-5)>0. Since the reduced costs are nonnegative, this is an optimal solution. As explained before, since $\underline{C}_{12}$=0, another optimal solution can be obtained by entering X_{12} into the basis.

- The optimal production plan #1 is: produce 35 units in January to satisfy the January demand of 30, and store 5; 10 units of February's demand is satisfied by February production, and 5 units of demand is satisfied by the inventory stored at the end of January, leaving 15 units of February demand unsatisfied; In March, produce 35 units, to satisfy the 20 units of demand in March and the

15 units of February's demand which was backlogged. This plan will result in 20 units of unused production capacity in February.

• Optimal solution #2 is: produce 30 units in January to satisfy the January demand of 30, with 5 units of production capacity unused; produce 15 units in February to partially satisfy the February demand, with 15 units of capacity unused and 15 units of demand backlogged; produce 35 units in March, 20 of which will satisfy the March demand and the other 15 of which will be used for the February demand which was backlogged.

2. Assignment Problem: (Exercise 2 of Review Problems, page 371 of text by Winston) Define a "dummy" job (job #5) to get a square cost matrix:

Column reduction (columns 1,2,3,4)

$$\begin{bmatrix} 10 & 15 & 10 & 15 & 0 \\ 12 & 8 & 20 & 16 & 0 \\ 12 & 9 & 12 & 18 & 0 \\ 6 & 12 & 15 & 16 & 0 \\ 16 & 12 & 8 & 12 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 7 & 2 & 3 & 0 \\ 6 & 0 & 12 & 4 & 0 \\ 6 & 1 & 4 & 6 & 0 \\ 0 & 4 & 7 & 4 & 0 \\ 10 & 4 & 0 & 0 & 0 \end{bmatrix}$$

Now each row and each column has at least one zero, which we try to cover with the minimum number of lines. One of several which require only 4 lines is:

$$\begin{bmatrix} 4 & 7 & 2 & 3 & 0 \\ 6 & 0 & 12 & 4 & 0 \\ 6 & 1 & 4 & 6 & 0 \\ 0 & 4 & 7 & 4 & 0 \\ 10 & 4 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 7 & 0 & 1 & 0 \\ 6 & 0 & 10 & 2 & 0 \\ 6 & 1 & 2 & 4 & 0 \\ 0 & 4 & 5 & 2 & 0 \\ 12 & 6 & 0 & 0 & 2 \end{bmatrix}$$

Since the number of lines is less than five, we perform another reduction, subtracting 2 from each number without a line, and adding 2 to the three intersections of lines. We then again cover the zeroes with the snallest number of lines, and discover that it cannot be done with less than five lines. It then follows that there is a zero-cost assignment using this cost matrix. Since columns 1, 2, and 4 have single zeroes, we make the corresponding assignments (4,1), (2,2), and (5,4). Row 3 also has a single zero, so we make the corresponding assignment (3,5). All workers & jobs are now assigned except worker #4 and job #3, so we make the assignment (4,3):

4	7	0	1	0
6	0	10	2	0
6	1	2	4	0
0	4	5	2	0
12	6	0	0	2

Assignments:

Worker #	to	Job #
1		3
2		2
3		5 ("dummy" job)
4		1
5		4

The cost of this assignment is 36 (hrs.) Note that worker #3 is actually assigned to do nothing!

Since you might cover the zeroes above with 4 lines in several ways, you might obtain different cost matrices.... however, the optimal solution will be the same. For example, suppose we cover the zeroes after the first step with lines in rows 2,4, and 5 and a line in column 5:

4	7	2	3	0
6	0	12	4	0
6	1	4	6	0
0	4	7	4	0
10	4	0	0	0

$\Rightarrow$

3	6	1	2	0
6	0	12	4	1
5	0	3	5	0
0	4	7	4	1
10	4	0	0	1

When we reduce the matrix (as indicated, by subtracting 1 from each number without a line and adding 1 to the three intersections), the resulting matrix has zeroes distributed differently than at this step in the earlier solution. Again, four lines are sufficient to cover the zeroes:

3	6	1	2	0
6	0	12	4	
5	0	3	5	0
0	4	7	4	
10	4	0	0	

$\Rightarrow$

3	6	0	1	0
6	0	11	3	1
5	0	2	4	0
0	4	6	3	1
11	5	0	0	2

Attempting to cover the zeroes in the matrix on the right above will soon convince you that it cannot be done in fewer than five lines. Therefore, we should be able to make zero-cost assignments in this matrix to obtain a solution to the problem:

3	6	0	1	0
6	0	11	3	1
5	0	2	4	0
0	4	6	3	1
11	5	0	0	2

Although the non-zero costs are different from the final matrix obtained before, the same assignment can be made.

Homework #6

1. Project Scheduling. An equipment maintenance building is to be erected near a large construction site. An electric generator and a large water storage tank are to be installed a short distance away and connected to the building. The activity descriptions and estimated durations for the project are:

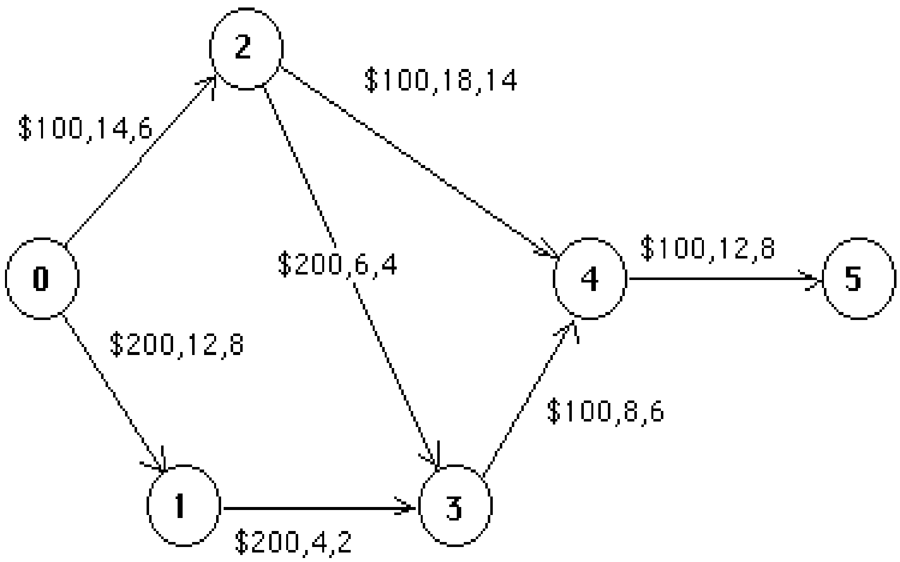
Activity	Description	Predecessor(s)	Duration (days)
A	Clear & level site	none	2
B	Erect building	A	6
C	Install generator	A	4
D	Install water tank	A	2
E	Install maintenance equipment	B	4
F	Connect generator & tank to building	B,C,D	5
G	Paint & finish work on building	B	3
H	Facility test & checkout	E,F	2

a. Draw the AON (activity-on-node) network representing this project.
b. Draw the AOA (activity-on-arrow) network representing this project. Explain the necessity for any "dummy" activities which you have included.
c. Label the nodes of the AOA network, so that $i<j$ if there is an activity with node i as its start and node j as its end node.
d. Perform the forward pass through the AOA network to obtain for each node i, ET(i) = earliest possible time for event i.
e. What is the earliest completion time for this project?
f. Perform the backward pass through the AOA network to obtain, for each node i, LT(i) = latest possible time for event i (assuming the project is to be completed in the time which you have specified in (e).)
g. For each activity, compute:
   - ES = earliest start time
   - LS = latest start time
   - EF = earliest finish time
   - LF = latest finish time
   - TF = total float (slack)
   - FF = "free" float (slack)

h. Which activities are "critical", i.e., have zero float?
i. Schedule this project by entering the AON network into MacProject II (found on several of the Macintosh II computers in the computer lab on 3rd floor of the Engineering Building.) Specify that the start time for the project will be November 1, 1992. What is the earliest completion time for the project? (Note that 5-day work weeks are assumed by default.)
j. Formulate an LP model to find the shortest possible completion time for the project. (You need not use LINDO to solve it.)

2. Crashing a Project. Consider a project whose AOA (activity-on-arrow) network is as shown below, where the duration of each activity can be reduced. On the network are shown, for each activity, three quantities: (i.) Cost/week of reducing the duration, (ii.) normal duration (in weeks), and (iii.) the minimum possible duration (in weeks). For example, the activity (2,4) has a normal duration of 18 weeks, but the duration can be shortened to as few as 14 weeks at a cost of $100/week.

a. What is the completion time of the project if no activity's duration is reduced ("crashed")?
b. Which activities are critical in (a.)?



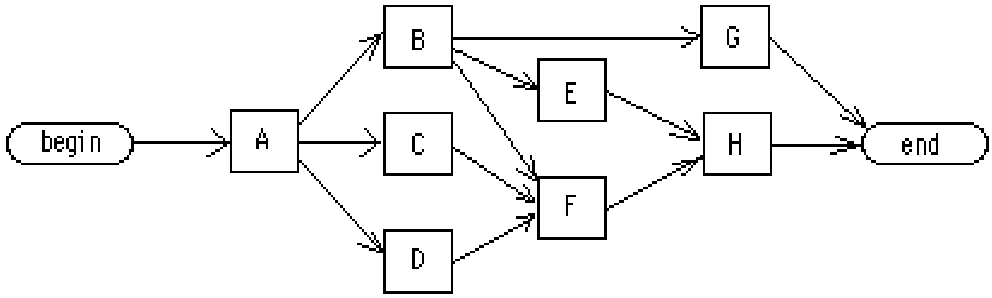
c. An LP model (LINDO input) to optimally "crash" the activities to meet a deadline of 34 weeks is shown below. (In this LP formulation, Y13 is the starting time for activity (1,3), R13 is the reduction in the duration of activity (1,3), etc. Y01 and Y02 are assumed to be constants, both zero. The variable COMPLETE is the completion time of the project. The "SUB" command was used to set upper bounds on the variables R01, etc.) Revise the LP to find the minimum completion time which can be achieved with a budget of $1800, and solve with LINDO.

```
MIN 200R01 +100R02 +200R13 +200R23 +100R24 +100R34 +100R45
ST
Y13+R01>12
Y23+R02>14
Y24+R02>14
Y34-Y13+R13>4
Y34-Y23+R23>6
Y45-Y24+R24>18
Y45-Y34-R34>8
COMPLETE-Y45+R45>12
COMPLETE<34
END
SUB R01 4.0
SUB R02 8.0
SUB R13 2.0
SUB R23 2.0
SUB R24 4.0
SUB R34 2.0
SUB R45 4.0
```

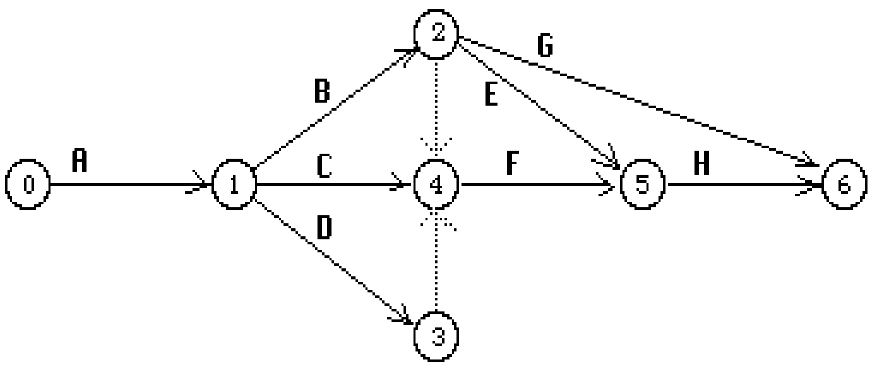
mmmmmm Homework #6 Solution mmmmmm

1. Project Scheduling.

a. *Draw the AON (activity-on-node) network representing this project.*



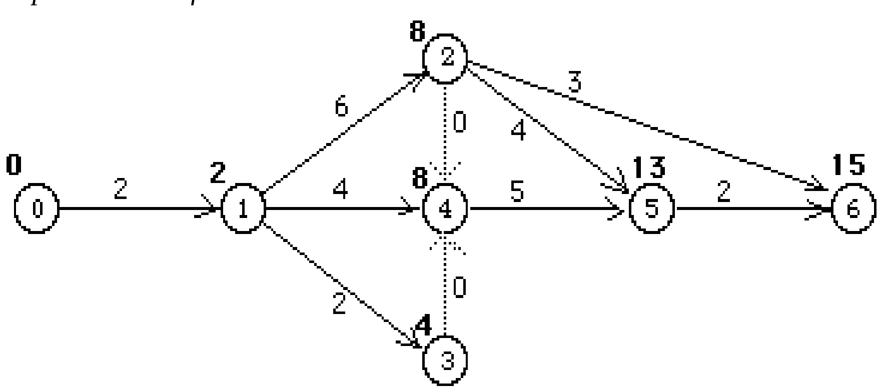
b. *Draw the AOA (activity-on-arrow) network representing this project. Explain the necessity for any "dummy" activities which you have included.*



The activities (2,4) and (3,4) (i.e., between node 2 and node 4, and between node 3 and node 4) are "dummy" activities with zero duration. Activity (3,4) is required because we wish to have only one activity (1,4) from node 1 to node 4. Activity (2,4) is required in order that activities E, F, and G might have B as predecessor while activity F also has as predecessor activity C (and D) as well.

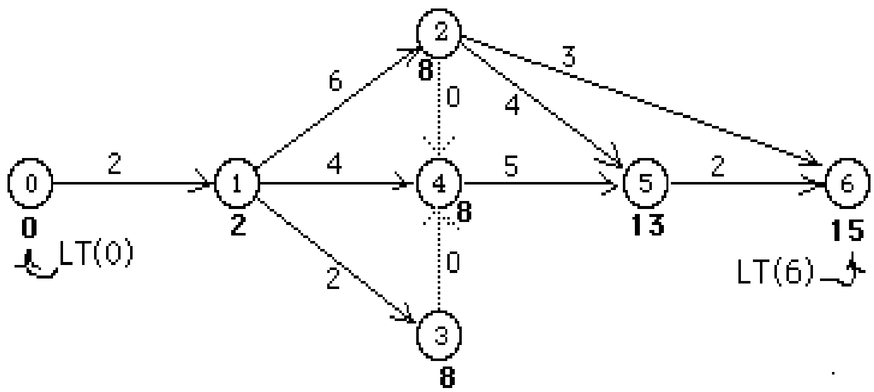
c. *Label the nodes of the AOA network, so that i<j if there is an activity with node i as its start and node j as its end node.* See the labelled network above. Note, however, that node labels 2 & 3 might be interchanged.

d. *Perform the forward pass through the AOA network to obtain for each node i, ET(i) = earliest possible time for event i.*



e. *What is the earliest completion time for this project?* 15 days, which is ET(6)

f. *Perform the backward pass through the AOA network to obtain, for each node i, LT(i) = latest possible time for event i (assuming the project is to be completed in the time which you have specified in (e).)*



(The Late Time for each node is written in bold type below the node.)

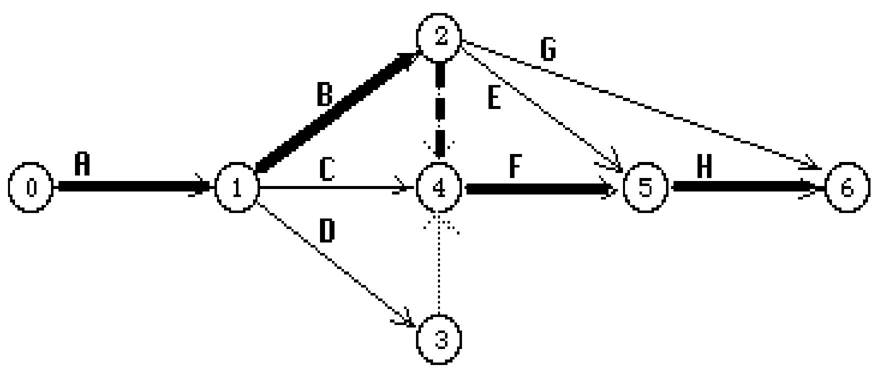
g. *For each activity, compute:*

ES = earliest start time — *LS = latest start time*
EF = earliest finish time — *LF = latest finish time*
TF = total float (slack) — *FF = "free" float (slack)*

Activity	Duration	ES	LS	EF	LF	TF	FF	Critical?
A	2	0	0	2	2	0	0	Yes
B	6	2	2	8	8	0	0	Yes
C	4	2	4	6	8	2	2	No
D	2	2	6	4	8	4	2	No
E	4	8	9	12	13	1	1	No
F	5	8	8	13	13	0	0	Yes
G	3	8	12	11	15	4	4	No
H	2	13	13	15	15	0	0	Yes

(Note that, in this example, total float = free float for every activity.)

h. *Which activities are "critical", i.e., have zero float?*



The activities having zero float are: A, B, F, and H (and dummy activity (2,4)). These are drawn in bold above.

i. Schedule this project by entering the AON network into MacProject II (found on several of the Macintosh II computers in the computer lab on 3rd floor of the Engineering Building.) Specify that the start time for the project will be November 1, 1992. What is the earliest completion time for the project? (Note that 5-day work weeks are assumed by default.)

j. *Formulate an LP model to find the shortest possible completion time for the project. (You need not use LINDO to solve it.)*

Define Variables: Y_A = start time for activity A, Y_B = start time for activity B, ...etc.
$Y_{complete}$ = completion time of project

Minimize $Y_{complete} - Y_A$

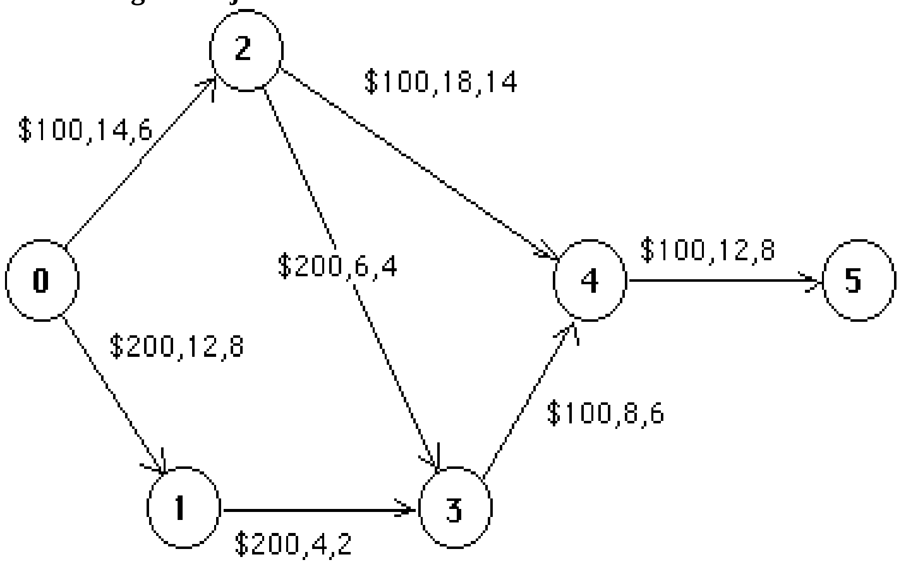
subject to

$Y_B \geq Y_A + 2$ $\quad Y_F \geq Y_B + 6$ $\quad Y_H \geq Y_E + 4$
$Y_C \geq Y_A + 2$ $\quad Y_F \geq Y_C + 4$ $\quad Y_H \geq Y_F + 5$
$Y_D \geq Y_A + 2$ $\quad Y_F \geq Y_D + 2$ $\quad Y_{complete} \geq Y_H + 2$
$Y_E \geq Y_B + 6$ $\quad Y_G \geq Y_B + 6$

$Y_A, Y_B, \ldots$ unrestricted in sign

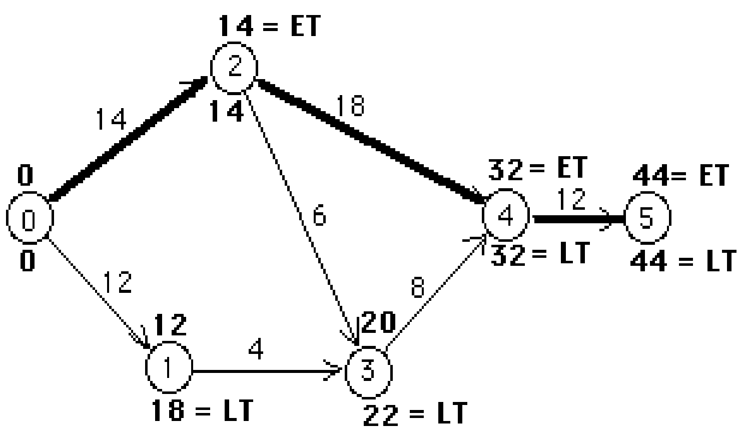
Note that we could, if we chose, restrict the variables to be non-negative and/or assign a fixed value to Y_A.

2. Crashing a Project.



a. *What is the completion time of the project if no activity's duration is reduced ("crashed")?*

The ET & LT of each node is indicated in the network below. The earliest completion time is 44 weeks.



b. *Which activities are critical in (a.)?* The activities drawn in bold are critical, i.e., activities (0,2), (2,4), and (4,5).

c. *An LP model (LINDO input) to optimally "crash" the activities to meet a deadline of 34 weeks is shown below. (In this LP formulation, Y13 is the starting time for activity (1,3), R13 is the reduction in the duration of activity (1,3), etc. Y01 and Y02 are assumed to be constants, both zero. The variable COMPLETE is the completion time of the project. The "SUB" command was used to set upper bounds on the variables R01, etc.) Revise the LP to find the minimum completion time which can be achieved with a budget of $1800, and solve with LINDO.*

```
MIN 200R01 +100R02 +200R13 +200R23 +100R24 +100R34 +100R45
```

```
ST
Y13+R01>12
Y23+R02>14
Y24+R02>14
Y34-Y13+R13>4
Y34-Y23+R23>6
Y45-Y24+R24>18
Y45-Y34-R34>8
COMPLETE-Y45+R45>12
COMPLETE<34
END
SUB R01 4.0
SUB R02 8.0
SUB R13 2.0
SUB R23 2.0
SUB R24 4.0
SUB R34 2.0
SUB R45 4.0
```

The reformulated LP model & LINDO output follows. Note that row (10) is the budget constraint, and that the new objective is to minimize the completion time:

```
 MIN     COMPLETE
 SUBJECT TO
        2)   Y13 + R01 >=   12
        3)   Y23 + R02 >=   14
        4)   R02 + Y24 >=   14
        5) - Y13 + Y34 + R13 >=   4
        6) - Y23 + Y34 + R23 >=   6
        7) - Y24 + Y45 + R24 >=   18
        8) - Y34 + Y45 - R34 >=   8
        9)   COMPLETE - Y45 + R45 >=   12
       10)   200 R01 + 100R02 + 200R13 + 200R23 + 100R24 + 100R34
      + 100R45 <=   1800
 END
 SUB      R01        4.00000
 SUB      R02        8.00000
 SUB      R13        2.00000
 SUB      R23        2.00000
 SUB      R24        4.00000
 SUB      R34        2.00000
 SUB      R45        4.00000

LP OPTIMUM FOUND AT STEP     13

        OBJECTIVE FUNCTION VALUE

        1)    30.0000000

  VARIABLE        VALUE          REDUCED COST
```

COMPLETE	30.000000	.000000
Y13	12.000000	.000000
R01	.000000	0.000000
Y23	8.000000	.000000
R02	6.000000	.000000
Y24	8.000000	.000000
Y34	14.000000	.000000
R13	2.000000	.000000
R23	.000000	.666667
Y45	22.000000	.000000
R24	4.000000	.000000
R34	.000000	1.000000
R45	4.000000	-.666667

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	.000000	-.666667
3)	.000000	.000000
4)	.000000	-.333333
5)	.000000	-.666667
6)	.000000	.000000
7)	.000000	-.333333
8)	.000000	-.666667
9)	.000000	-1.000000
10)	.000000	.003333

*The computations in problems 1 & 2 may be done with the APL workspace named "DECISION", if you wish. This would require that you obtain from me a copy of the APL*PLUS interpreter, which may then be used on one of the Mac Classics or Mac SE (not Mac II).*

1. Decision Trees. (Exercise 14, page 711-712 of textbook by W. Winston) A patient enters the hospital with severe abdominal pains. Based on past experience, Dr. Craig believes there is a 28% chance that the patient has appendicitis and a 72% chance that the patient has nonspecific abdominal pains (NAP). Dr. Craig may operate on the patient now or wait 12 hours in order to gain a more accurate diagnosis. In 12 hours, Dr. Craig will surely know whether the patient has appendicitis. The problem is that in the intervening 12 hours, the patient's appendix may perforate (if he has appendicitis), thereby making the operation much more dangerous. Again based on past experience, Dr. Craig believes that if hew waits 12 hours, there is a 6% chance that the patient will end up with a perforated appendix, a 22% chance that the patient will end up with "normal" appendicitis, and a 72% chance that the patient will end up with NAP. From past experience, Dr. Craig assesses the following probabilities of the patient's dying:

SITUATION	PROBABILITY THAT PATIENT WILL DIE
Operation on patient with appendicitis	.0009
Operation on patient with NAP	.0004
Operation on perforated appendix	.0064
No operation on patient with NAP	0.0

Dr. Craig's objective is to maximize the probability that the patient will survive.

a. Draw and properly label the decision tree. *(Hint: Let the "payoff" be 1.0 if the patient survives, and 0 if the patient dies.)*
b. Evaluate ("fold back") the nodes of the decision tree.
c. Determine the strategy that maximizes the patient's probability of surviving.

2. Decision Trees. (Exercise 7, page 717 of text by W. Winston) Pat Sajork has two drawers. One drawer contains three gold coins, and the other contains one gold coin and two silver coins. We are allowed to choose one drawer, and we will be paid $500 for each gold coin in that drawer and $100 for each silver coin in that drawer. Before choosing a drawer, we may pay Pat $200, and he will draw a randomly selected coin (i.e., each of the six coins has an equal chance of being chosen), and tell us whether it is gold or silver. For instance, Pat may say that he drew a gold coin from drawer #1.

a. Draw and properly label the decision tree.
b. Evaluate ("fold back") the nodes of the decision tree.
c. Determine the strategy that maximizes the payoff. (Should we pay Pat $200 to draw a coin?)
d. What is the expected value of sample information (EVSI), i.e., knowing the coin drawn from a drawer
e. What is the expected value of perfect informatio (EVPI)?

3. Integer Programming Model-Building. (Exercise 12, page 470-471 of text by W. Winston) A company is considering opening warehouses in four cities: New York, Los Angeles, Chicago, and Atlanta. Each warehouse can ship 100 units per week. the weekly fixed cost of keeping each warehouse open is $400 for NY, $500 for LA, $300 for Chicago, and $150 for Atlanta. Region 1 of the country requries 80 units per week, region 2 requires 70 units per week, and region 3 requires 40 units per week. The costs (including production and shipping costs) of sending 1 unit from a plant to a region are:

		TO	
FROM	Region 1	Region 2	Region 3
New York	$20	$40	$50
Los Angeles	$48	$15	$26
Chicago	$26	$35	$18
Atlanta	$24	$50	$35

We wish to meet weekly demands at minimum cost, subject to the preceding information and the following restrictions:

1. If the NY warehouse is opened, then the LA warehouse must be opened.
2. At most 2 warehouses can be opened.
3. Either the Atlanta or the LA warehouse must be opened.

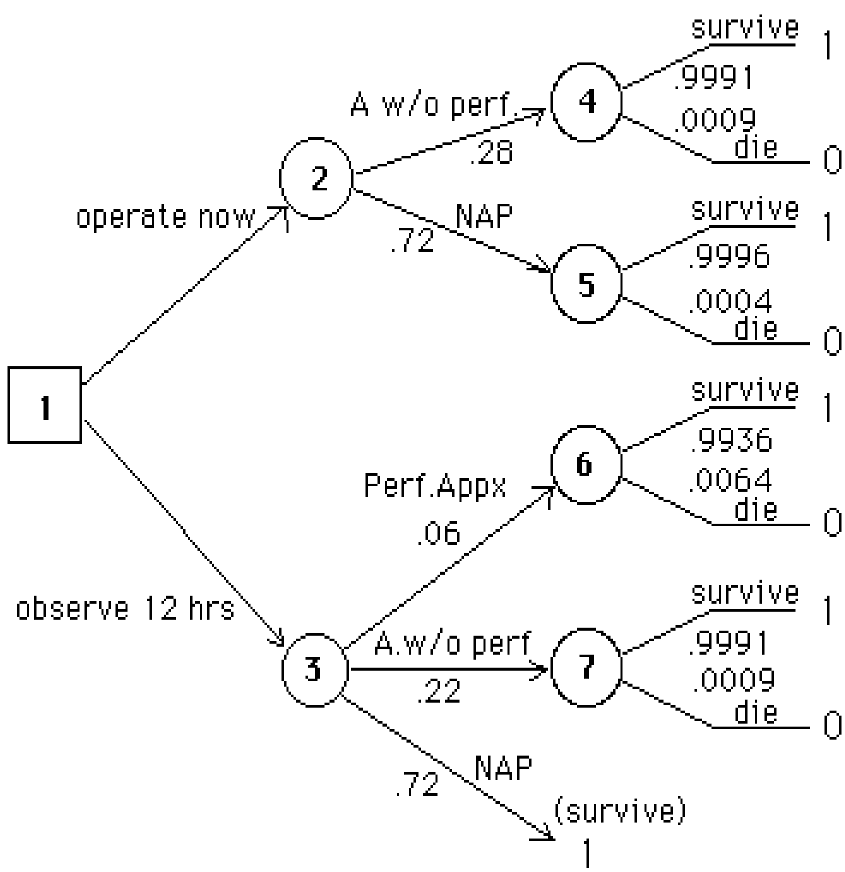
Formulate an IP (integer programming) model that can be used to minimize the weekly costs of meeting demand, and solve with LINDO.

Homework #7 Solution

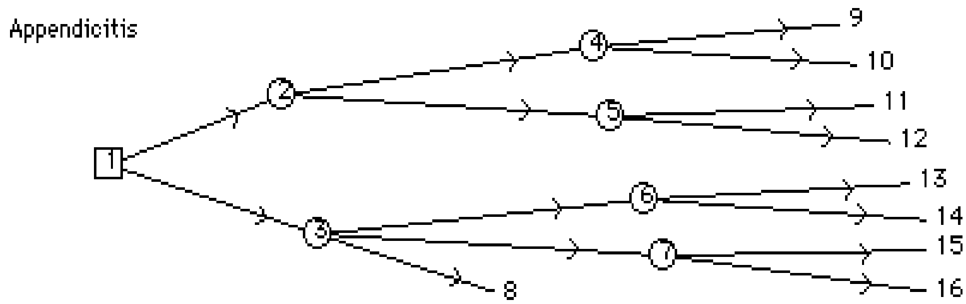
1. Decision Trees. (Exercise 14, page 711-712 of textbook by W. Winston)

a. Draw and properly label the decision tree.

Let the "payoff" be 1.0 if the patient survives, and 0 if the patient dies.



Using the "DECISION" APL-workspace, the tree was entered with the terminals of the branches numbered as follows:



The table of nodes, probabilities, & payoffs is:

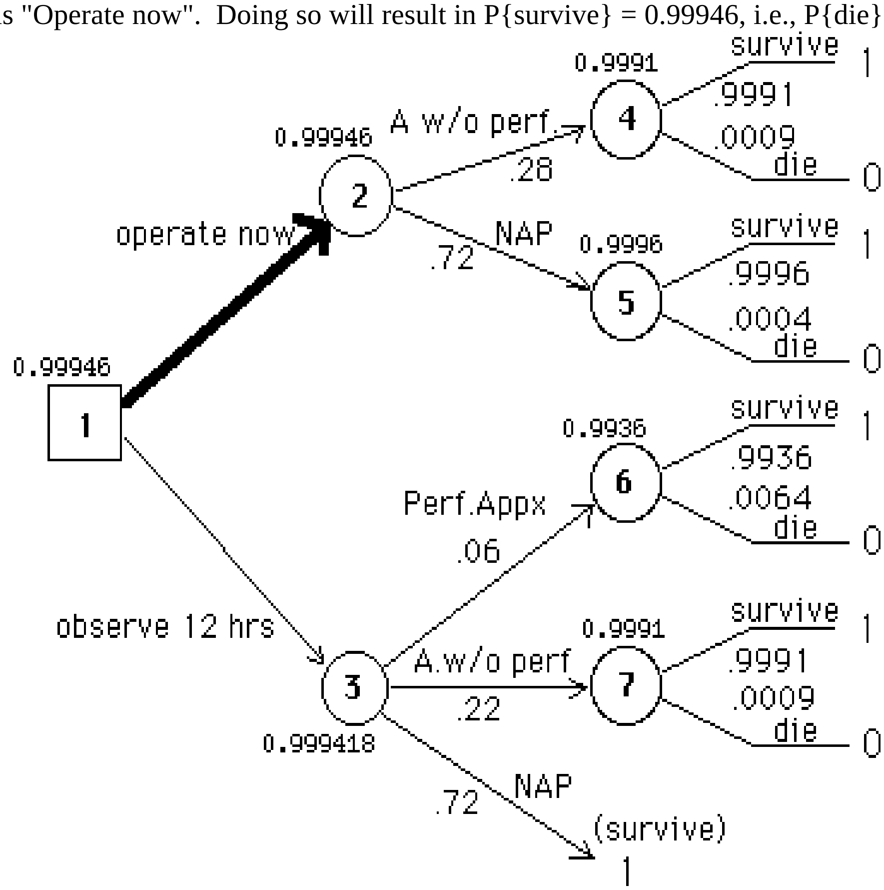
Node	Name	Type	P{outcome}	Value
1	A	Decision		
2	B	Chance		
3	C	Chance		
4	D	Chance	0.280000	
5	E	Chance	0.720000	
6	F	Chance	0.060000	
7	G	Chance	0.220000	
8	H	Terminal	0.720000	1.00
9	I	Terminal	0.999100	1.00
10	J	Terminal	0.000900	
11	K	Terminal	0.999600	1.00
12	L	Terminal	0.000400	
13	M	Terminal	0.993600	1.00
14	N	Terminal	0.006400	
15	O	Terminal	0.999100	1.00
16	P	Terminal	0.000900	

b. Evaluate ("fold back") the nodes of the decision tree.

```
Folding back node #4:D (Chance node)
     Outcome:      9       10
     P{outcome}:   0.9991   0.0009
     Value:        1        0
     Value of node 4: 0.9991
Folding back node #5:E (Chance node)
     Outcome:     11       12
     P{outcome}:   0.9996   0.0004
     Value:        1        0
     Value of node 5: 0.9996
Folding back node #2:B (Chance node)
     Outcome:     4        5
     P{outcome}:  0.28     0.72
     Value:       0.9991  0.9996
     Value of node 2: 0.99946
Folding back node #6:F (Chance node)
     Outcome:     13       14
     P{outcome}:   0.9936   0.0064
     Value:        1        0
     Value of node 6: 0.9936
Folding back node #7:G (Chance node)
     Outcome:     15       16
     P{outcome}:   0.9991   0.0009
     Value:        1        0
     Value of node 7: 0.9991
```

```
Folding back node #3:C (Chance node)
     Outcome:     6        7        8
     P{outcome}:  0.06     0.22     0.72
     Value:       0.9936   0.9991   1
     Value of node 3: 0.999418
Folding back node #1:A (Decision node)
     Alternative:  2          3
     Value:        0.99946    0.999418
     Optimal value: 0.99946
     Optimal decision: 2
```

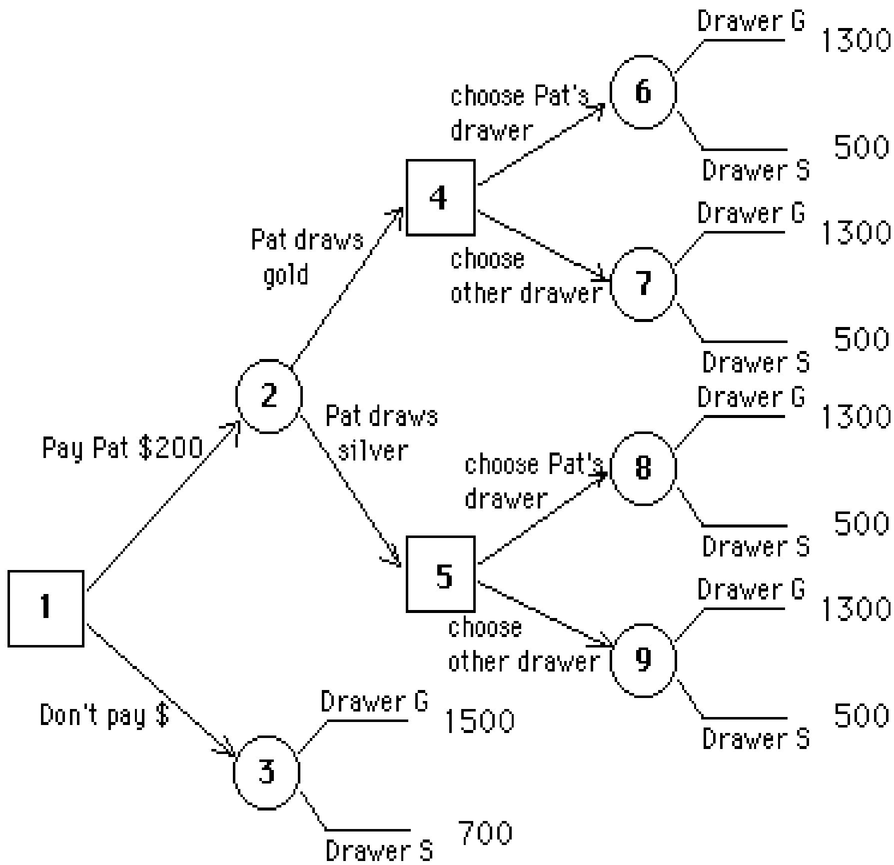
c. *Determine the strategy that maximizes the patient's probability of surviving.* At the decision node (#1), the optimal decision is 2, i.e., the branch which leads to node #2, which is "Operate now". Doing so will result in P{survive} = 0.99946, i.e., P{die} = 0.00054.



2. Decision Trees. (Exercise 7, page 717 of text by W. Winston)

a. *Draw and properly label the decision tree.*

Let's refer to the drawer with 3 gold coins as Drawer "G" and the other as Drawer "S". After Pat draws a coin (and, I assume, returns it to the drawer from which it was taken), we must decide whether to select that same drawer from which he drew the coin (we'll call it "Pat's Drawer"), or the other drawer. We now draw the decision tree:



Calculation of probabilities: The probability that Pat will draw a gold coin is $^4/_6$, and $^2/_6$ for silver (since each of the six coins has an equal probability of being selected).

We must use Bayes' Rule to compute the posterior probabilities, i.e., the probabilities of selecting either drawer G (3 gold coins) or S (1 gold & 2 silver coins), after we have the results of Pat's sampling a coin. The output provided by the function in the DECISION workspace is:

Likelihoods P{j|i}

i \ j	1	2
1	1	0
2	0.3333333333	0.6666666667

Joint probability P{i&j}

i \ j	1	2
1	0.5	0.1666666667
2	0	0.3333333333

Probability of Outcome

j=	1	2
P{j}=	0.6666666667	0.3333333333

Posterior Probability P{i|j}

i \ j	1	2
1	0.75	0
2	0.25	1

At node 2: P{Sample=G}=P{G}= $^4/_6$ = 0.6667, P{Sample=S}=P{S}= 0.3333

At node 3: P{Drawer G}=P{Drawer S}=0.50 ("prior probabilities")

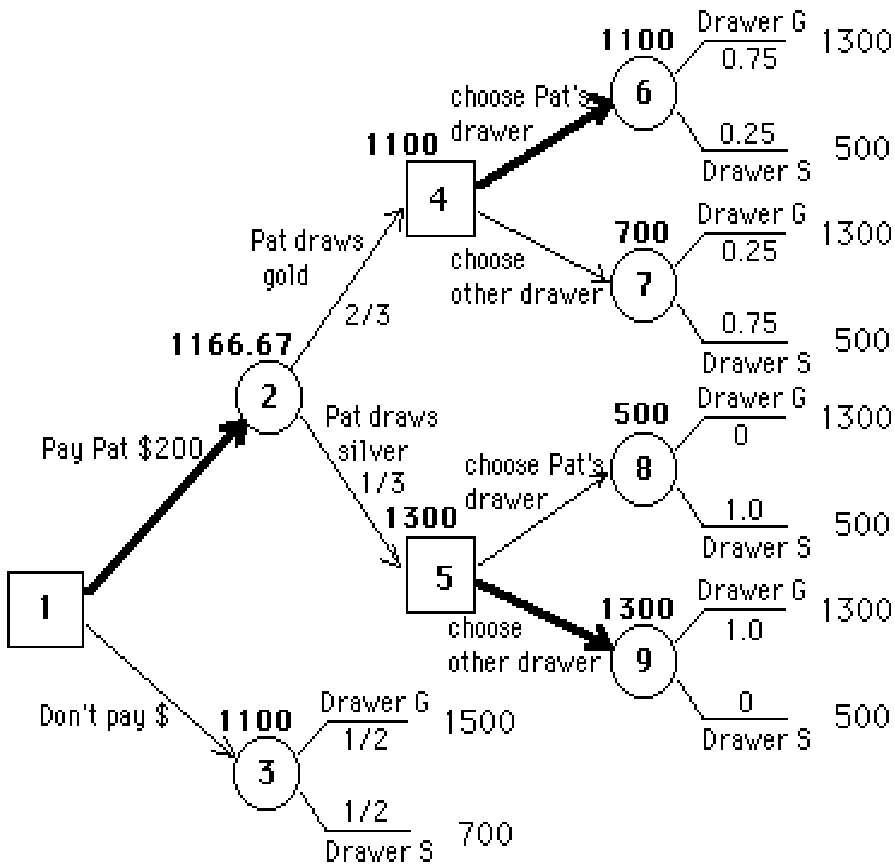
At node 6: P{Drawer G | Sample G} = 0.75, P{Drawer S | Sample G}=0.25
i.e., using Bayes' rule:

$$P\{\text{Drawer G} \mid \text{Sample G}\} = \frac{P\{\text{Sample G} \mid \text{Drawer G}\}P\{\text{Drawer G}\}}{P\{\text{Sample G}\}} = \frac{1\times 1/2}{2/3} = \frac{3}{4}$$

At node 7: P{Drawer G | Sample G} = 0.25, P{Drawer S | Sample S}=0.75
At node 8: P{ Drawer G | Sample S} = 0, P{Drawer S | Sample S} = 1
At node 9: P{ Drawer G | Sample S} = 1, P{Drawer S | Sample S} = 0

That is, if Pat draws a silver coin, then you have perfect information about the contents of the two drawers. If he draws a gold coin, the drawer from which he has removed the coin is 3 times as likely to contain the three gold coins than is the other drawer.

The new decision tree with the probabilities inserted is:



b. *Evaluate ("fold back") the nodes of the decision tree.* (see above)
c. *Determine the strategy that maximizes the payoff.* The optimal strategy is to pay Pat the $200, and if he selects a gold coin, we should choose the drawer from which he selected the coin. If, on the other hand, he selects a silver coin, we should choose the other drawer.
d. *What is the expected value of sample information (EVSI)?* If we were to do the computation without accounting for the $200 paid to Pat, the expected payoff in that branch of the tree should be increased by $200 to $1366.67. Therefore, EVSI = EVWSI - EVWOI = $1366.67 - $1100.00 = $266.67.
e. *What is the expected value of perfect informatio (EVPI)?* If Pat were to give us "perfect information", i.e., if he were to select a drawer and tell us all the coins in that drawer, then we would be certain of a payoff of $1500 (assuming we did not pay for this information). Therefore, EVPI = EVWPI - EVWOI = $1500 - $1100 = $400.

3. *Integer Programming Model-Building.* *(Exercise 12, page 470-471 of text by W. Winston) We wish to meet weekly demands at minimum cost, subject to the preceding information and the following restrictions:*

1. *If the NY warehouse is opened, then the LA warehouse must be opened.*
2. *At most 2 warehouses can be opened.*
3. *Either the Atlanta or the LA warehouse must be opened.*

Formulate an IP (integer programming) model that can be used to minimize the weekly costs of meeting demand, and solve with LINDO.

Define binary decision variables

Y_{NY} = 1 if warehouse in NY is opened, zero otherwise
Y_{LA} = 1 if warehouse in LA is opened, zero otherwise
Y_C = 1 if warehouse in Chicago is opened, zero otherwise
Y_A = 1 if warehouse in Atlanta is opened, zero otherwise

Define continuous decision variables

$X_{NY,1}$ = quantity shipped from NY warehouse to region #1, etc.

Then the objective will be

Minimize 400YNY + 500YLA + 300YC + 150YA + 20XNY1 + 40XNY2 + 50XNY3 + 48XLA1 + 15XLA2 + 26XLA3 + 26XC1 + 35XC2 + 18XC3 + 24XA1 + 50XA2 + 35XA3

and the constraints:

XNY1 + XLA1 + XC1 + XA1 ≥ 80 *(reqmt. of region #1 must be met)*

XNY2 + XLA2 + XC2 + XA2 ≥ 70 *(reqmt. of region #2 must be met)*

XNY3 + XLA3 + XC3 + XA3 ≥ 40 *(reqmt. of region #3 must be met)*

XNY1 + XNY2 + XNY3 ≤100YNY *(quantity shipped from NY warehouse is no more than 100 if the NY warehouse is used, and 0 otherwise.)*
XLA1 + XLA2 + XLA3 ≤100YLA *(quantity shipped from LA warehouse is no more than 100 if the LA warehouse is used, and 0 otherwise.)*
XC1 + XC2 + XC3 ≤100YC *(quantity shipped from Chicago warehouse is no more than 100 if the Chicago warehouse is used, and 0 otherwise.)*
XA1 + XA2 + XA3 ≤100YA *(quantity shipped from Atlanta warehouse is no more than 100 if the Atlanta warehouse is used, and 0 otherwise.)*

As an alternative, you could use a set of 4 constraints for each of the above 4. For NY, for example, you could use the constraints

XNY1 + XNY2 + XNY3 ≤100, XNY1≤YNY, XNY2≤YNY, and XNY3≤YNY.

Homework #8

*The computations in problems 2 & 3 may be done with the APL workspace named "MARKOV", if you wish. This would require that you obtain from me a copy of the APL*PLUS interpreter, which may then be used on one of the Mac Classics or Mac SE (not Mac II).*

1. Integer Programming Model. (Exercise 6, §9.2, page 470 of textbook by W. Winston) In order to graduate from Basketweavers University with a major in Operations Research, a student must complete at least two math courses, at least two operations research (O.R.) courses, and at least two computer science (C.S.) courses. Some courses can be used to fulfill more than one requirement:

Course	can fulfill	Requirements
Calculus		Math
Operations Research	Math & O.R.	
Data Structures		C.S. & Math
Business Statistics		Math & O.R.
Computer Simulation		O.R. & C.S.
Intro. to Computer Prog.		C.S.
Forecasting		O.R. & Math

Some courses are prerequisites for others:

Course	is prerequisite for	Course
Calculus		Business Statistics
Intro. to Computer Prog.		Computer Simulation
Intro. to Computer Prog.		Data Structures
Business Statistics		Forecasting

Formulate an integer LP model that minimizes the number of courses needed to satisfy the major requirements. Use LINDO to solve the problem.

2. Markov Chains. (Variation on Exercise 1, §19.3, page 918 of text by W. Winston) Each American family is classified as living in an urban, rural, or suburban location. During a given year, 15% of all urban families move to a suburban location and 5% move to a rural location; also, 6% of all suburban families move to an urban location, and 4% move to a rural location; finally, 4% of all rural families move to an urban location, and 6% to a suburban location.

a. Draw a transition diagram for a Markov chain model of a single American family.
b. If a family now lives in an urban location, what is the probability that it will live in an urban area ***three*** years from now? in a suburban area? a rural area?
c. Suppose that at present, 40% of all families live in an urban area, 35% live in a suburban area, and 25% live in a rural area. ***Three*** years from now, what percentage of American families will live in an urban area?
d. Write down the system of linear equations which should be solved in order to compute the steady-state probability distribution of the population.
e. Find the steady-state probability distribution.

3. Markov Chains. (Exercise 4, §19.5, page 927 of text by W. Winston) At the beginning of each year, my car is in *good*, *fair*, or *broken-down* condition. A *good* car will be *good* at the beginning of the next year with probability 85%, *fair* , with probability 10%, and *broken-down* with probability 5%. A *fair* car will be a *fair* car at the beginning of the next year with probability 70% and *broken-down* with probability 30%. It costs $6000 to purchase a *good* car; a *fair* car can be traded in for $2000, and a *broken-down* car has no trade-in value and must immediately be replaced by a *good* car. It costs $1000 per year to operate a *good* car and $1500 to operate a *fair* car. Assume that the cost of operating a car during a year depends on the type of car on hand at the beginning of the year (after a new car, if any, arrives.)

(a.) Should I replace my car as soon as it becomes a *fair* car, or should I drive my car until it breaks down? *Hint: Formulate two Markov chain models, one in which a fair car is replaced with a good car, and the other in which a fair car is kept until it breaks down. For each model, compute the steady-state distribution in order to compute the average cost/year. Assume that any break-downs occur immediately before the replacement decision is made at the beginning of the year.*

If at the beginning of 1992 my car is *good,* then under the replacement policy which you recommend in (a.), ...

(b.) ...what will be the expected time until the next replacement?
(c.) ...what is the probability that my car is in *good* condition at the beginning of 1995?
(d.) what is the expected time between break-downs?

Homework #8 Solution

1. *Integer Programming Model.* *(Exercise 6, §9.2, page 470 of textbook by W. Winston)*

Define the seven binary (zero/one) decision variables, one for each course:
Y1 = 1 if Calculus is taken, 0 otherwise
Y2 = 1 if O.R. is taken, 0 otherwise
Y3 = 1 if Data Structures is taken, 0 otherwise
Y4 = 1 if Business Statistics is taken, 0 otherwise
Y5 = 1 if Computer Simulation is taken, 0 otherwise
Y6 = 1 if Intro. to Computing is taken, 0 otherwise
Y7 = 1 if Forecasting is taken, 0 otherwise
Objective: Minimize Y1 + Y2 + Y3 + Y4 + Y5 + Y6 + Y7
Constraints: Y1 + Y2 + Y3 + Y4 + Y7 ≥ 2 (Math reqmt.)
Y2 + Y4 + Y5 + Y7 ≥ 2 (O.R. reqmt.)
Y3 + Y5 + Y6 ≥ 2 (Computer Science reqmt.)
Y4 ≤ Y1 (Calculus is prerequisite for Business Statistics)
Y5 ≤ Y6 (Intro. to Computing is prereq. for Com. Simulation)
Y3 ≤ Y6 (Intro. to Computing is prerequisite for Data Structures)
Y7 ≤ Y4 (Business Statistics is prerequisite for Forecasting)
(plus the integer constraints on Y1 through Y7).
The optimal solution, found by LINDO, is:

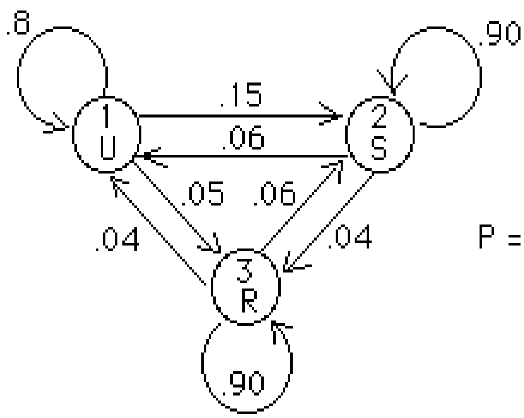
```
MIN     Y1 + Y2 + Y3 + Y4 + Y5 + Y6 + Y7
 SUBJECT TO
        2)   Y1 + Y2 + Y3 + Y4 + Y7 >=   2
        3)   Y2 + Y4 + Y5 + Y7 >=   2
        4)   Y3 + Y5 + Y6 >=   2
        5)   Y1 - Y6 >=   0
        6) - Y5 + Y6 >=   0
        7) - Y3 + Y6 >=   0
        8)   Y4 - Y7 >=   0
 END
 INTE     7


 NEW INTEGER SOLUTION OF   4.000000     AT BRANCH      0 PIVOT
7

        OBJECTIVE FUNCTION VALUE

        1)    4.00000000
```

VARIABLE	VALUE	REDUCED COST
Y1	1.000000	2.000000
Y2	1.000000	.000000
Y3	.000000	.000000
Y4	.000000	.000000
Y5	1.000000	.000000
Y6	1.000000	.000000
Y7	.000000	.000000

Note that the variables are restricted to be (binary) integer-valued by the command "INTEGER 7" (abbreviated INTE 7). However, it happens that the LP solution of the problem has only integer values, so that branch-&-bound was not required. The optimal set of courses is: Calculus, O.R., Computer SImulation, and Intro. to Computing.

2. Markov Chains. (Variation on Exercise 1, §19.3, page 918 of text by W. Winston)

a. Draw a transition diagram for a Markov chain model of a single American family. Define the states: (1) Urban, (2) Suburban, and (3) Rural. The transition diagram is:



$$P = \begin{bmatrix} 0.8 & 0.15 & 0.05 \\ 0.06 & 0.9 & 0.04 \\ 0.04 & 0.06 & 0.9 \end{bmatrix}$$

b. *If a family now lives in an urban location, what is the probability that it will live in an urban area* ***three*** *years from now? in a suburban area? a rural area?* This is denoted by $p_{11}^{(3)}$, the value in row 1 & column 1 of P^3. P^3 is

$$P^3 = \begin{bmatrix} 0.53992 & 0.33531 & 0.12477 \\ 0.135164 & 0.7593 & 0.105536 \\ 0.096696 & 0.162204 & 0.7411 \end{bmatrix}$$

and so

P{state 1 (urban) at stage 3 | state 1 (urban) at stage 0} = 0.53992.
P{state 2 (suburban) at stage 3 | state 1 (urban) at stage 0} = 0.33531
P{state 3 (rural) at stage 3 | state 1 (urban) at stage 0} = 0.12477

c. *Suppose that at present, 40% of all families live in an urban area, 35% live in a suburban area, and 25% live in a rural area.* ***Three*** *years from now, what percentage of American families will live in an urban area?* Let r = [0.40, 0.35, 0.25] be the initial distribution of the population. Then the product rP^3 will give the distribution of the population 3 years hence. This product is rP^3 = [0.2874494, 0.44043, 0.27212], i.e., 28.74% will reside in urban areas, 44.04% in suburban areas, and 27.21% in rural areas.

d. *Write down the system of linear equations which should be solved in order to compute the steady-state probability distribution of the population.*

$$\pi_1 = 0.8\pi_1 + 0.06\pi_2 + .04\pi_3,$$

$$\pi_2 = 0.15\pi_1 + 0.9\pi_2 + 0.06\pi_3$$

$$\pi_3 = 0.05\pi_1 + 0.04\pi_2 + 0.9\pi_3$$

$\pi_1 + \pi_2 + \pi_3 = 1$

The first set of 3 constraints is simply $\pi = \pi P$. One of these first three constraints is redundant... any one of these may be discarded, leaving us with 3 equations with 3 unknowns.

e. *Find the steady-state probability distribution.*

Other interesting conclusions could be drawn from the *Mean First Passage Times:*

$$M = \begin{bmatrix} 4.8157895 & 8.3333333 & 22.727273 \\ 18.421053 & 2.0333333 & 23.636364 \\ 21.052632 & 13.333333 & 3.3272727 \end{bmatrix}$$

For example, this shows that an "average" urban family will, after 8.333 years, move to a suburban area, and an "average" rural family will move to an urban area in 21.05 years.

3. Markov Chains. (Exercise 4, §19.5, page 927 of text by W. Winston)

(a.) *Should I replace my car as soon as it becomes a fair car, or should I drive my car until it breaks down? Hint: Formulate two Markov chain models, one in which a fair car is replaced with a good car, and the other in which a fair car is kept until it breaks down. For each model, compute the steady-state distribution in order to compute the average cost/year. Assume that any break-downs occur immediately before the replacement decision is made at the beginning of the year.*

The state of the system (i.e., my car) is the condition in which it is classified at the beginning of the year, <u>before</u> I do any replacement. These states are (1) "Good", (2) "Fair", and (3) "Broken-down". This model will assume that cars are replaced only at the beginning of each year.

Policy "A": replace the car only if it is in state (3), i.e., "Broken-down".

The transition probability matrix is:

$$P = \begin{bmatrix} 0.85 & 0.10 & 0.05 \\ 0 & 0.70 & 0.30 \\ 0.85 & 0.10 & 0.05 \end{bmatrix}$$

(Note that the last row is the same as the first; if the car is "broken-down" at the beginning of the year and is immediately replaced with a "good" car, the probability that it will still be "good" one year hence is only 85%.) The cost of each state is: (1) \$1000 for a "good" car, (2) \$1500 for a "fair" car, and (3) \$7000 (=\$6000 for replacement and \$1000 for operating costs) if the car is found to be "broken-down". If the steady-state probabilities are computed, then we can compute the expected cost/year, which is found to be \$1800:

<u>Auto replacement policy A</u>

i	State	π_i	C	$\pi \times C$
1	Good	0.6375	1000	637.5
2	Fair	0.25	1500	375
3	Broken-down	0.1125	7000	787.5

The average cost/period in steady state is 1800

Policy "B": Replace a car at the beginning of the year if it is in "fair" condition. The transition probabilities are now

$$P = \begin{bmatrix} 0.85 & 0.10 & 0.05 \\ 0.85 & 0.10 & 0.05 \\ 0.85 & 0.10 & 0.05 \end{bmatrix}$$

Obviously, the steady-state probabilities are p = [0.85, 0.10, 0.05]. The cost of state (2) is now \$5000 (\$6000 for replacement car minus \$2000 trade-in value, plus \$1000 operating cost for the replacement.) The average cost per year is now \$1700:

Auto replacement policy B

i	State	π_i	C	$\pi \times C$
1	Good	0.85	1000	850
2	Fair	0.1	5000	500
3	Broken-down	0.05	7000	350

The average cost/period in steady state is 1700

The optimal policy, then, is to **replace a car whenever it is found to be in "fair" condition** at the beginning of the year.

If at the beginning of 1992 my car is good, then under the replacement policy which you recommend in (a.), ...

(b.) *...what will be the expected time until the next replacement?*

Auto replacement policy B
Mean First Passage Times

from \ to	1	2	3
1	1.1764706	10	20
2	1.1764706	10	20
3	1.1764706	10	20

The probability that a car is replaced, with this policy, is 15%, and so the expected number of years between replacements is 1/0.15 = 6.667. For this particular policy (B), the car is always in good condition at the beginning of the year (after any replacement if necessary), we can conclude that, independent of the beginning state, the expected time until the next replacement is 6.667 years.

(c.) *...what is the probability that my car is in good condition at the beginning of 1995?*

This will be given by the value in row 1, column 1, of P^3. For this particular matrix P (in which all rows are identical), $P=P^3$, and the probability that the car is in good condition in any year is 85%, independent of the initial condition of the car.

(d.) *what is the expected time between break-downs?*

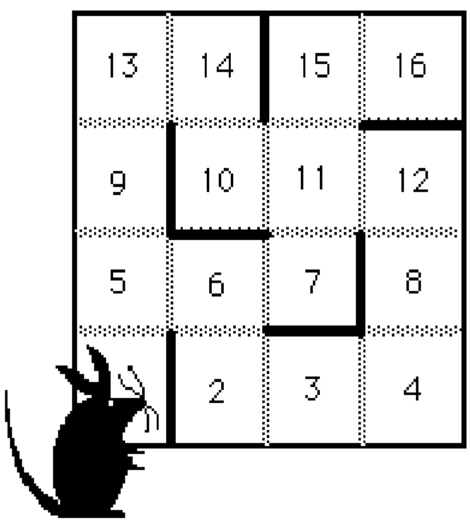
The expected time between visits to state 3, under policy B, is m_{33} = 20 years.

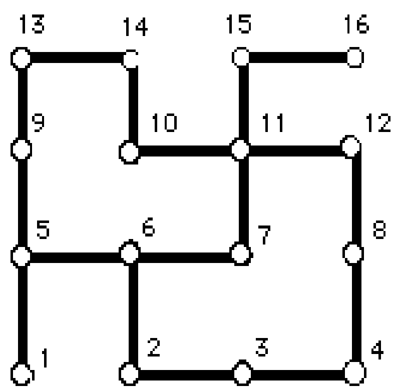
Homework #9

(1.) A machine has two critical parts that are subject to failure. The machine can continue to operate if one part has failed. Only in the case where *both* parts are no longer intact does a repair need to be done. A repair takes exactly one day, and after a repair both parts are intact again. At the beginning of each day, the machine is examined to determine whether or not a repair is required. If at the beginning of a day a part is intact, then it will fail during the day with probability 0.25. Each repair costs $50. For each day the machine is running, it generates $100 in profit. (For the sake of simplicity, assume that all failures occur very late in the day, so that if the machine is operating at the beginning of the day, it will generate the full $100 in profit, and repairs will not begin until the following morning.)

a. Define a Markov chain model of this process. What are the states? What are the transition probabilities? Draw a diagram of your model.
b. Write the system of linear equations which must be solved to compute the steady-state distribution.
c. Find the steady-state distribution for your Markov chain.
d. Compute the average profit per day for this machine.
e. What is the expected number of days which the machine will run after a repair is completed?

(2.) We wish to model the passage of a rat through a maze. Sketch a maze in the form of a 4x4 array of boxes, such as the one below on the left:





The solid lines represent walls, the shaded lines represent doors. We will assume that a rat is placed into box #1. While in any box, the rat is assumed to be equally likely to choose each of the doors leaving the box (including the one by which he entered the box). For example, when in box #2 above, the probability of going next to boxes 3 and 6 are each $^1/_2$, regardless of the door by which he entered the box. *This assumption implies that no learning takes place if the rat tries the maze several times.*

Once you have sketched a 4x4 maze (**not** identical to the one above!), load the **APL** workspace named **RAT** and enter the maze which you have designed. (Be sure that your maze is <u>connected</u>, i.e. each box is reachable from any other box.) The functions in this "RAT" workspace construct a Markov chain model of the movement of the rat through the maze.

a. On the diagram representing your Markov chain, write the transition probabilities on each transition in each direction.

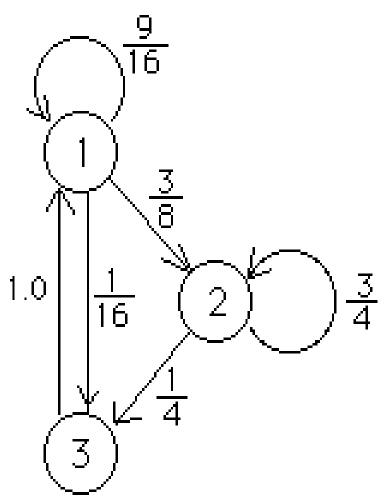
b. Is your Markov chain ergodic? Is it regular?
c. Compute the steady-state distribution of the rat's location. Which box will be visited most frequently by the rat?
d. Select some box far from box #1 in which a reward (e.g. food) might be placed for the rat. What is the expected number of moves of the rat required to reach this reward?
e. Count the minimum number of moves (M) required to reach the reward. What is the probability that the rat reaches the reward in **exactly** this number of moves?
f. What is the probability that the rat reaches the reward with no more than 4 unnecessary moves?
g. Simulate twice the first 2M moves made by the rat. Did he reach the reward in either simulation?
h. Briefly discuss the utility of this model in testing a hypothesis that a *real* rat is able to learn, thereby finding the reward in relatively few moves after he has made several trial runs through the maze.
i. Briefly discuss how you might modify your Markov chain model if the rat will never exit a box by the same door through which he entered, unless he has reached a "dead end".

Homework #9 Solution

(1.)

a. *Define a Markov chain model of this process. What are the states? What are the transition probabilities? Draw a diagram of your model.*

There are three states for this system:
(1) Both parts intact
(2) One part intact
(3) Both parts failed



$$P = \begin{bmatrix} \frac{9}{16} & \frac{3}{8} & \frac{1}{16} \\ 0 & \frac{3}{4} & \frac{1}{4} \\ 1 & 0 & 0 \end{bmatrix}$$

The transition probabilities are found as follows:
p_{12} = P{one part of two fails} = 2(0.25) (0.75)
p_{13} = P{two parts fail} = (0.25)(0.25)
$p_{11} = 1 - p_{12} - p_{13}$
etc.

Note that if the morning inspection finds that both parts have failed, that day is spent in repairing the machine, so that at the beginning of the next day it will be restored to its original condition, so that $p_{31} = 1.0$

b. *Write the system of linear equations which must be solved to compute the steady-state distribution.*

$$\pi = \pi \begin{bmatrix} \frac{9}{16} & \frac{3}{8} & \frac{1}{16} \\ 0 & \frac{3}{4} & \frac{1}{4} \\ 1 & 0 & 0 \end{bmatrix}$$

and $\sum_{i=1}^{3} \pi_i = 1$

That is,

$$\begin{cases} \pi_1 = \frac{9}{16}\pi_1 + \pi_3 \\ \pi_2 = \frac{3}{8}\pi_1 + \frac{3}{4}\pi_2 \\ \pi_3 = \frac{1}{16}\pi_1 + \frac{1}{4}\pi_2 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

(There is one redundant equation among the first three, which may be discarded.)

c. *Find the steady-state distribution for your Markov chain.*

The solution of the above equations is $\pi_1 = 0.34042553$, $\pi_2 = 0.5106383$, $\pi_3 = 0.14893617$.

d. *Compute the average profit per day for this machine.*

The machine will generate \$100/day profits when in states 1 or 2, and will incur a cost of \$50 when in state 3. The average profit/day is therefore $100\pi_1 + 100\pi_2 - 50\pi_3 = \$77.66$

i	π_i	C	$\pi_i \times C_i$
1	0.34042553	100	34.042553
2	0.5106383	100	51.06383
3	0.14893617	¯50	¯7.4468085

The average cost/period in steady state is 77.659574

e. *What is the expected number of days which the machine will run after a repair is completed?*

The expected number of days which the machine will run after a repair is completed is the "mean first passage time", $m_{13} = 5.71$ days. (The table below was computed using the MARKOV workspace.) This value can also be found by solving the system of equations:

$$m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj} \qquad i,j$$

for j fixed at 3 and i=1,2,&3. This will be 3 equations in 3 unknowns:

$$m_{13} = 1 + p_{11}m_{13} + p_{12}m_{23}$$
$$m_{23} = 1 + p_{21}m_{13} + p_{22}m_{23}$$
$$m_{33} = 1 + p_{31}m_{13} + p_{32}m_{23}$$

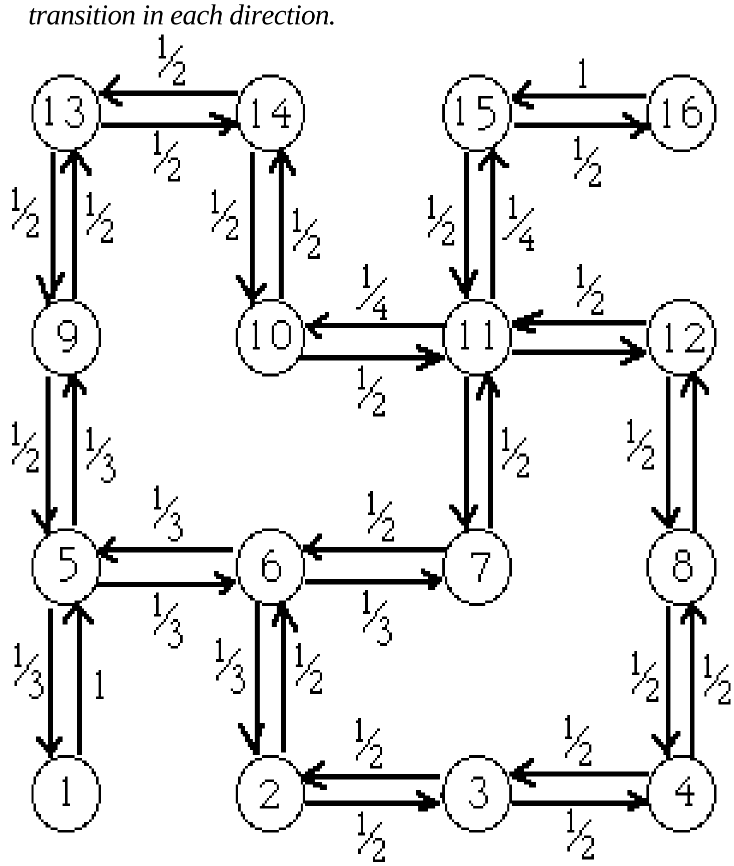
Mean First Passage Times

from \ to	1	2	3
1	2.9375	2.8333333	5.7142857
2	5	1.9583333	4
3	1	3.8333333	6.7142857

(2.) The solution below is for the maze which is shown above. The matrix of transition probabilities is:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0.5	0	0	0.5	0	0	0	0	0	0	0	0	0	0
3	0	0.5	0	0.5	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0.5	0	0	0	0	0.5	0	0	0	0	0	0	0	0
5	0.333	0	0	0	0	0.333	0	0	0.333	0	0	0	0	0	0	0
6	0	0.333	0	0	0.333	0	0.333	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0.5	0	0	0	0	0.5	0	0	0	0	0
8	0	0	0	0.5	0	0	0	0	0	0	0	0.5	0	0	0	0
9	0	0	0	0	0.5	0	0	0	0	0	0	0	0.5	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0.5	0	0	0.5	0	0
11	0	0	0	0	0	0	0.25	0	0	0.25	0	0.25	0	0	0.25	0
12	0	0	0	0	0	0	0	0.5	0	0	0.5	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0.5	0	0	0	0	0.5	0	0
14	0	0	0	0	0	0	0	0	0	0.5	0	0	0.5	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0.5	0	0	0	0	0.5
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0

a. *On the diagram representing your Markov chain, write the transition probabilities on each transition in each direction.*



(Note that the assumptions imply that the mouse is as equally likely to exit a box by the door which he entered as any of the other exiting doors.)

b. *Is your Markov chain ergodic? Is it regular?* The Markov chain is both regular and ergodic.

c. *Compute the steady-state distribution of the rat's location. Which box will be visited most frequently by the rat?* The steadystate probability distribution exists because the chain is regular, and is:

Steady State Distribution

i	P{i}
1	0.0294
2	0.0588
3	0.0588
4	0.0588
5	0.0882
6	0.0882
7	0.0588
8	0.0588
9	0.0588
10	0.0588
11	0.118
12	0.0588
13	0.0588
14	0.0588
15	0.0588
16	0.0294

Box #11 is most likely to be visited by the rat.

d. *Select some box far from box #1 in which a reward (e.g. food) might be placed for the rat. What is the expected number of moves of the rat required to reach this reward?* Suppose that the reward is placed in box #16. The mean first passage time matrix (M) is

M	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	34	31	43.6	49.1	1	11.3	23.1	47.6	19.6	33	23.3	39	31.1	35.6	54.3	87.3
2	59.7	17	18.3	29.5	26.7	8.67	20.5	33.6	39.6	36	20.7	30.7	45.5	44.3	51.7	84.7
3	65.2	11.2	17	15.7	32.2	15.3	23.7	24.4	44	37	20.5	26	48.7	46.4	51.5	84.5
4	68.7	20.4	13.7	17	35.7	20	25	13.2	46.4	36	18.4	19.3	50	46.5	49.4	82.4
5	33	30	42.6	48.1	11.3	10.3	22.1	46.6	18.6	32	22.3	38	30.1	34.6	53.3	86.3
6	52.1	20.8	34.5	41.2	19.1	11.3	15.2	40.8	33.2	33	18.8	33.3	40.2	40.1	49.8	82.8
7	60.7	29.4	39.7	43	27.7	12	17	39.2	38.4	28	10.4	28.3	42	38.5	41.4	74.4
8	70.3	27.6	25.5	16.3	37.3	22.7	24.3	17	46.8	33	14.3	10.7	49.3	44.7	45.3	78.3
9	43.9	35.2	46.7	51.1	10.9	16.7	25.1	48.4	17	27	21.9	38.7	16.1	25.1	52.9	85.9
10	64.5	38.8	46.9	47.9	31.5	23.7	21.9	41.8	34.2	17	8.47	28.7	29.9	18.5	39.5	72.5
11	67.3	36	42.9	42.8	34.3	22	16.8	35.6	41.6	21	8.5	21.3	41.8	34.9	31	64
12	69.8	32.8	35.2	30.5	36.8	23.3	21.5	18.8	45.2	28	8.13	17	46.5	40.8	39.1	72.1
13	52.7	38.4	48.7	52	19.7	21	26	48.2	13.4	20	19.4	37.3	17	13.5	50.4	83.4
14	59.6	39.6	48.8	50.9	26.6	23.3	24.9	46	24.8	11	14.9	34	15.9	17	45.9	78.9
15	70.3	39	45.9	45.8	37.3	25	19.8	38.6	44.6	24	3	24.3	44.8	37.9	17	33
16	71.3	40	46.9	46.8	38.3	26	20.8	39.6	45.6	25	4	25.3	45.8	38.9	1	34

The expected number of moves before reaching box #16, starting from box #1, is therefore

$m_{1,16} = 87.3$

e. *Count the minimum number of moves (M) required to reach the reward. What is the probability that the rat reaches the reward in* ***exactly*** *this number of moves?* The shortest path from box 1 to box 16 is: 1->5->6->7->11->15->16, or 6 moves. The matrix P^6 is

6-th Power

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0.151	0	0.0802	0	0	0.276	0	0.0208	0.243	0	0.101	0	0	0.12	0	0.00694
2	0	0.22	0	0.206	0.198	0	0.125	0	0	0.0448	0	0.111	0.0523	0	0.0431	0
3	0.0401	0	0.279	0	0	0.22	0	0.227	0.0471	0	0.154	0	0	0.02	0	0.013
4	0	0.206	0	0.299	0.0758	0	0.0992	0	0	0.0521	0	0.193	0.0148	0	0.0599	0
5	0	0.132	0	0.0505	0.365	0	0.117	0	0	0.0855	0	0.0357	0.182	0	0.0322	0
6	0.0922	0	0.147	0	0	0.256	0	0.0903	0.137	0	0.169	0	0	0.0772	0	0.0301
7	0	0.125	0	0.0992	0.176	0	0.141	0	0	0.113	0	0.116	0.0879	0	0.142	0
8	0.0104	0	0.227	0	0	0.135	0	0.266	0.0182	0	0.25	0	0	0.0469	0	0.0469
9	0.122	0	0.0471	0	0	0.206	0	0.0182	0.256	0	0.136	0	0	0.203	0	0.0113
10	0	0.0448	0	0.0521	0.128	0	0.113	0	0	0.198	0	0.12	0.186	0	0.159	0
11	0.0253	0	0.077	0	0	0.127	0	0.125	0.0678	0	0.343	0	0	0.123	0	0.112
12	0	0.111	0	0.193	0.0535	0	0.116	0	0	0.12	0	0.198	0.0503	0	0.159	0
13	0	0.0523	0	0.0148	0.273	0	0.0879	0	0	0.186	0	0.0503	0.278	0	0.0582	0
14	0.0602	0	0.02	0	0	0.116	0	0.0469	0.203	0	0.247	0	0	0.26	0	0.0469
15	0	0.0431	0	0.0599	0.0483	0	0.142	0	0	0.159	0	0.159	0.0582	0	0.331	0
16	0.00694	0	0.026	0	0	0.0903	0	0.0938	0.0226	0	0.448	0	0	0.0938	0	0.219

The probability that the rat reaches box 16 in <u>exactly</u> 6 moves is 0.00694.

*f. What is the probability that the rat reaches the reward with no more than 4 unneces*sary moves? The first passage probabilities for the first 10 stages, from box 1 to box 16, are

First Visit Probabilities
to State 16
from State 1

$f^{(n)}_{1,16}$

n	P
1	0
2	0
3	0
4	0
5	0
6	0.00694
7	0
8	0.0126
9	0
10	0.0165

The probability that the rat reaches box 16 within 10 moves is therefore 0.00694 +0.0126 +0.0165 = 0.03604, i.e., slightly greater than 3%.

g. *Simulate twice the first 2M moves made by the rat. Did he reach the reward in either simulation?* For this maze, M=6. The results of twice simulating the first **30** (=5M) moves of the rat are

```
Simulation results
```

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1	5	1	5	6	2	3	2	3	4	3	2	6	7	6	7	11	10	11	7	6	5	1	5	1	5	6	5	9	13	9
1	5	9	5	9	5	9	5	9	5	1	5	9	5	6	5	6	2	6	7	11	10	14	13	14	10	14	10	14	13	9

```
The array RUN has now been globally defined in the workspace.
Each row of the array represents a repetition of the simulation.
Note: Column 1 represents stage 0, i.e. the initial state.
```

In neither simulation did the rat reach box 16.

h. *Briefly discuss the utility of this model in testing a hypothesis that a real rat is able to learn, thereby finding the reward in relatively few moves after he has made several trial runs through the maze.* Suppose that after several tries, the rat is able to reach the food in K moves. The probability that the rat reaches the food within K moves under the "null hypothesis" can be computed by using the Markov model, i.e., $\sum_{n=1}^{K} f_{1,16}^{(n)}$. If this is sufficiently small, the null hypothesis might be rejected.

i. *Briefly discuss how you might modify your Markov chain model if the rat will never exit a box by the same door through which he entered, unless he has reached a "dead end".* The present model cannot easily be modified to handle this, because of the "memoryless" property of Markov chains. In order to model this behavior, the state information must contain not only which box the rat is in, but also by which door he entered. Thus, there would be 3 states corresponding to box #5: (5,N), (5,S), and (5,E) representing the rat's entering by the <u>N</u>orth, <u>S</u>outh, and <u>E</u>ast door, respectively. Then the probability of going from (5,S) to (1,N) can be assigned the value zero. The resulting Markov chain will have considerable more than 16 states. (Perhaps around 35 or 40 states?)

Homework #10

(1.) The Green Valley Christmas Tree Farm owns a plot of land with 5000 evergreen trees. Each year they allow individuals to select and cut their own Christmas trees. However, they protect small trees (usually less than 4 feet tall) so that they will grow and be available for sale in future years. Currently 1500 trees are classified as protected trees, while the remaining 3500 are available for cutting. However, even though a tree is available for cutting in a given year, it possibly might not be selected for cutting until future years. While most trees not cut in a given year live until the next year(protected or unprotected), approximately 15% are lost to disease. Each year, approximately 60% of the unprotected trees are cut, and 30% of the protected trees surviving from the previous year have matured sufficiently to be made available for cutting.

- (a.) Define a (discrete-time) Markov chain model of the system consisting of a *single* tree. Sketch the transition diagram and write down the probability matrix.
- (b.) What are the absorbing states of this model?
- (c.) What is the probability that a tree which is now protected is eventually sold? ...that it eventually dies of disease?
- (d.) How many of the farm's 5000 trees are expected to be sold eventually, and how many will be lost to disease?
- (e.) If a tree is now protected, what is the expected number of years until it is either sold or dies?

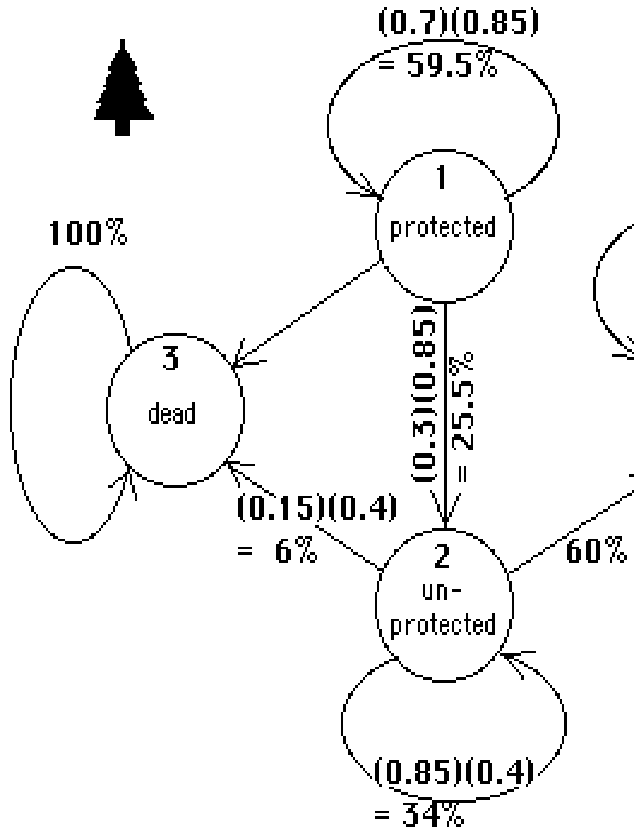
(2.) A certain machine has occasional breakdowns. After the first breakdown, it can be repaired at a cost of $1000. However, such a repair can only be done once, and consequently, the machine has to be replaced by a new one after the second breakdown. The replacement cost of the machine is $5000. The time until the first breakdown follows an exponential distribution with an expected value of 5 years, and the time between the first breakdown (repair) and the second breakdown (replacement) is also exponentially distributed, but with an expected value of 4 years. The time to perform the repairs is negligible.

- (a.) Formulate the problem as a continuous-time Markov process, and draw its transition diagram. (Only 2 states are necessary. Let state 1 represent the condition before the first repair is made, and state 2 the condition after the first repair but before the second breakdown.)
- (b.) Write the balance equations and find the steady-state distribution of the state of the system, π_1 and π_2.
- (c.) We want to find the *average cost per year of repairs and replacements.* To do this, first find:
  - (i.) (Conditional) rate at which repairs are made <u>when in state 1</u>
  - (ii.) Rate at which repairs are made (value from (i) times π_1)
  - (iii.) (Conditional) rate at which replacements are made <u>when in state 2</u>
  - (iv.) Rate at which replacements are made (value from (iii) times π_2)
  - (v.) Average cost per year (sum of the rates of repairs (ii) and replacements (iv) times the appropriate costs).

Homework #10 Solution

(1.) Discrete-Time Markov Chains. Define a Markov chain in which the state of the tree is observed each year immediately ***before*** the Christmas season begins.

a. The four states of the system, and transition probabilities, are indicated below:



Transition Probability Matrix

from \ to	1	2	3	4
1	.595	.255	.15	0
2	0	.34	.06	.6
3	0	0	1	0
4	0	0	0	1

b. The absorbing states are #3: "dead" and #4: "cut ".

Using the MARKOV workspace, and selecting "Absorption Analysis" on the menu produces the following output:

Analysis of Markov Chain with Absorbing States

Enter list of absorbing states
⎕:
3 4

E = Expected No. Visits to Transient States

from \ to	1	2
1	2.6666667	0.90909091
2	0	1.5151515

A = Absorption Probabilities

from \ to	3	4
1	0.45454545	0.54545455
2	0.090909091	0.90909091

This computation could also be done manually as follows:

$$Q = \begin{bmatrix} .595 & .255 \\ 0 & .34 \end{bmatrix},\ E=(I-Q)^{-1} = \begin{bmatrix} .405 & -.255 \\ 0 & .66 \end{bmatrix}^{-1} = \begin{bmatrix} 2.66667 & .90909 \\ 0 & 1.51515 \end{bmatrix}.$$

$$A = ER = \begin{bmatrix} 2.66667 & .90909 \\ 0 & 1.51515 \end{bmatrix} \begin{bmatrix} .15 & 0 \\ .06 & .6 \end{bmatrix} = \begin{bmatrix} 4/9 & 5/9 \\ 1/9 & 8/9 \end{bmatrix}$$

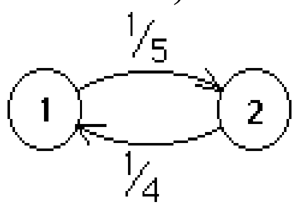
c. According to the A matrix above, a protected tree has probability 54.54% (=a_{14}) of eventually being cut & sold, and probability 45.45% (=a_{13}) of eventually being lost to disease.

d. The number which is expected to be eventually sold is 1500x0.4545 + 3500x0..0909 = 3182. The remaining 1818 are expected to be lost to disease.

e. If the tree is initially protected, the expected number of visits to the transient state #1 is e_{11}=2.66667 and to transient state #3 is e_{12}=0.90909. Note, however, that e11 counts the initial visit to state #1, so that the number of additional visits to this state is 1.66667 (the expected number of years that it will be protected), i.e., the expected lifetime of the tree will be 1.6667 + 0.90909 = 2.5757 years.

(2.) **Continuous-Time Markov Chains.**

(a.) *Formulate the problem as a continuous-time Markov process, and draw its transition diagram. (Only 2 states are necessary. Let state 1 represent the condition before the first repair is made, and state 2 the condition after the first repair but before the second breakdown.)*



(b.) *Write the balance equations and find the steady-state distribution of the state of the system, π_1 and π_2.* The balance equation for this C-T Markov Chain is

$$\frac{1}{5}\pi_1 = \frac{1}{4}\pi_2$$

This equation, together with the restriction on the sum of the probabilities, gives us a system of 2 equations with 2 unknowns:

$$\begin{cases} \frac{1}{5}\pi_1 = \frac{1}{4}\pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

The optimal solution is

$$\begin{cases} \pi_1 = 5/9 \\ \pi_2 = 4/9 \end{cases}$$

(c.) We want to find the *average cost per year of repairs and replacements.* To do this, first find:

(i.) (Conditional) rate at which repairs are made <u>when in state 1</u> = $^1/_5$

(ii.) Rate at which repairs are made (value from (i) times π_1) = $(^1/_5)\pi_1 = ^1/_9$

(iii.) (Conditional) rate at which replacements are made <u>when in state 2</u> = $^1/_4$

(iv.) Rate at which replacements are made (value from (iii) times π_2) = $(^1/_4)\pi_2 = ^1/_9$

(v.) Average cost per year (sum of the rates of repairs (ii) and replacements (iv) times the appropriate costs). $(^1/_9)$($1000)+$(^1/_9)$($5000) = $666.67/year

Homework #11

(1.) A neighborhood grocery store has only one check-out counter. Customers arrive at the check-out at a rate of one per 2 minutes. The grocery store clerk requires an average of one minute and 30 seconds to serve each customer. However, as soon as the waiting line exceeds 2 customers, including the customer being served, the manager helps by packing the groceries, which reduces the average service time to one minute. Assume a Poisson arrival process and exponentially-distributed service times.

- (a.) Draw the flow diagram for a birth-death model of this system.
- (b.) Either manually or using the **Birth/Death** workspace, compute the steady-state distribution of the number of customers at the check-out.
- (c.) What fraction of the time will the check-out clerk be idle?
- (d.) What is the expected number of customers in the check-out area?
- (e.) What is the expected length of time that a customer spends in the check-out area?
- (f.) Suppose that the store is being remodeled, and space is being planned so that the waiting line does not overflow the space allocated to it more than 1 percent of the time, and that 4 feet must be allocated per customer (with cart). How much space should be allocated for the waiting line?
- (g.) What fraction of the time will the manager spend at the check-out area?

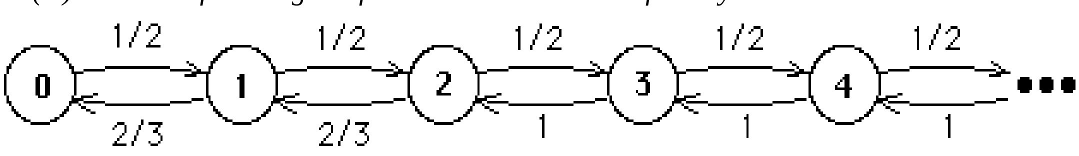
(2.) A local takeout Chinese restaurant has space to accommodate at most five customers. During the frigid Iowa winter, it is noticed that when customers arrive and the restaurant is full, virtually no one waits outside in the subfreezing weather, but goes next door to Luigi's Pizza Palace. Customers arrive at the restaurant at the average rate of 15 per hour, according to a Poisson process. The restaurant serves customers one at a time, first-come, first-served, in an average of 4 minutes each (the actual time being exponentially distributed.)

- (a.) What is the steady-state distribution of the number of customers in the restaurant?
- (b.) What is the average number of customers in the Chinese restaurant at any time?
- (c.) What is the average arrival rate, considering that when there are 5 customers in the restaurant, the arrival rate is zero?
- (d.) According to Little's Law, what is the expected amount of time that a customer spends in the restaurant?
- (e.) What is the fraction of potential customers who are lost to the pizza establishment?

Homework #11 Solution

(1.)

(a.) Draw the flow diagram for a birth-death model of this system.



The "birth" rate is 1/2 per minute for all states, while the "death" rate is 2/3 per minute in states 1 and 2, and 1/minute for higher-numbered states.

*(b.) Either manually or using the **Birth/Death** workspace, compute the steady-state distribution of the number of customers at the check-out.*

Manual computation:

$$\frac{1}{\pi_0} = 1 + \frac{1/2}{2/3} + \left(\frac{1/2}{2/3}\right)^2 + \left(\frac{1/2}{2/3}\right)^2\left(\frac{1/2}{1}\right) + \left(\frac{1/2}{2/3}\right)^2\left(\frac{1/2}{1}\right)^2 + \left(\frac{1/2}{2/3}\right)^2\left(\frac{1/2}{1}\right)^3 + \ldots$$

$$\frac{1}{\pi_0} = 1 + \frac{1/2}{2/3} + \left(\frac{1/2}{2/3}\right)^2 \left\{1 + \left(\frac{1/2}{1}\right) + \left(\frac{1/2}{1}\right)^2 + \left(\frac{1/2}{1}\right)^3 + \cdots \right\}$$

The infinite series within the braces is a geometric series with sum

$$\frac{1}{1-1/2} = 2$$

and so

$$\frac{1}{\pi_0} = 1 + \frac{1/2}{2/3} + \left(\frac{1/2}{2/3}\right)^2 \{2\} = 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 \times 2 = 1 + 0.75 + 2(0.5625) = 2.875$$

Therefore, $\pi_0 = 1/2.875 = 0.347826$ and $\pi_2 = 0.75\pi_0$, $\pi_2 = 0.5625\pi_0$, etc.

Use of Birth/Death workspace:

Entering the birth & death rates, and asking for computation of the steadystate probabilities up to π_{10}, we get:

Steady-State Distribution

i	Lambda	Mu	ρ	Pi	CDF
0	0.5000	0.0000	1.000000	0.347826	0.347826
1	0.5000	0.6667	0.750000	0.260870	0.608696
2	0.5000	0.6667	0.750000	0.195652	0.804348
3	0.5000	1.0000	0.500000	0.097826	0.902174
4	0.5000	1.0000	0.500000	0.048913	0.951087
5	0.5000	1.0000	0.500000	0.024457	0.975543
6	0.5000	1.0000	0.500000	0.012228	0.987772
7	0.5000	1.0000	0.500000	0.006114	0.993886
8	0.5000	1.0000	0.500000	0.003057	0.996943
9	0.5000	1.0000	0.500000	0.001529	0.998471
10	0.5000	1.0000	0.500000	0.000764	0.999236

The computation of π_0 is done by truncating the series after the 11th term, so that the probabilities above are approximations. However, from the CDF (cumulative distribution function), we see that p{# of customers ≤ 10} > 99.9%, so that the approximated probabilities should be very near to the actual probabilities.

(c.) What fraction of the time will the check-out clerk be idle?

π_0 = 34.78% of the time, the check-out clerk will be idle.

(d.) What is the expected number of customers in the check-out area?

$$L = \sum_{i=0}^{\infty} i\pi_i = 0 + 0.608696 + 2\times 0.804348 + 3\times 0.902174 + \cdots$$

According to the Birth/Death workspace, we get L=1.4267 customers (including the one being served, if any).

The Mean Population Size is 1.4267

(e.) What is the expected length of time that a customer spends in the check-out area?

To compute the average time in the system, W, we use Little's Law: $L = \underline{\lambda}W$, where $\underline{\lambda}$ is the average arrival rate, in this case 1/2. Therefore, $W = L/\underline{\lambda} = 1.4267/(0.5/\text{minute}) = 2.8534$ minutes.

(f.) Suppose that the store is being remodeled, and space is being planned so that the waiting line does not overflow the space allocated to it more than 1 percent of the time, and that 4

feet must be allocated per customer (with cart). How much space should be allocated for the waiting line?

By examining the cumulative probabilities (CDF) in the table above, we see that P{# in system ≤ 7} = 0.993886, i.e., P{# in system exceeds 7} = 0.6114%. Therefore, if we allocate enough space for 7 customers (6 waiting plus one being served), i.e., 24 feet plus space for the customer being served, the customers will overflow the space less than 1% of the time.

(g.) What fraction of the time will the manager spend at the check-out area?

$1-\pi_0 - \pi_1 - \pi_2 = 1 - 0.804348 = 19.5652\%$.

(2.)

(a.) What is the steady-state distribution of the number of customers in the restaurant?

This is a M/M/1/5 queue, with arrival rate λ = service rate μ = 15/hour. For this special case, $\pi_i = 1/6$ for each i=0,1,2,3,4,5.

(b.) What is the average number of customers in the Chinese restaurant at any time?

$L = 0\pi_0 + 1\pi_1 + 2\pi_2 + 3\pi_3 + 4\pi_4 + 5\pi_5 = (1/6)(0+1+2+3+4+5)= 15/6 = 2.5$, i.e., half-full.

(c.) What is the average arrival rate, considering that when there are 5 customers in the restaurant, the arrival rate is zero?

The average arrival rate is

$\underline{\lambda} = 15\pi_0 + 15\pi_1 + 15\pi_2 + 15\pi_3 + 15\pi_4 + 0\pi_5 = 15(1-\pi_5) = 15(5/6) = 12.5$/hour.

(d.) According to Little's Law, what is the expected amount of time that a customer spends in the restaurant?

$W = L/\underline{\lambda} = 2.5/12.5 = 1/5$ hour, i.e., 12 minutes.

(e.) What is the fraction of potential customers who are lost to the pizza establishment?

The fraction of the time that potential customers are turned away is $\pi_5 = 1/6$, so that one-sixth of the customers will be lost.