56:171 O.R.
HW - Fall '98

❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ Homework #1 ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍

1. *(Exercise 7, page 91, O.R., W. Winston)* Eli Daisy uses chemicals 1 and 2 to produce two drugs. Drug A must be at least 70% chemical 1 and drug B must be at least 60% chemical 2. Up to 40 oz. of drug A can be sold at $6 per oz; up to 30 oz. of drug B can be sold at $5 per oz. Up to 45 oz. of chemical 1 can be purchased at $6 per oz. and up to 40 oz. of chemical 2 can be purchased at $4 per oz. Formulate and solve (using LINDO) an LP to maximize Daisy's profits.

Solution:

Define variables:

A = ounces of drug A to be produced
B = ounces of drug B to be produced
C1 = ounces of chemical 1 purchased (& used)
C2 = ounces of chemical 2 purchased (& used)
X1A = ounces of chemical **1** used to produce drug **A**
X2A = ounces of chemical **2** used to produce drug **A**
X1B = ounces of chemical **1** used to produce drug **B**
X2B = ounces of chemical **2** used to produce drug **B**

Objective:

Maximize 6A + 5B - 6C1 - 4C2

Constraints:

Drug A is composed entirely of chemicals 1 & 2: A = X1A + X2A
Drug B is composed entirely of chemicals 1 & 2: B = X1B + X2B
Usage of chemical 1 is limited by the amount purchased: X1A + X1B ≤ C1
Usage of chemical 2 is limited by the amount purchased: X2A + X2B ≤ C2
Chemical 1 must be at least 70% of drug A: X1A ≥ 0.7A
Chemical 2 must be at least 60% of drug B: X2B ≥ 0.6B

Upper Bound Constraints:

A maximum of 40 ounces of drug A can be produced: A ≤ 40
A maximum of 40 ounces of drug B can be produced: B ≤ 30
A maximum of 45 ounces of chemical 1 can be purchased: C1 ≤ 45
A maximum of 40 ounces of chemical 2 can be purchased: C2 ≤ 40

All variables are restricted to be nonnegative.

Note 1: The variables A, B, C1, and C2 could have been eliminated from the model! Doing so would limit the sensitivity analysis which is possible, however. (More on sensitivity analysis later.)

Note 2: The four upper bounds (on A, B, C1, and C2) will be handled by the "simple upper bound" command (SUB) of LINDO, , instead of by "regular" constraints.

LINDO output:

```
 MAX     6 A + 5 B - 6 C1 - 4 C2
  SUBJECT TO
        2)   A - X1A - X2A =    0
        3)   B - X1B - X2B =    0
        4) - C1 + X1A + X1B <=   0
        5) - C2 + X2A + X2B <=   0
        6) - 0.7 A + X1A >=   0
        7) - 0.6 B + X2B >=   0
  END
  SUB        A       40.00000
  SUB        B       30.00000
  SUB       C1       45.00000
  SUB       C2       40.00000

: GO
 LP OPTIMUM FOUND AT STEP      4

       OBJECTIVE FUNCTION VALUE
       1)      52.00000

  VARIABLE        VALUE          REDUCED COST
         A        40.000000         -0.300000
         B        28.000000          0.000000
        C1        28.000000          0.000000
        C2        40.000000         -1.000000
       X1A        28.000000          0.000000
       X2A        12.000000          0.000000
       X1B         0.000000          1.000000
       X2B        28.000000          0.000000

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000          5.000000
        3)         0.000000          5.000000
        4)         0.000000          6.000000
        5)         0.000000          5.000000
        6)         0.000000         -1.000000
        7)        11.200000          0.000000

 NO. ITERATIONS=       4
```

Description of solution:
purchase 28 ounces of chemical 1 and 40 ounces of chemical 2.
40 ounces of drug A are to be produced, by blending 28 ounces of chemical 1 and 12 ounces of chemical 2.
28 ounces of drug B are to be produced, consisting entirely of chemical 2.
The resulting profit is $52.

2. *(Exercise 5, page 104, O.R., W. Winston)* During the next two months, General Cars must meet (on time) the following demands for trucks and cars: Month 1: 400 trucks, 800 cars; Month 2: 300 trucks, 300 cars. During each month, at most 1000 vehicles can be produced. Each truck uses 2 tons of steel, and each car uses 1 ton of steel. During month 1, steel costs $400 per ton; during month 2, steel costs $600 per ton. At most 1500 tons of steel may be purchased each month (steel may only be used during the month in which it is purchased). At the beginning of month 1, 100 trucks and 200 cars are in inventory. At the end of each month, a holding cost of \$150 per vehicle is assessed. Each car gets 20 mpg (miles per gallon), and each truck gets 10 mpg. During each month, the vehicles produced by the company must average at least 16 mpg. Formulate and solve (using LINDO) an LP to meet the demand and mileage requirements at minimum cost (including steel costs and holding costs).

Solution:

Define variables:

- C1 = number of cars to be produced in month 1
- C2 = number of cars to be produced in month 2
- T1 = number of trucks to be produced in month 1
- T2 = number of truckes to be produced in month 2
- S1 = tons of steel used in month 1
- S2 = tons of steel used in month 2
- IC1 = number of cars in inventory at end of month 1
- IT1 = number of trucks in inventory at end of month 1
- IC2 = number of cars in inventory at end of month 2
- IT2 = number of trucks in inventory at end of month 2

Objective:

Minimize 400 S1 + 600 S2 + 150 IC1 + 150 IT1 + 150 IC2 + 150 IT2

Constraints:

Production capacity constraints: $C1 + T1 \leq 1000$, $C2 + T2 \leq 1000$

Steel usage: $C1 + 2\,T1 = S1$, $C2 + 2\,T2 = S2$

Material balance equations: $200 + C1 = 800 + IC1$, $100 + T1 = 400 + IT1$

$IC1 + C2 = 300 + IC2$, $IT1 + T2 = 300 + IT2$

Demand constraints: $C2 \geq 300$, $T2 \geq 300$

Gasoline economy constraints:

$$\frac{20C1 + 10T1}{C1 + T1} \geq 16 \Rightarrow 4C1 - 6T1 \geq 0$$

$$\frac{20C2 + 10T2}{C2 + T2} \geq 16 \Rightarrow 4C2 - 6T2 \geq 0$$

Limitation on steel purchases: $S1 \leq 1500$, $S2 \leq 1500$

LINDO output:

```
MIN     400 S1 + 600 S2 + 150 IC1 + 150 IT1 + 150 IC2 + 150 IT2
SUBJECT TO
       2)   C1 + T1 <=   1000
       3)   C2 + T2 <=   1000
       4) - S1 + C1 + 2 T1 =    0
       5) - S2 + C2 + 2 T2 =    0
       6) - IC1 + C1 =    600
       7) - IT1 + T1 =    300
       8)   IC1 + C2 - IC2 =    300
       9)   IT1 + T2 - IT2 =    300
      10)   4 C1 - 6 T1 >=   0
      11)   4 C2 - 6 T2 >=   0
END
```

```
  SUB       S1     1500.00000
  SUB       S2     1500.00000

: go

 LP OPTIMUM FOUND AT STEP      3

        OBJECTIVE FUNCTION VALUE

        1)      995000.0

  VARIABLE        VALUE          REDUCED COST
        S1      1400.000000          0.000000
        S2       700.000000          0.000000
       IC1         0.000000          0.000000
       IT1       100.000000          0.000000
        C1       600.000000          0.000000
        T1       400.000000          0.000000
        C2       300.000000          0.000000
        T2       200.000000          0.000000
       IC2         0.000000          0.000000
       IT2         0.000000       2475.000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000        130.000000
        3)       500.000000          0.000000
        4)         0.000000        400.000000
        5)         0.000000        600.000000
        6)         0.000000        300.000000
        7)         0.000000      -2175.000000
        8)         0.000000        150.000000
        9)         0.000000      -2325.000000
       10)         0.000000       -207.500000
       11)         0.000000       -187.500000

 NO. ITERATIONS=       3
```

Description of solution:

Purchase 1400 tons of steel in month 1 and 700 tons in month 2.
Produce 600 cars and 400 trucks in month 1.
With the initial inventory of 200 cars and 100 trucks, this will meet the demands for vehicles in month 1, with 100 trucks remaining in inventory at the end of month 1.
Produce 300 cars (meeting the demand for cars in month 2) and 200 trucks in month 2 (which, with the 100 trucks in inventory, meets the demand for trucks in month 2).
The cost of this production plan is $995,000.

3. *(Exercise 5, page 107, O.R., W. Winston)* A small toy store, Toyco, projects the following monthly cash flows (in thousands of dollars) during the year 2000:

Month	Cash flow
January	-12

February	-10
March	- 8
April	-10
May	- 4
June	+ 5
July	- 7
August	- 2
September	+15
October	+1
November	- 7
December	+45

A negative cash flow means that cash outflows exceed cash inflows to the business. To pay their bills, Toyco will need to borrow money early in the year. Money can be borrowed in two ways:

a. Taking out a long-term one-year loan in January. Interest of 1% is charged each month, and the loan must be paid back at the end of December.
b. Each month, money can be borrowed from a short-term line of credit. Here, a monthly interest rate of 1.5% is charged. All short-term loans must be paid off at (or before) the end of December.

At the end of each month, excess cash earns 0.4% interest. Formulate an LP whose solution will help Toyco maximize their cash position at the beginning of January, 2001.

Solution:

Define variables:

A = amount of one-year loan borrowed in January & repaid in December

Bt = amount of one-month loan borrowed in month t & repaid in month t+1 (t=1,2, ... 12)

Rt = amount of cash held in reserve after meeting obligations in month t (t=1, 2, ... 11)

Objective:

Maximize cash reserve at beginning of January

= cash reserve (with interest) from December

= 1.004 R12

Constraints: *(Material balance equations)*

New loans + cash reserve (with interest) from previous month + cash inflow =

Short-term loans repaid (with interest) + interest on long-term loan

+ cash outflow + cash reserve

Example (February): B2 + 1.004R1 = 1.015B1 + 0.01A + 10 + R2

LINDO output:

```
MAX      1.004 R12
  SUBJECT TO
         2)   A + B1 - R1 =    12
         3) - 0.009999999 A - 1.015 B1 + 1.004 R1 + B2 - R2 =    10
         4) - 0.009999999 A - 1.015 B2 + 1.004 R2 + B3 - R3 =    8
         5) - 0.009999999 A - 1.015 B3 + 1.004 R3 + B4 - R4 =    10
         6) - 0.009999999 A - 1.015 B4 + 1.004 R4 + B5 - R5 =    4
         7) - 0.009999999 A - 1.015 B5 + 1.004 R5 + B6 - R6 =  - 5
         8) - 0.009999999 A - 1.015 B6 + 1.004 R6 + B7 - R7 =    7
```

```
         9) - 0.009999999 A - 1.015 B7 + 1.004 R7 + B8 - R8 =     2
        10) - 0.009999999 A - 1.015 B8 + 1.004 R8 + B9 - R9 =  - 15
        11) - 0.009999999 A - 1.015 B9 + 1.004 R9 + B10 - R10 =  - 12
        12) - 0.009999999 A - 1.015 B10 + 1.004 R10 + B11 - R11 =    7
        13) - R12 - 1.01 A - 1.015 B11 + 1.004 R11 =  - 45
 END

: GO
 LP OPTIMUM FOUND AT STEP     17

        OBJECTIVE FUNCTION VALUE
        1)      12.62700

  VARIABLE        VALUE          REDUCED COST
       R12        12.576693          0.000000
         A        32.102283          0.000000
        B1         0.000000          0.012277
        R1        20.102283          0.000000
        B2         0.000000          0.012228
        R2         9.861670          0.000000
        B3         0.000000          0.012179
        R3         1.580093          0.000000
        B4         8.734610          0.000000
        R4         0.000000          0.011999
        B5        13.186651          0.000000
        R5         0.000000          0.011822
        B6         8.705474          0.000000
        R6         0.000000          0.011647
        B7        16.157080          0.000000
        R7         0.000000          0.011475
        B8        18.720457          0.000000
        R8         0.000000          0.011306
        B9         4.322287          0.000000
        R9         0.000000          0.011139
       B10         0.000000          0.011094
       R10         7.291856          0.000000
       B11         0.000000          0.010499
       R11         0.000000          0.000545

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000         -1.120555
        3)         0.000000         -1.116091
        4)         0.000000         -1.111644
        5)         0.000000         -1.107216
        6)         0.000000         -1.090853
        7)         0.000000         -1.074732
        8)         0.000000         -1.058849
        9)         0.000000         -1.043201
       10)         0.000000         -1.027784
       11)         0.000000         -1.012595
       12)         0.000000         -1.008561
       13)         0.000000         -1.004000

 NO. ITERATIONS=      17
```

Description of solution: Obtain a long-term loan of 32.102283 thousand in January. In month 4 (April), borrow 8.734610 thousand, etc. The cash position in January of the following year will be 12.627 thousand dollars (including interest on the cash reserve held from December).

4. a. Draw the feasible region of the following LP:

$$
\begin{array}{ll}
\text{Maximize} & 3X_1 + X_2 \\
\text{subject to} & 4X_1 + 7X_2 \leq 28 \\
& X_1 + X_2 \leq 5 \\
& 3X_1 + X_2 \leq 8 \\
& X_1 \geq 0,\ X_2 \geq 0
\end{array}
$$

b. Use the simplex algorithm to find the optimal solution of the above LP. *(Show the initial and each succeeding tableau.)*

c. On the sketch of the feasible region in (a), indicate the initial basic solution and the basic solution at each succeeding iteration.

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Solve the LP problems in (1) and (2) below, using **LINDO**. Be sure to state precisely the definitions of your decision variables, and briefly explain the purpose of each type of constraint. State the optimal solution in "plain English" that the person who is to implement the solution might understand. *(All exercises are from the O.R. text by W. Winston. See the appendix of chapter 4 for instructions on using LINDO.)*

1. ***LP Model Formulation:*** Exercise #1, page 76 *(postal worker scheduling)*

> "In the post office example (Example 3-7, §3-5), suppose that each full-time employee works 8 hours per day. Thus, Monday's requirement of 17 workers may be viewed as a requirement of 8(17)=136 hours. The post office may meet its daily labor requirements by using both full-time and part-time employees. During each week, a full-time employee works 8 hours a day for five consecutive days, and a part-time employee works 4 hours a day for five consecutive days. A full-time employee costs the post office $15 per hour, whereas a part-time employee (with reduced fringe benefits) costs the post office only $10 per hour. Union requirements limit part-time labor to 25% of weekly labor requirements. Formulate an LP to minimize the post office's weekly labor costs."

Solution:

Definition of variables:

Xt = # of full-time workers beginning work on day t (t=1, 2, ... 7)

Yt = # of part-time workers beginning work on day t (t=1, 2, ... 7)

Objective:

$$Minimize\ \ 80\sum_{t=1}^{7} X_t + 60\sum_{t=1}^{7} Y_t$$

(weekly salary, in $)

Constraints:

For each day t (t=1,2,...7), 8(# full-time workers on duty) + 4(# part-time workers on duty) ≥ man-hour requirement

Total # part-time hours per week ≤ 25% (840 man-hours per week)

Xt ≥ 0 & integer; Yt ≥ 0 & integer

If the integer restrictions are ignored, LINDO obtains an LP solution which is non-integer:

LINDO output:

```
: look all

  MIN     80 X1 + 80 X2 + 80 X3 + 80 X4 + 80 X5 + 80 X6 + 80 X7 + 60 Y1
       + 60 Y2 + 60 Y3 + 60 Y4 + 60 Y5 + 60 Y6 + 60 Y7
  SUBJECT TO
  2)   8 X1 + 8 X4 + 8 X5 + 8 X6 + 8 X7 + 4 Y1 + 4 Y4 + 4 Y5 + 4 Y6
           + 4 Y7 >=   136
  3)   8 X1 + 8 X2 + 8 X5 + 8 X6 + 8 X7 + 4 Y1 + 4 Y2 + 4 Y5 + 4 Y6
           + 4 Y7 >=   104
  4)   8 X1 + 8 X2 + 8 X3 + 8 X6 + 8 X7 + 4 Y1 + 4 Y2 + 4 Y3 + 4 Y6
           + 4 Y7 >=   120
  5)   8 X1 + 8 X2 + 8 X3 + 8 X4 + 8 X7 + 4 Y1 + 4 Y2 + 4 Y3 + 4 Y4
           + 4 Y7 >=   152
  6)   8 X1 + 8 X2 + 8 X3 + 8 X4 + 8 X5 + 4 Y1 + 4 Y2 + 4 Y3 + 4 Y4
           + 4 Y5 >=   112
  7)   8 X2 + 8 X3 + 8 X4 + 8 X6 + 4 Y2 + 4 Y3 + 4 Y4 + 12 Y5 + 4 Y6
                 >=   128
  8)   8 X3 + 8 X4 + 8 X7 + 4 Y3 + 4 Y4 + 12 Y5 + 12 Y6 + 4 Y7 >=   88
  9)   4 Y1 + 4 Y2 + 4 Y3 + 4 Y4 + 4 Y5 + 4 Y6 + 4 Y7 <=   210
  END

: picture
```

```
           X X X X X X X Y Y Y Y Y Y Y
           1 2 3 4 5 6 7 1 2 3 4 5 6 7

        1: B B B B B B B B B B B B B B MIN
        2: 8     8 8 8 8 4   ' 4 4 4 4 > C
        3: 8 8'  ' 8'8 8 4'4 '  '4 4 4'> C
        4: 8 8 8 '   8 8 4 4 4     4 4 > C
        5: 8 8 8 8     8 4 4 4 4   ' 4 > C
        6: 8 8'8 8 8'  ' 4'4 4 4'4 '  '> C
        7:   8 8 8   8 '   4 4 4 B 4   > C
        8:     8 8     8     4 4 B B 4 > B
        9: '  '  '  '  ' 4'4 4 4'4 4 4'< C
```

```
: go

 LP OPTIMUM FOUND AT STEP      3
        OBJECTIVE FUNCTION VALUE
        1)      1786.667

  VARIABLE        VALUE          REDUCED COST
        X1         2.666667          0.000000
        X2         5.333333          0.000000
        X3         0.000000          0.000000
        X4         7.333333          0.000000
        X5         0.000000         53.333332
        X6         3.333333          0.000000
        X7         3.666667          0.000000
        Y1         0.000000         20.000000
        Y2         0.000000         20.000000
        Y3         0.000000         20.000000
        Y4         0.000000         20.000000
        Y5         0.000000          6.666667
        Y6         0.000000         20.000000
        Y7         0.000000         20.000000

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000         -3.333333
        3)        16.000000          0.000000
        4)         0.000000         -3.333333
        5)         0.000000         -3.333333
        6)        10.666667          0.000000
        7)         0.000000         -3.333333
        8)         0.000000          0.000000
        9)       210.000000          0.000000
```

Note that the optimal LP solution does not use part-time labor at all, which would be expected since part-time labor has a higher cost per hour! If we add the integer restrictions (by using the command: "GIN 14"), then we obtain a solution in which all the variables are integer-valued:

```
 ENUMERATION COMPLETE. BRANCHES=    15 PIVOTS=      77

 LAST INTEGER SOLUTION IS THE BEST FOUND
 RE-INSTALLING BEST SOLUTION...

        OBJECTIVE FUNCTION VALUE

        1)      1800.000
```

```
  VARIABLE        VALUE          REDUCED COST
        X1         4.000000         80.000000
        X2         2.000000         80.000000
        X3         3.000000         80.000000
        X4         6.000000         80.000000
        X5         0.000000         80.000000
        X6         2.000000         80.000000
        X7         4.000000         80.000000
        Y1         0.000000         60.000000
        Y2         0.000000         60.000000
        Y3         0.000000         60.000000
        Y4         0.000000         60.000000
        Y5         2.000000         60.000000
        Y6         0.000000         60.000000
        Y7         0.000000         60.000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000          0.000000
        3)         0.000000          0.000000
        4)         0.000000          0.000000
        5)         0.000000          0.000000
        6)        16.000000          0.000000
        7)         0.000000          0.000000
        8)        40.000000          0.000000
        9)       202.000000          0.000000
```

Note that in the integer solution, 2 part-time workers should be used, beginning on day 5, and working days 5, 6, 7, 1, & 2. The total weekly salaries will be $1800.

Comparison:

	LP Solution	Integer Solution
X1	2.666667	4.000000
X2	5.333333	2.000000
X3	0.000000	3.000000
X4	7.333333	6.000000
X5	0.000000	0.000000
X6	3.333333	2.000000
X7	3.666667	4.000000
Y5	0.000000	2.000000
OBJECTIVE	1786.667	1800.000

Notice that the optimal integer solution cannot be obtained simply by rounding the LP solution!

2. *LP Model Formulation:* Exercise #28, page 118 *(Waste Disposal)*

"City 1 produces 500 tons of waste per day, and city 2 produces 400 tons of waste per day. Waste must be incinerated at incinerator 1 or incinerator 2, and each incinerator can process up to 500 tons of waste per day. The cost to incinerate waste is $40/ton at incinerator 1 and $30/ton at incinerator 2. Incineration reduces each ton of waste to 0.2 tons of debris, which must be dumped at one of two landfills. Each landfill can receive at most 200 tons of debris per day. It costs $3 per mile to transport a ton of material (either debris or waste). Distances (in miles) between locations are shown in the table below. Formulate an LP that can be used to minimize the total cost of disposing of the waste of both cities.

	Incin.1	Incin.2
City 1	30	5
City 2	36	42

	Landfill 1	Landfill 2
Incin.1	5	8

Incin.2	9	6

Solution:

Definition of variables:

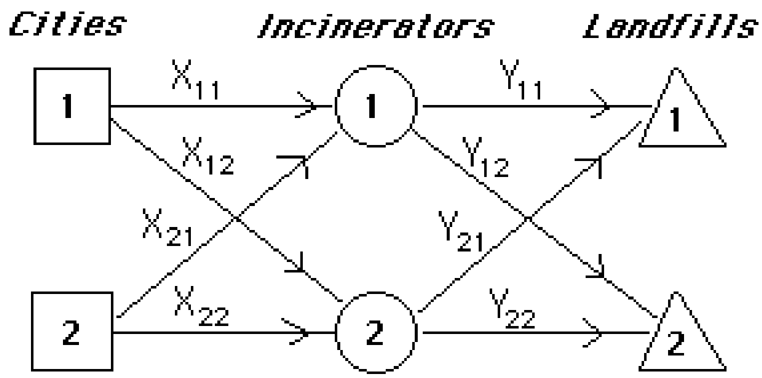
X_{ij} = tons of City #i waste that is sent to Incinerator #j

i=1,2; j=1,2

Y_{jk} = tons of debris sent from Incinerator #j to Landfill #k

j=1,2; k=1,2

(There are a total of 8 variables.)



Model formulation:

```
 MIN  130 X11 + 45 X12 + 148 X21 + 156 X22 + 15 Y11 + 24 Y12 +27 Y21
    + 18 Y22
SUBJECT TO
      2)   X11 + X12 =    500
      3)   X21 + X22 =    400
      4) - 0.2 X11 - 0.2 X21 + Y11 + Y12 =    0
      5) - 0.2 X12 - 0.2 X22 + Y21 + Y22 =    0
      6)   Y11 + Y21 <=   200
      7)   Y12 + Y22 <=   200
      8)   X11 + X21 <=   500
      9)   X12 + X22 <=   500
END
```

Explanation: Rows 2&3 state that all waste is sent from cities 1 & 2, respectively. Rows 4 & 5 state that all debris created at an incinerator will leave the incinerator. Rows 6 & 7 state the capacities of the landfill, while rows 8 & 9 state the capacities of the incinerators. The cost coefficient of X_{ij} includes both transportation from city i to incineratory j and the cost of incineration. (For example, the cost of X_{11} = (\$3/ton-mile)(30 miles)+\$40/ton = \$130/ton.) The cost coefficient of Y_{jk} is the cost of transportation from incinerator j to landfill k only. (For example, the cost of Y_{12} is (\$3/ton-mile)(8 miles) = \$24/ton.)

LINDO output:

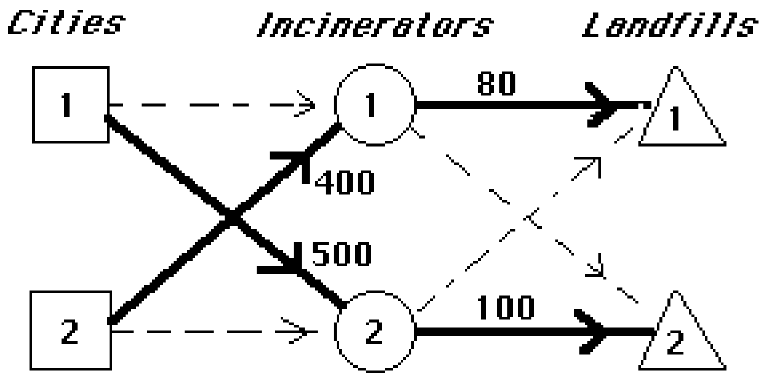
```
             OBJECTIVE FUNCTION VALUE
        1)     84700.0000

  VARIABLE        VALUE          REDUCED COST
       X11           .000000         84.400001
       X12        500.000000           .000000
       X21        400.000000           .000000
       X22           .000000          8.600000
       Y11         80.000000           .000000
       Y12           .000000          9.000000
```

```
Y21          .000000           9.000000
Y22       100.000000            .000000

ROW   SLACK OR SURPLUS     DUAL PRICES
 2)          .000000         -48.600000
 3)          .000000        -151.000000
 4)          .000000         -15.000000
 5)          .000000         -18.000000
 6)       120.000000            .000000
 7)       100.000000            .000000
 8)       100.000000            .000000
 9)          .000000            .000000
```

Thus, City #1 sends its waste to Incinerator #2, while City #2 sends its waste to Incinerator #1. Incinerator #1 sends all its debris to Landfill #1 and Incinerator #2 sends all its debris to Landfill #2.



✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠

3. ***Simplex Algorithm:*** (Exercise 15, page 190)
Suppose that you have obtained the tableau below for a *maximization* problem.

-z	x_1	x_2	x_3	x_4	x_5	x_6	rhs
1	$\mathbf{c_1}$	$\mathbf{c_2}$	0	0	0	0	-10
0	4	$\mathbf{a_1}$	1	0	$\mathbf{a_2}$	0	**b**
0	-1	-5	0	1	-1	0	2
0	$\mathbf{a_3}$	-3	0	0	-4	1	3

(Note that I have altered the tableau which appears in the book to correspond with the lecture notes' use of -z as the basic variable rather than z.) State conditions (for example, restrictions on the signs) on the quantities $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$, **b**, $\mathbf{c_1}$, and $\mathbf{c_2}$ that are required to make the following statements true:

(a) The current solution is optimal, and there are alternative optimal solutions.

(b) The current basic solution is not a basic feasible solution.

(c) The current basic solution is a degenerate basic feasible solution.

(d) The current basic solution is feasible, but the LP is unbounded

(e) The current basic solution is feasible, but the objective function value can be improved by replacing x_6 as a basic variable with x_1.

Quantity:	a_1	a_2	a_3	c_1	c_2	b
(a)				≤0	≤0	≥0
(b)						<0
(c)						=0
(d)	≤0				>0	≥0
(e)			≥ 12/b	>0		≥0

a.) $b \geq 0$ is necessary for feasibility, and $C_1 \leq 0$ and $C_2 \leq 0$ is necessary for optimality. For an alternative optimal solution to exist, one (or more) of the nonbasic variables must have a zero relative profit. Clearly X_5 has a zero relative profit, so that alternative optimal solutions <u>do</u> exist for this tableau. If $a_2 > 0$ we can pivot in X_5 and obtain an alternative optimal basic solution, while if $a2 \leq 0$, X_5 can be assigned any positive value to obtain an alternative (nonbasic) optimal solution.
In addition, other optimal solutions <u>may</u> exist. If $C_1=0$,we can pivot in X_1 to obtain an alternative optimum. If $C_2=0$ and $a_1>0$, we can pivot in X_2 and obtain an alternative optimal basic solution, while if $C_2=0$ and $a_1 \leq 0$, X_2 can be assigned any positive value to obtain an alternative (nonbasic) optimal solution.

b.) Only if $b<0$ will the basic solution be infeasible.

c.) Only if $b=0$ will the basic solution be degenerate.

d.) $b \geq 0$ makes the solution feasible. If $C_2>0$ and $a_1 \leq 0$ we can make X_2 as large as desired and obtain an unbounded solution.

e.) $b \geq 0$ makes the current basic solution feasible. For X_6 to replace X_1, we need $C_1>0$ (this ensures that increasing X_1 will increase Z) and we need row 3 to win the ratio test when entering X_1. This requires $3/a_3 \leq b/4$, i.e., $a_3 \geq 12/b$.

❄✎✉ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ Homework #3 ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍

1. LP Duality*(Problems 3&4, page 274)* : Find the dual of each of the following LP problems:

(a) Max $z = 4X_1 - X_2 + 2X_3$

$$
\begin{array}{llll}
\text{s.t.} & X_1 + X_2 & & \le 5 \\
& 2X_1 + X_2 & & \le 7 \\
& 2X_2 + X_3 & & \ge 6 \\
& X_1 + X_3 & & = 4 \\
& X_1 \ge 0 \ (X_2 \& X_3 \text{ unrestricted in sign}) & &
\end{array}
$$

Solution: The dual problem is:

$$
\begin{aligned}
&Min\ \ z = 5Y_1 + 7Y_2 + 6Y_3 + 4Y_4 \\
&subject\ to \\
&Y_1 + 2Y_2 + Y_4 \ge 4 \\
&Y_1 + Y_2 + 2Y_3 = -1 \\
&Y_3 + Y_4 = 2 \\
&Y_1 \ge 0,\ Y_2 \ge 0,\ Y_3 \le 0,\ Y_4\ urs
\end{aligned}
$$

(b) Min $w = 4Y_1 + 2Y_2 - Y_3$

$$
\begin{array}{lll}
\text{s.t.} & Y_1 + 2Y_2 & \le 6 \\
& Y_1 - Y_2 + 2Y_3 & = 8 \\
& Y_1 \ge 0,\ Y_2 \ge 0 \ (Y_3 \text{ unrestricted in sign}) &
\end{array}
$$

Solution: The dual problem is:

$$
\begin{aligned}
&Max\ z = 6X_1 + 8X_2 \\
&subject\ to \\
&X_1 + X_2 \le 4 \\
&2X_1 - X_2 \le 2 \\
&2X_2 = -1 \\
&X_1 \le 0.\ X_2\ \ urs
\end{aligned}
$$

2. Below are several simplex tableaus. Assume that the objective in each case is to be **minimized**. Classify each tableau by writing to the right of the tableau a letter **A** through **G**, according to the descriptions below. Also answer the question accompanying each classification, if any. ("Solution" here does not imply *optimal* solution, but only a solution of the constraint equations in the tableau.)

(**A**) Nonoptimal, nondegenerate tableau (with no indication of unbounded solution). *Circle a pivot element which would improve the objective.*

(**B**) Nonoptimal, degenerate tableau (with no indication of unbounded solution). *Circle an appropriate pivot element. Would the objective improve with this pivot?*

(**C**) Unique optimum.

(**D**) Optimal tableau, with alternate optimum. *Circle a pivot element which would lead to another optimal basic solution.*

(**E**) Objective unbounded (below). *Specify a variable which, when going to infinity, will make the objective arbitrarily low.*

(F) Tableau with infeasible primal but feasible dual solution.

(G) Tableau with both primal and dual solutions infeasible.

Warning: Some of these classifications might be used for several tableaus, while others might not be used at all!

-z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
1	-3	0	1	1	0	0	2	3	-84	
0	0	0	-4	0	0	1	0	0	13	___A___
0	$\boxed{4}$	1	2	-5	0	0	1	1	8	
0	-6	0	3	-2	1	0	2	3	5	

-z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
1	3	0	1	1	0	0	0	12	-84	
0	0	0	-4	0	0	1	3	0	13	___D____
0	4	1	2	-5	0	0	$\boxed{2}$	1	8	
0	-6	0	3	-2	1	0	-4	3	15	

-z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
1	3	0	1	3	0	0	3	0	-84	
0	0	0	-4	0	0	1	3	0	13	___F____
0	4	1	2	-5	0	0	2	1	-8	
0	-6	0	3	-2	1	0	-4	3	15	

-z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
1	3	0	1	1	0	0	3	5	-84	
0	0	0	-4	0	0	1	3	0	3	___C____
0	4	1	2	-5	0	0	2	1	8	
0	-6	0	3	-2	1	0	-4	3	15	

-z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
1	3	0	1	-3	0	0	3	0	-84	
0	0	0	-4	0	0	1	3	0	13	____E_____
0	4	1	-2	-5	0	0	2	1	8	(unbounded
0	-6	0	3	-2	1	0	-4	3	15	as $X_4 \rightarrow \infty$)

-z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
1	3	0	1	1	0	0	-2	0	-84	
0	0	0	-4	0	0	1	3	0	13	____B_____
0	4	1	2	-5	0	0	$\boxed{2}$	1	8	Degenerate
0	-6	0	3	2	1	0	-4	3	0	Improvement!

-z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
1	3	0	1	4	0	0	-2	2	-84	
0	0	0	-4	0	0	1	-3	0	13	____G____
0	4	1	2	-5	0	0	2	1	-8	
0	-6	0	3	-2	1	0	-4	3	15	

-z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS	
1	3	0	1	3	0	0	2	-2	-84	
0	0	0	-4	0	0	1	3	0	13	A
0	-4	1	2	-5	0	0	-2	1	8	
0	-6	0	3	-2	1	0	-4	$\boxed{3}$	5	

3. Consider the vehicle production problem in Homework #1 *(Exercise 5, page 104, O.R., W. Winston).* Consult the LINDO output to answer the questions below.

> "During the next two months, General Cars must meet (on time) the following demands for trucks and cars: Month 1: 400 trucks, 800 cars; Month 2: 300 trucks, 300 cars. During each month, at most 1000 vehicles can be produced. Each truck uses 2 tons of steel, and each car uses 1 ton of steel. During month 1, steel costs $400 per ton; during month 2, steel costs $600 per ton. At most 1500 tons of steel may be purchased each month (steel may only be used during the month in which it is purchased). At the beginning of month 1, 100 trucks and 200 cars are in inventory. At the end of each month, a holding cost of $150 per vehicle is assessed. Each car gets 20 mpg (miles per gallon), and each truck gets 10 mpg. During each month, the vehicles produced by the company must average at least 16 mpg. Formulate and solve (using LINDO) an LP to meet the demand and mileage requirements at minimum cost (including steel costs and holding costs)."

Define variables:

- C1 = number of cars to be produced in month 1
- C2 = number of cars to be produced in month 2
- T1 = number of trucks to be produced in month 1
- T2 = number of truckes to be produced in month 2
- S1 = tons of steel used in month 1
- S2 = tons of steel used in month 2
- IC1 = number of cars in inventory at end of month 1
- IT1 = number of trucks in inventory at end of month 1
- IC2 = number of cars in inventory at end of month 2
- IT2 = number of trucks in inventory at end of month 2

LINDO output:

```
  MIN     400 S1 + 600 S2 + 150 IC1 + 150 IT1 + 150 IC2 + 150 IT2
  SUBJECT TO
         2)   C1 + T1 <=   1000
         3)   C2 + T2 <=   1000
         4) - S1 + C1 + 2 T1 =    0
         5) - S2 + C2 + 2 T2 =    0
         6) - IC1 + C1 >=   600
         7) - IT1 + T1 >=   300
         8)   IC1 - IC2 + C2 >=   300
         9)   IT1 - IT2 + T2 >=   300
        10)   4 C1 - 6 T1 >=   0
        11)   4 C2 - 6 T2 >=   0
  END
  SUB       S1     1500.00000
  SUB       S2     1500.00000
```

There is a single optimal solution of the primal, which is degenerate; more than one dual optimal solution exists. The dual solution which you obtain will determine the sensitivity analysis which you can perform, as described below.

Solution #1:

```
 LP OPTIMUM FOUND AT STEP      8

        OBJECTIVE FUNCTION VALUE
```

```
       1)      995000.0

 VARIABLE        VALUE          REDUCED COST
       S1      1400.000000          0.000000
       S2       700.000000          0.000000
      IC1         0.000000          0.000000
      IT1       100.000000          0.000000
      IC2         0.000000        750.000000
      IT2         0.000000       1350.000000
       C1       600.000000          0.000000
       T1       400.000000          0.000000
       C2       300.000000          0.000000
       T2       200.000000          0.000000

      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)         0.000000        130.000000
       3)       500.000000          0.000000
       4)         0.000000        400.000000
       5)         0.000000        600.000000
       6)         0.000000       -450.000000
       7)         0.000000      -1050.000000
       8)         0.000000       -600.000000
       9)         0.000000      -1200.000000
      10)         0.000000        -20.000000
      11)         0.000000          0.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:

                           OBJ COEFFICIENT RANGES
VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
      S1      400.000000        92.857147         INFINITY
      S2      600.000000         INFINITY        92.857147
     IC1      150.000000       216.666656       200.000000
     IT1      150.000000       200.000000         INFINITY
     IC2      150.000000         INFINITY       750.000000
     IT2      150.000000         INFINITY      1350.000000
      C1        0.000000       216.666656       200.000000
      T1        0.000000       200.000000         INFINITY
      C2        0.000000       200.000000       216.666656
      T2        0.000000         INFINITY       200.000000

                           RIGHTHAND SIDE RANGES
     ROW         CURRENT        ALLOWABLE        ALLOWABLE
                   RHS           INCREASE         DECREASE
       2     1000.000000        71.428574         0.000000
       3     1000.000000         INFINITY       500.000000
       4        0.000000      1400.000000       100.000000
       5        0.000000       700.000000       800.000000
       6      600.000000         0.000000         0.000000
       7      300.000000         0.000000       200.000000
       8      300.000000       500.000000         0.000000
       9      300.000000         0.000000       200.000000
      10        0.000000         0.000000         0.000000
      11        0.000000         0.000000         INFINITY

THE TABLEAU

     ROW  (BASIS)        S1         S2         IC1        IT1        IC2        IT2
       1 ART          0.000      0.000      0.000      0.000    750.000   1350.000
       2     IC1      0.000      0.000      1.000      0.000      0.000      0.000
       3 SLK  3       0.000      0.000      0.000      0.000      1.000      1.000
       4      S1      1.000      0.000      0.000      0.000      0.000      0.000
```

5	S2	0.000	1.000	0.000	0.000	-1.000	-2.000
6	C1	0.000	0.000	0.000	0.000	0.000	0.000
7	T1	0.000	0.000	0.000	0.000	0.000	0.000
8	C2	0.000	0.000	0.000	0.000	-1.000	0.000
9	SLK 11	0.000	0.000	0.000	0.000	-4.000	6.000
10	IT1	0.000	0.000	0.000	1.000	0.000	0.000
11	T2	0.000	0.000	0.000	0.000	0.000	-1.000

ROW	C1	T1	C2	T2	SLK 2	SLK 3	SLK 6
1	0.000	0.000	0.000	0.000	130.000	0.000	450.000
2	0.000	0.000	0.000	0.000	0.600	0.000	1.000
3	0.000	0.000	0.000	0.000	1.000	1.000	1.000
4	0.000	0.000	0.000	0.000	1.400	0.000	0.000
5	0.000	0.000	0.000	0.000	-1.400	0.000	-1.000
6	1.000	0.000	0.000	0.000	0.600	0.000	0.000
7	0.000	1.000	0.000	0.000	0.400	0.000	0.000
8	0.000	0.000	1.000	0.000	-0.600	0.000	-1.000
9	0.000	0.000	0.000	0.000	0.000	0.000	-4.000
10	0.000	0.000	0.000	0.000	0.400	0.000	0.000
11	0.000	0.000	0.000	1.000	-0.400	0.000	0.000

ROW	SLK 7	SLK 8	SLK 9	SLK 10	SLK 11	
1	0.10E+04	0.60E+03	0.12E+04	20.	0.00E+00	-0.10E+07
2	0.000	0.000	0.000	-0.100	0.000	0.000
3	1.000	1.000	1.000	0.000	0.000	500.000
4	0.000	0.000	0.000	0.100	0.000	1400.000
5	-2.000	-1.000	-2.000	-0.100	0.000	700.000
6	0.000	0.000	0.000	-0.100	0.000	600.000
7	0.000	0.000	0.000	0.100	0.000	400.000
8	0.000	-1.000	0.000	0.100	0.000	300.000
9	6.000	-4.000	6.000	1.000	1.000	0.000
10	1.000	0.000	0.000	0.100	0.000	100.000
11	-1.000	0.000	-1.000	-0.100	0.000	200.000

Solution #2:

LP OPTIMUM FOUND AT STEP 6

OBJECTIVE FUNCTION VALUE

1) 995000.0

VARIABLE	VALUE	REDUCED COST
S1	1400.000000	0.000000
S2	700.000000	0.000000
IC1	0.000000	300.000000
IT1	100.000000	0.000000
IC2	0.000000	0.000000
IT2	0.000000	2475.000000
C1	600.000000	0.000000
T1	400.000000	0.000000
C2	300.000000	0.000000
T2	200.000000	0.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	310.000000
3)	500.000000	0.000000
4)	0.000000	400.000000
5)	0.000000	600.000000
6)	0.000000	0.000000

```
       7)          0.000000       -2175.000000
       8)          0.000000         150.000000
       9)          0.000000       -2325.000000
      10)          0.000000        -177.500000
      11)          0.000000        -187.500000

 NO. ITERATIONS=       6

DO RANGE(SENSITIVITY) ANALYSIS?
yes

 RANGES IN WHICH THE BASIS IS UNCHANGED:

                           OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
       S1      400.000000       221.428574         INFINITY
       S2      600.000000         INFINITY       221.428574
      IC1      150.000000         INFINITY       300.000000
      IT1      150.000000       775.000000         INFINITY
      IC2      150.000000         INFINITY       300.000000
      IT2      150.000000         INFINITY      2475.000000
       C1        0.000000       516.666626      1775.000000
       T1        0.000000       775.000000         INFINITY
       C2        0.000000         INFINITY       516.666626
       T2        0.000000         INFINITY       775.000000

                           RIGHTHAND SIDE RANGES
      ROW         CURRENT        ALLOWABLE        ALLOWABLE
                    RHS          INCREASE         DECREASE
        2     1000.000000         0.000000         0.000000
        3     1000.000000         INFINITY       500.000000
        4        0.000000      1400.000000       100.000000
        5        0.000000       700.000000       800.000000
        6      600.000000         0.000000         INFINITY
        7      300.000000       100.000000         0.000000
        8      300.000000         0.000000         INFINITY
        9      300.000000       200.000000         0.000000
       10        0.000000      1000.000000         0.000000
       11        0.000000      2000.000000         0.000000

THE TABLEAU

      ROW  (BASIS)          S1         S2        IC1        IT1        IC2        IT2
        1 ART            0.000      0.000    300.000      0.000      0.000   2475.000
        2 SLK    6       0.000      0.000      1.000      0.000      0.000      0.000
        3 SLK    3       0.000      0.000      0.000      0.000      0.000      2.500
        4        S1      1.000      0.000      0.000      0.000      0.000      0.000
        5        S2      0.000      1.000      0.000      0.000      0.000     -3.500
        6        C1      0.000      0.000      0.000      0.000      0.000      0.000
        7       IT1      0.000      0.000      0.000      1.000      0.000      0.000
        8        C2      0.000      0.000      0.000      0.000      0.000     -1.500
        9        T2      0.000      0.000      0.000      0.000      0.000     -1.000
       10        T1      0.000      0.000      0.000      0.000      0.000      0.000
       11       IC2      0.000      0.000     -1.000      0.000      1.000     -1.500


      ROW         C1         T1         C2         T2      SLK 2      SLK 3      SLK 6
        1      0.000      0.000      0.000      0.000    310.000      0.000      0.000
        2      0.000      0.000      0.000      0.000      0.600      0.000      1.000
        3      0.000      0.000      0.000      0.000      1.000      1.000      0.000
        4      0.000      0.000      0.000      0.000      1.400      0.000      0.000
        5      0.000      0.000      0.000      0.000     -1.400      0.000      0.000
```

```
   6      1.000      0.000      0.000      0.000      0.600      0.000      0.000
   7      0.000      0.000      0.000      0.000      0.400      0.000      0.000
   8      0.000      0.000      1.000      0.000     -0.600      0.000      0.000
   9      0.000      0.000      0.000      1.000     -0.400      0.000      0.000
  10      0.000      1.000      0.000      0.000      0.400      0.000      0.000
  11      0.000      0.000      0.000      0.000     -0.600      0.000      0.000


 ROW      SLK 7     SLK 10     SLK 11
   1  0.22E+04  0.18E+03  0.19E+03 -0.10E+07
   2      0.000     -0.100      0.000      0.000
   3      2.500      0.250      0.250    500.000
   4      0.000      0.100      0.000   1400.000
   5     -3.500     -0.350     -0.250    700.000
   6      0.000     -0.100      0.000    600.000
   7      1.000      0.100      0.000    100.000
   8     -1.500     -0.150     -0.250    300.000
   9     -1.000     -0.100      0.000    200.000
  10      0.000      0.100      0.000    400.000
  11     -1.500     -0.150     -0.250      0.000
```

a. Suppose that the cost of steel in month 1 were to increase by \$50/ton. Would the production plan need to be revised?

Solution: Since 50 ≤ allowable increase (=92.857147), there will be no change in basis, and therefore no change in the production plan.

What if the cost were to increase by \$100/ton?

Solution: Since 100 ≥ allowable increase, the basis will change, and therefore the basic solution (and the production plan) will change.

b. Suppose that the holding cost of vehicles is increased to \$160/month. Should the production plan be revised?

Solution: No change in the basis, and therefore no change in basic solution.

c. If the demand for trucks in month 1 were to increase by 10, what would be the effect on the total cost?

Solution #1: An increase in the demand for trucks in month 1 would result in a change (increase) in the right-hand-side of row # 7. The allowable increase for row #7 is, however, zero, and so the dual price (-1050) provides us with no useful information about the effect of an increase in the demand. The basic solution found by LINDO is *degenerate*. (Note that IC1 and the surplus variable in row #11 are both zero but also have zero reduced costs.)

Solution #2: An increase in the demand for trucks in month 1 would result in a change (increase) in the right-hand-side of row # 7. The increase (10) is less than the allowable increase (100) for row #7, and therefore the basis (& dual variables) will not change. The "dual price" for row #7 is -2175.00 (\$/unit demand) and therefore the objective function (total cost) will "improve" by (-2175.00)(10 units of demand) = -21750.00 dollars, i.e., the cost will "deteriorate", i.e. increase, by \$21750.00.

d. By using the *substitution rates* in the tableau, determine what would be the effect on the production plan if the demand for trucks in month 1 were to increase by 10.

Solution: In order to determine the effect of an increase of 10 trucks/month in right-hand-side #7, we reason as follows:

Row #7, after being converted into equation form by LINDO by subtracting a surplus variable, is

```
- IT1 + T1 - SLK_7 =   300.
```

In the current soluton, -IT1 + T1 is 300 and SLK_7 is 0, i.e.,

```
300 - 0 = 300.
```

If the left-hand-side of the inequality in row#7 (-IT1 + T1) were to increase by 10 to 310, i.e., 10 additional trucks are produced, then in order to balance the equation, the "surplus" variable SLK_7 must increase by 10 trucks, i.e.,

```
310  - 10 =   300.
```

If you obtained ***Solution #1:*** Because the "allowable increase" in the right-hand-side of row #7 is zero, we cannot answer this question. (The substitution rate of SLK_7 for SLK_11 is, according to the substitution rates, +6:

```
ROW  (BASIS)   SLK 7
  1  ART      0.10E+04
  2  IC1      0.000
  3  SLK 3    1.000
  4  S1       0.000
  5  S2      -2.000
  6  C1       0.000
  7  T1       0.000
  8  C2       0.000
  9  SLK 11   6.000
 10  IT1      1.000
 11  T2      -1.000
```

But SLK_11, although basic, has the value zero, i.e., the basic solution is degenerate. Therefore, any positive increase in SLK_7 would decrease SLK_11 to a negative value so that (unless the basis is changed) would be infeasible! Consequently, we cannot answer the question based upon this output.)

If you obtained ***Solution #2:*** The proposed increase (10) of SLK_7 is less than the allowable increase in the right-hand-side of row #7 (100). Hence, we refer to the substitution rates for SLK_7:

```
ROW  (BASIS)     SLK 7
  1 ART         0.10E+04
  2   IC1       0.000
  3 SLK 3       1.000
  4    S1       0.000
  5    S2      -2.000
  6    C1       0.000
  7    T1       0.000
  8    C2       0.000
  9 SLK 11      6.000
 10     IT1      1.000
 11      T2     -1.000
```

Recall that a positive substitution rate indicates that as the nonbasic variable (in this case, SLK_7) increases, the basic variable will decrease, while a negative substitution rate indicates that the basic variable will increase. According to the substitution rates, then, if SLK_7 increases by 10 units (trucks), then

- SLK_3 will decrease by 10 units, i.e., there will be a decrease of the unused capacity in month 2,
- S2 will increase by 20, i.e., an additional 20 tons of steel will be purchased in month #2,
- SLK_11 will decrease by 60,
- IT1 will decrease by 10, i.e., ten fewer trucks will be kept in inventory at the end of month 1, and
- T2 will increase by 20, i.e., an additional 20 trucks will be produced in month 2.

Note that the substitution rate for T1 is zero, indicating that there will be no change in the number of trucks produced in month 1.

❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ Homework #4 ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍

1. **Sensitivity Analysis**: Consult the LP model & LINDO output for the Gasoline Blending Problem which is in the lecture notes and was discussed in class.

LINDO output: *(Note that the formulation is somewhat different than that in the notes, in that I have changed rows 5-8 from equations to inequalities. More on this later!)*

```
MAX     14.13 X11 + 12 X21 + 8.8 X31 + 6.4 X41 + 11.93 X12 + 9.8 X22
     + 6.6 X32 + 4.2 X42 + 9.97 X13 + 7.84 X23 + 4.64 X33 + 2.24 X43
     + 5.83 Y1 + 3.7 Y2 + 2.6 Y3 + 0.2 Y4
SUBJECT TO
       2) - 27 X11 - 9 X21 - 4 X31 + 4 X41 >=   0
       3) - 22 X12 - 4 X22 + X32 + 9 X42 >=   0
       4) - 17 X13 + X23 + 6 X33 + 14 X43 >=   0
       5)   X11 + X12 + X13 + Y1 <=   4000
       6)   X21 + X22 + X23 + Y2 <=   5050
       7)   X31 + X32 + X33 + Y3 <=   7100
       8)   X41 + X42 + X43 + Y4 <=   4300
       9)   X11 + X21 + X31 + X41 <=   10000
      10)   X13 + X23 + X33 + X43 >=   15000
END

       OBJECTIVE FUNCTION VALUE

       1)      140216.5

  VARIABLE        VALUE          REDUCED COST
       X11         0.000000          0.000000
       X21         0.000000          0.000000
       X31      2453.703613          0.000000
       X41      2453.703613          0.000000
       X12         0.000000          0.000000
       X22         0.000000          0.542424
       X32         0.000000          0.693098
       X42         0.000000          0.934175
       X13      3457.407471          0.000000
       X23      5050.000000          0.000000
       X33      4646.296387          0.000000
       X43      1846.296265          0.000000
        Y1       542.592590          0.000000
        Y2         0.000000          5.533333
        Y3         0.000000          4.970370
        Y4         0.000000          7.429630

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000         -0.307407
        3)         0.000000         -0.277273
        4)         0.000000         -0.307407
        5)         0.000000          5.830000
        6)         0.000000          9.233334
        7)         0.000000          7.570370
        8)         0.000000          7.629630
        9)      5092.592773          0.000000
       10)         0.000000         -1.085926

 RANGES IN WHICH THE BASIS IS UNCHANGED:
                           OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
      X11       14.130000         0.000000         INFINITY
      X21       12.000000         0.000000         INFINITY
      X31        8.800000         INFINITY         0.000000
      X41        6.400000         0.000000         1.627273
      X12       11.930000         2.283539         2.983334
      X22        9.800000         0.542424         INFINITY
      X32        6.600000         0.693098         INFINITY
      X42        4.200000         0.934175         INFINITY
      X13        9.970000         1.627273         0.000000
```

X23	7.840000	INFINITY	0.000000
X33	4.640000	0.000000	1.207331
X43	2.240000	1.627273	0.000000
Y1	5.830000	6.100000	2.932000
Y2	3.700000	5.533334	INFINITY
Y3	2.600000	4.970370	INFINITY
Y4	0.200000	7.429630	INFINITY

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	0.000000	17096.773438	14650.000000
3	0.000000	0.000000	11937.037109
4	0.000000	93350.000000	14650.000000
5	4000.000000	INFINITY	542.592590
6	5050.000000	5538.888672	1627.777710
7	7100.000000	4334.782227	3662.500000
8	4300.000000	3662.500000	4274.193359
9	10000.000000	INFINITY	5092.592773
10	15000.000000	1465.000000	5864.706055

THE TABLEAU

ROW	(BASIS)	X11	X21	X31	X41	X12	X22
1	ART	0.000	0.000	0.000	0.000	0.000	0.542
2	X31	3.875	1.625	1.000	0.000	0.000	0.333
3	X12	0.000	0.000	0.000	0.000	1.000	0.182
4	X13	1.000	0.000	0.000	0.000	0.000	-0.333
5	X33	-3.875	-1.625	0.000	0.000	0.000	-0.333
6	X23	0.000	1.000	0.000	0.000	0.000	1.000
7	X41	-2.875	-0.625	0.000	1.000	0.000	0.333
8	X43	2.875	0.625	0.000	0.000	0.000	-0.333
9	SLK 9	0.000	0.000	0.000	0.000	0.000	-0.667
10	Y1	0.000	0.000	0.000	0.000	0.000	0.152

ROW	X32	X42	X13	X23	X33	X43	Y1
1	0.693	0.934	0.000	0.000	0.000	0.000	0.000
2	0.426	0.574	0.000	0.000	0.000	0.000	0.000
3	-0.045	-0.409	0.000	0.000	0.000	0.000	0.000
4	-0.148	0.148	1.000	0.000	0.000	0.000	0.000
5	0.574	-0.574	0.000	0.000	1.000	0.000	0.000
6	0.000	0.000	0.000	1.000	0.000	0.000	0.000
7	0.426	0.574	0.000	0.000	0.000	0.000	0.000
8	-0.426	0.426	0.000	0.000	0.000	1.000	0.000
9	-0.852	-1.148	0.000	0.000	0.000	0.000	0.000
10	0.194	0.261	0.000	0.000	0.000	0.000	1.000

ROW	Y2	Y3	Y4	SLK 2	SLK 3	SLK 4	SLK 5
1	5.533	4.970	7.430	0.307	0.277	0.307	5.830
2	0.333	0.426	0.574	0.144	0.000	0.019	0.000
3	0.000	0.000	0.000	0.000	0.045	0.000	0.000
4	-0.333	-0.148	0.148	0.037	0.000	0.037	0.000
5	-0.333	0.574	-0.574	-0.144	0.000	-0.019	0.000
6	1.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.333	0.426	0.574	-0.106	0.000	0.019	0.000
8	-0.333	-0.426	0.426	0.106	0.000	-0.019	0.000
9	-0.667	-0.852	-1.148	-0.037	0.000	-0.037	0.000
10	0.333	0.148	-0.148	-0.037	-0.045	-0.037	1.000

ROW	SLK 6	SLK 7	SLK 8	SLK 9	SLK 10	
1	9.2	7.6	7.6	0.00E+00	1.1	0.14E+06
2	0.333	0.426	0.574	0.000	0.315	2453.704
3	0.000	0.000	0.000	0.000	0.000	0.000
4	-0.333	-0.148	0.148	0.000	-0.370	3457.407
5	-0.333	0.574	-0.574	0.000	-0.315	4646.296
6	1.000	0.000	0.000	0.000	0.000	5050.000
7	0.333	0.426	0.574	0.000	0.315	2453.704
8	-0.333	-0.426	0.426	0.000	-0.315	1846.296
9	-0.667	-0.852	-1.148	1.000	-0.630	5092.593
10	0.333	0.148	-0.148	0.000	0.370	542.593

a. In the optimal solution, raw gasoline type #2 is not sold on the market, even though it can be sold for more than the price paid by the refinery.

• What increase in the selling price of this gasoline would be required in order to make its sale optimal? **Solution:** $ 5.53 /barrel (Allowable increase in the objective coefficient of the variable Y2)

• If it could be sold at this price, how much would be sold?

Solution: 1629.41 barrels/day. Perform the minimum ratio test to determine how much of the variable Y2 would enter the solution. There are four positive elements in the Y2 column of the tableau (rows 2, 6, 7, and 10), and the corresponding ratios are 2453.704/0.333, 5050/1, 2453.704/0.333, and 542.593/0.333. The minimum ratio is 542.593/0.333 = 1629.41 in row 10. Therefore, if the profit coefficient of Y2 were to increase to the point that it would enter the solution, it would increase to 1629.41 barrels/day.

```
ROW  (BASIS)       Y2         RHS
  1    ART      5.533     0.14E+06
  2    X31      0.333     2453.704
  3    X12      0.000        0.0
  4    X13     -0.333     3457.407
  5    X33     -0.333     4646.296
  6    X23      1.000     5050.000
  7    X41      0.333     2453.704
  8    X43     -0.333     1846.296
  9  SLK 9     -0.667     5092.593
 10     Y1      0.333      542.593
```

• What would be the effect on the quantities of the blends produced? *(Hint: use substitution rates!)*

Solution: Using the substitution rates for Y2, we see that each unit (barrel/day) of Y2 will replace (substitute for) 0.333 barrel/day each of variables X31, X41, and Y1, and 1 barrel/day of X23; on the other hand, it will require an increase of 0.333 barrel/day in each of X13, X33, and X43, and 0.667 barrel/day of SLK 9. Multiplying by 1629.41 barrels/day yields the changes below:

```
ROW  (BASIS)     RHS           CHANGE
  1    ART      0.14E+06
  2    X31      2453.704        +542.593
  3    X12         0.0             0
  4    X13      3457.407        -542.593
  5    X33      4646.296        -542.593
  6    X23      5050.000       +1629.41
  7    X41      2453.704        +542.593
  8    X43      1846.296        -542.593
  9  SLK 9      5092.593       -1086.816
 10     Y1       542.593        -542.593
```

Notice that (because the minimum ratio occurred in row 10, the pivot row), variable Y1 decreases to zero and leaves the basis, replaced by variable Y2. Summing, we see that

the change in blend #1 is the change in X31+X41

= +542.593+542.593

= +1085 barrels/day,

while the change in blend #3 is the change in X13+X23+X33

= -542.593-542.593+1629.41

= +844.224 barrels/day

b. 4300 barrels/day of raw gas type #4 is now available for $38.75/barrel.

• If more would be available, would the refinery be able to increase their profit?

Solution: Yes Row 8 limits the purchase of raw gas type #4 to 4300 barrels/day. The dual price for this row is 7.62963 $/barrel, indicating that an increase in the right-hand-side of this row will increase the profit at the rate of $7.62963/barrel.

• What is the maximum price/barrel that the refinery should be willing to pay for the type #4 gasoline?

Solution: If the refinery were to pay an extra $7.62963/barrel (a total of approximately $38.75+$7.63 = $46.38/barrel), then they would "break even", with the extra profit and extra cost of raw gas #4 canceling, while if the cost were anything less than $46.38, they would have a net gain in profit.

• What is the quantity of gasoline that they should be willing to buy at that price?

Solution: The dual price ($7.62963/barrel) is valid unless the basis changes, and the ALLOWABLE

INCREASE in the right-hand-side of row 8 is 3662.5 barrels/day.

• If there were an additional 10 barrels/day available at the original price ($38.75/barrel), how would it be used, i.e., how would the optimal solution be changed? *(Hint: use substitution rates!)*

Solution: Recall that row 8 is

```
(X41 + X42 + X43 + Y4)  + SLK8 = 4300
```

with slack equal to zero in the optimal solution:

```
4300             +   0  = 4300
```

If the quantity purchased were 4310, then the equation would be

```
4310             + (-10)= 4300
```

That is, the effect of purchasing an additional 10 barrels/day would be the effect of changing the variable SLK8 from 0 to -10. (Of course, this would violate the original constraint, since the slack variable must satisfy the nonnegativity condition.) We therefore need substitution rates for SLK8. *(Note: in the model in the notes, row 8 was stated as an equation, and so no column for SLK8 appears in the tableau. However, the variable Y4 has exactly the same constraint column as SLK8 would have, i.e., a +1 in row 8 and zeroes elsewhere, so that we can use the substitution rates for Y4 instead.)*

ROW	(BASIS)	Y4	RHS
1	ART	7.430	0.14E+06
2	X31	0.574	2453.704
3	X12	0.000	0.0
4	X13	0.148	3457.407
5	X33	-0.574	4646.296
6	X23	0.000	5050.000
7	X41	0.574	2453.704
8	X43	0.426	1846.296
9	SLK 9	-1.148	5092.593
10	Y1	-0.148	542.593

According to the substitution rates of Y4, then, a decrease of 10 in the value of SLK8 would result in

X31	increase 5.74
X12	no change
X13	increase 1.48
X33	decrease 5.74
X23	no change
X41	increase 5.74
X43	increase 4.26
SLK9	decrease 11.48
Y1	decrease 1.48

That is, of the extra 10 barrels, 5.74 barrels would be added to blend 1 and 4.26 barrels to blend 3. In addition, 5.74 barrels of raw gas #3 would be diverted from blend #3 to blend #1, and 1.48 barrels of raw gas #1 from blend #3 to sale on the market.

2. LP formulation: Recent federal regulations strongly encourage the assignment of students to schools in a city so that the racial composition of any school approximates the racial composition of the entire city. Consider the case of the Greenville city schools. The city can be considered as composed of five areas with the following characteristics:

Area	Percent minority	Number of students
1	20%	1200
2	10%	900
3	85%	1700
4	60%	2000
5	90%	2500

The ruling handed down for Greenville is that a school can have neither more than 75% nor less than 30% minority enrollment. There are three schools in Greenville with the following capacities:

School	Capacity
Bond	3900
Pocahontas	3100
Pierron	2100

The objective is to design an assignment of students to schools so as to stay within the capacity of each school and satisfy the composition constraints, while minimizing the total distance traveled by students (and therefore the average distance traveled by students). The distances in kilometers between areas and schools are:

	Area				
School	1	2	3	4	5
Bond	2.7	1.4	2.4	1.1	0.5
Pocahontas	0.5	0.7	2.9	0.8	1.9
Pierron	1.6	2.0	0.1	1.3	2.2

There is an additional condition that no student can be transported more than 2.6 kilometers. Find the number of students which should be assigned to each school from each area. Assume that any group of students from an area have the same ethnic mix as the whole area.

a. Formulate a linear programming model for this problem. *Be sure to define your variables!*

Solution:

Define variables:

xij = number of students from area *i* assigned to school *j*

where *i* =1, 2, 3, 4, 5 and *j* =1, 2, 3, except for the cases i=1&j=1 and i=3&j=2 (because the distances traveled exceed the maximum allowed.)

LINDO output:

```
 MIN     1.4 X21 + 2.4 X31 + 1.1 X41 + 0.5 X51 + 0.5 X12
  + 0.7 X22 + 0.8 X42 + 1.9 X52 + 1.6 X13 + 2 X23 + 0.1 X33
  + 1.3 X43 + 2.2 X53
  SUBJECT TO
    2)   X12 + X13 =    1200        (all students from area 1 must be assigned to a school)
    3)   X21 + X22 + X23 =    900   (all students from area 2  must be assigned to a school)
    4)   X31 + X33 =    1700        (all students from area 3 must be assigned to a school)
    5)   X41 + X42 + X43 =    2000  (all students from area 4  must be assigned to a school)
    6)   X51 + X52 + X53 =    2500  (all students from area 5 must be assigned to a school)
    7)   X21 + X31 + X41 + X51 <=   3900              (capacity of school 1)
    8)   X12 + X22 + X42 + X52 <=   3100              (capacity of school 2)
    9)   X13 + X23 + X33 + X43 + X53 <=   2100        (capacity of school 3)
   10) - 0.55 X21 + 0.1 X31 - 0.15 X41 + 0.15 X51 <=   0 (
   11)   0.2 X21 - 0.55 X31 - 0.3 X41 - 0.6 X51 <=   0
   12) - 0.55 X12 - 0.65 X22 - 0.15 X42 + 0.15 X52 <=   0
   13)   0.1 X12 + 0.2 X22 - 0.3 X42 - 0.6 X52 <=   0
   14) - 0.55 X13 - 0.65 X23 + 0.1 X33 - 0.15 X43 + 0.15 X53 <=   0
   15)   0.1 X13 + 0.2 X23 - 0.55 X33 - 0.3 X43 - 0.6 X53 <=   0
  END
```

b. Solve the problem, using LINDO (or LP software of your choice). What is the optimal solution? Enter the numbers of students transported below:

Solution:

```
LP OPTIMUM FOUND AT STEP     12

        OBJECTIVE FUNCTION VALUE

        1)      5014.364

  VARIABLE        VALUE          REDUCED COST
       X21      542.727295          0.000000
       X31        0.000000          2.212727
       X41      148.181824          0.000000
       X51     2500.000000          0.000000
```

X12	890.909119	0.000000
X22	357.272736	0.000000
X42	1851.818237	0.000000
X52	0.000000	1.460000
X13	309.090912	0.000000
X23	0.000000	0.032727
X33	1700.000000	0.000000
X43	0.000000	0.069091
X53	0.000000	1.830909

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	-0.680000
3)	0.000000	-0.880000
4)	0.000000	-0.267273
5)	0.000000	-0.980000
6)	0.000000	-0.620000
7)	709.090881	0.000000
8)	0.000000	0.180000
9)	90.909088	0.000000
10)	0.000000	0.800000
11)	1435.909058	0.000000
12)	1000.000000	0.000000
13)	395.000000	0.000000
14)	0.000000	1.672727
15)	904.090881	0.000000

NO. ITERATIONS= 12

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X21	1.400000	0.600000	0.316666
X31	2.400000	INFINITY	2.212727
X41	1.100000	0.073077	0.138462
X51	0.500000	1.460000	INFINITY
X12	0.500000	0.920000	0.027692
X22	0.700000	0.034615	0.600000
X42	0.800000	0.138462	0.400000
X52	1.900000	INFINITY	1.460000
X13	1.600000	0.027692	0.920000
X23	2.000000	INFINITY	0.032727
X33	0.100000	2.212727	INFINITY
X43	1.300000	INFINITY	0.069091
X53	2.200000	INFINITY	1.830909

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	1200.000000	709.090881	113.986023
3	900.000000	607.692322	113.986023
4	1700.000000	76.923080	1699.999878
5	2000.000000	709.090881	113.986023
6	2500.000000	493.939392	1809.090942
7	3900.000000	INFINITY	709.090881
8	3100.000000	113.986023	709.090881
9	2100.000000	INFINITY	90.909088
10	0.000000	271.363647	74.090912
11	0.000000	INFINITY	1435.909058
12	0.000000	INFINITY	1000.000000
13	0.000000	INFINITY	395.000000
14	0.000000	170.000000	49.999996

15	0.000000	INFINITY	904.090881

THE TABLEAU

ROW	(BASIS)	X21	X31	X41	X51	X12	X22
1	ART	0.000	2.213	0.000	0.000	0.000	0.000
2	X12	0.000	-0.182	0.000	0.000	1.000	0.000
3	X22	0.000	0.145	0.000	0.000	0.000	1.000
4	X41	0.000	-0.036	1.000	0.000	0.000	0.000
5	SLK 9	0.000	-1.182	0.000	0.000	0.000	0.000
6	X13	0.000	0.182	0.000	0.000	0.000	0.000
7	X51	0.000	0.000	0.000	1.000	0.000	0.000
8	X42	0.000	0.036	0.000	0.000	0.000	0.000
9	SLK 7	0.000	1.182	0.000	0.000	0.000	0.000
10	X21	1.000	-0.145	0.000	0.000	0.000	0.000
11	SLK 11	0.000	-0.532	0.000	0.000	0.000	0.000
12	SLK 12	0.000	0.000	0.000	0.000	0.000	0.000
13	SLK 13	0.000	0.000	0.000	0.000	0.000	0.000
14	X33	0.000	1.000	0.000	0.000	0.000	0.000
15	SLK 15	0.000	0.532	0.000	0.000	0.000	0.000

ROW	X42	X52	X13	X23	X33	X43	X53
1	0.000	1.460	0.000	0.033	0.000	0.069	1.831
2	0.000	0.000	0.000	-1.182	0.000	-0.273	0.273
3	0.000	-0.600	0.000	0.945	0.000	0.218	-0.218
4	0.000	-1.600	0.000	-0.236	0.000	0.945	0.055
5	0.000	0.000	0.000	-0.182	0.000	0.727	1.273
6	0.000	0.000	1.000	1.182	0.000	0.273	-0.273
7	0.000	1.000	0.000	0.000	0.000	0.000	1.000
8	1.000	1.600	0.000	0.236	0.000	0.055	-0.055
9	0.000	0.000	0.000	0.182	0.000	-0.727	-1.273
10	0.000	0.600	0.000	0.055	0.000	-0.218	0.218
11	0.000	0.000	0.000	-0.082	0.000	0.327	0.573
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000
14	0.000	0.000	0.000	0.000	1.000	0.000	0.000
15	0.000	0.000	0.000	0.082	0.000	-0.327	-0.573

ROW	SLK 7	SLK 8	SLK 9	SLK 10	SLK 11	SLK 12	SLK 13
1	0.000	0.180	0.000	0.800	0.000	0.000	0.000
2	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3	0.000	-0.300	0.000	2.000	0.000	0.000	0.000
4	0.000	-1.300	0.000	2.000	0.000	0.000	0.000
5	0.000	0.000	1.000	0.000	0.000	0.000	0.000
6	0.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.000	0.000	0.000	0.000	0.000	0.000	0.000
8	0.000	1.300	0.000	-2.000	0.000	0.000	0.000
9	1.000	1.000	0.000	0.000	0.000	0.000	0.000
10	0.000	0.300	0.000	-2.000	0.000	0.000	0.000
11	0.000	-0.450	0.000	1.000	1.000	0.000	0.000
12	0.000	0.000	0.000	1.000	0.000	1.000	0.000
13	0.000	0.450	0.000	-1.000	0.000	0.000	1.000
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000

ROW	SLK 14	SLK 15	
1	1.673	0.000	-5014.364
2	1.818	0.000	890.909
3	0.545	0.000	357.273
4	2.364	0.000	148.182
5	1.818	0.000	90.909
6	-1.818	0.000	309.091
7	0.000	0.000	2500.000

```
  8    -2.364     0.000  1851.818
  9    -1.818     0.000   709.091
 10    -0.545     0.000   542.727
 11     0.818     0.000  1435.909
 12     1.000     0.000  1000.000
 13    -1.000     0.000   395.000
 14     0.000     0.000  1700.000
 15     0.182     1.000   904.091
```

The optimal LP solution is not integer, because areas 1, 2, & 4 each send a fraction a student each to two schools:

	Area					
School	1	2	3	4	5	total
Bond		542.73		148.18	2500	3900
Pocahontas	890.91	357.27		1851.82		2300
Pierron	309.09		1700			2100
Total	1200	900	1700	2000	2500	8300

Suppose that we round the solution to integer. Based upon the slack/surplus variables above, we see that neither the lower limit nor the upper limit of minority attendance is reached at school 2 (there is slack in both rows 12 & 13), while schools 1 & 3 are at their upper limits. Therefore, we could assign the student from area 1 to school 3 and the student from area 2 to school 1 (since areas 1&2 are relatively low in minorities), and the student from area 4 to school 2 (since area 4 is relatively high in minorities):

	Area					
School	1	2	3	4	5	total
Bond		543		148	2500	3191
Pocahontas	890	357		1852		3099
Pierron	310		1700			2010
Total	1200	900	1700	2000	2500	8300

As a result of this modification in the solution, the objective function is increased by 1.136 km. to 5015.5 km.

Here is the optimal integer solution: By issuing the command GIN 13 before solving the problem, we get the optimal integer solution (which requires a branch-and-bound procedure by LINDO, requiring much more computational effort!):

```
     OBJECTIVE FUNCTION VALUE

      1)      5015.100

 VARIABLE        VALUE          REDUCED COST
      X21       542.000000         1.400000
      X31         0.000000         2.400000
      X41       152.000000         1.100000
      X51      2500.000000         0.500000
      X12       905.000000         0.500000
      X22       346.000000         0.700000
      X42      1848.000000         0.800000
      X52         0.000000         1.900000
      X13       295.000000         1.600000
      X23        12.000000         2.000000
      X33      1700.000000         0.100000
      X43         0.000000         1.300000
      X53         0.000000         2.200000
```

```
ROW   SLACK OR SURPLUS     DUAL PRICES
 2)         0.000000          0.000000
 3)         0.000000          0.000000
 4)         0.000000          0.000000
 5)         0.000000          0.000000
 6)         0.000000          0.000000
 7)       706.000000          0.000000
 8)         1.000000          0.000000
 9)        93.000000          0.000000
10)         0.099973          0.000000
11)      1437.200073          0.000000
12)       999.850037          0.000000
13)       394.700012          0.000000
14)         0.050001          0.000000
15)       903.100037          0.000000
```

	Area					
School	1	2	3	4	5	total
Bond		542		152	2500	3194
Pocahontas	905	346		1848		3099
Pierron	295	12	1700			2007
Total	1200	900	1700	2000	2500	8300

The optimal integer solution was not that obtained earlier by rounding the continuous solution, but the total distance is only 0.4 km less than the rounded solution.

c. What is the average distance traveled by students, according to this plan?

Solution:

Solution:	Continuous	Rounded	Optimal
Average distance (km):	0.604140241	0.604277108	0.604228915

d. Suppose that a temporary classroom building are available which could be erected at any one of the school locations, which would increase the school's capacity by 100 students. At which school should the building be erected?

Solution: If we base our decision on the *Dual Prices* of the continuous LP solution, we see that only row 8 (the capacity restriction of school 2) is "tight" and has a positive dual price (0.18 km/unit of capacity), and the *ALLOWABLE INCREASE* in the right-hand-side of row 8 is 113.98, and so the obvious decision is to build the temporary classrooms at school 2, which would improve (decrease) the objective function (total km. traveled) by 100x0.18 = 18 km. *(Because we are considering continuous and not integer solutions, this is probably only a close approximation of the improvement.)*

How much decrease in the average distance traveled would result?

Solution: The (approximate) total distance is reduced to 4996.364 km. so that the new average distance traveled by a student is 0.601971566 km., an improvement of about <u>0.00226 km</u>.

e. If the building were erected at this location, use the substitution rates to determine the adjustments to your solution which would result (without re-solving the LP), and write them below (e.g., +25, -10, -15, etc.):

Solution: The effect of increasing the capacity of school 2 by 100 units can be estimated by determining the effect of decreasing the slack capacity in row 8 by 100 units (from 0 to -100), using the substitution rates from the tableau of the continuous LP solution:

ROW	SLK 7	SLK 8	
1	ART	0.180	
2	X12	0.000	*no change*
3	X22	-0.300	*decrease 30*
4	X41	-1.300	*decrease 130*

```
 5      SLK 9      0.000
 6      X13        0.000
 7      X51        0.000
 8      X42        1.300
 9      SLK 7      1.000
10      X21        0.300
11      SLK 11    -0.450
12      SLK 12     0.000
13      SLK 13     0.450
14      X33        0.000
15      SLK 15     0.000
```

5	SLK 9	0.000	*no change*
6	X13	0.000	*no change*
7	X51	0.000	*no change*
8	X42	1.300	*increase 130*
9	SLK 7	1.000	*increase 100*
10	X21	0.300	*increase 30*
11	SLK 11	-0.450	*decrease 45*
12	SLK 12	0.000	*no change*
13	SLK 13	0.450	*increase 45*
14	X33	0.000	*no change*
15	SLK 15	0.000	*no change*

That is, 30 students from area 2 should be sent to school 1 instead of school 2, while 130 students from area 4 should be sent to school 2 instead of school 1. If we make this modification in the optimal integer solution, we would have:

	Area					
School	1	2	3	4	5	total
Bond		572		22	2500	3194
Pocahontas	905	316		1978		3099
Pierron	295	12	1700			2007
Total	1200	900	1700	2000	2500	8300

For this solution, the total distance traveled is 5039.1 km, a reduction in total distance of 24 km. (compared to the reduction of only 18 km in the continuous solution!)

❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ Homework #5 ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍

1. ***Sensitivity Analysis****:* (Cornco, Inc., Problem 16, page 231) Cornco produces two products: PS and QT. The sales price for each product and the maximum quantity of each that can be sold during each of the next three months are:

	Month 1		Month 2		Month 3	
Product	Price	Demand	Price	Demand	Price	Demand
PS	\$40	50	\$60	45	\$55	50
QT	\$35	43	\$40	50	\$44	40

Each product must be processed through two assembly lines: 1 & 2. The number of hours required by each product on each assembly line are:

Product	Line 1	Line 2
PS	3 hours	2 hours
QT	2 hours	2 hours

The number of hours available on each assembly line during each month are:

Line	Month 1	Month 2	Month 3
1	200*	160	190
2	140*	150	110

* *there is apparently a typographical error in the textbook, which has values of 1200 & 2140 hours for lines 1 & line 2, respectively, in month #1.*

Each unit of PS requires 4 pounds of raw material while each unit of QT requires 3 pounds. Up to 710 units of raw material can be purchased at \$3 per pound. At the beginning of month 1, 10 units of PS and 5 units of QT are available. It costs \$10 to hold a unit of a unit of either product in inventory for a month.

a. Formulate a linear programming model to maximize Cornco's profit during this period. Be sure to define your decision variables!

Solution #1: Assume that the supply of 710 units of raw material is the total supply for all three months, and that the times on lines 1 & 2 in month 1 are 200 and 140, respectively, instead of the values shown in the textbook.

Define variables

Pt = # units of product PS produced in month t, t=1,2,3
Qt = # units of product QT produced in month t, t=1,2,3
R = (total) # units of raw material purchased
St = # units of product PS sold in month t, t=1,2,3
Tt = # units of product QT sold in month t, t=1,2,3
It = # units of product PS in inventory at end of month t, t=0,1,2
Jt = # units of product QT in inventory at end of month t, t=0,1,2

Objective: Maximize profit =

40S1 + 60S2 + 55S3	*(revenue from sale of PS)*
+35T1 + 40T2 + 44T3	*(revenue from sale of QT)*
- 3R	*(purchase of raw material)*
- 10I1 - 10I2	*(storage cost of PS)*
- 10J1 - 10 J2	*(storage cost of QT)*

Subject to the constraints:

R ≤ 710	*(limited availability of raw material)*
S1 ≤ 50, S2 ≤ 45, S3 ≤ 50	*(demand constraints for PS)*
T1 ≤ 43, T2 ≤ 50, T3 ≤ 40	*(demand constraints for QT)*
3P1 +2Q1 ≤ 200	*(hours available on line 1, month 1)*
3P2 +2Q2 ≤ 160	*(hours available on line 1, month 2)*
3P3 +2Q3 ≤ 190	*(hours available on line 1, month 3)*
2P1 +2Q1 ≤ 140	*(hours available on line 2, month 1)*

$2P2 + 2Q2 \leq 150$	*(hours available on line 2, month 2)*
$2P3 + 2Q3 \leq 110$	*(hours available on line 2, month 3)*
$P1 + I0 = 50 + S1 + I1$	*(material balance of PS, month 1)*
$P2 + I1 = 45 + S2 + I2$	*(material balance of PS, month 2)*
$P3 + I2 = 50 + S3$	*(material balance of PS, month 3)*
$Q1 + J0 = 43 + T1 + J1$	*(material balance of QT, month 1)*
$Q2 + J1 = 50 + T2 + J2$	*(material balance of QT, month 2)*
$Q3 + J2 = 40 + T3$	*(material balance of QT, month 3)*
$4P1+3Q1+4P2+3Q2+4P3+3Q3 \leq R$	*(consumption of raw material)*

Note: the upper bounds on R, St, Tt, etc. could be imposed either by using the "simple upper bound" (SUB) command or by adding a row to the problem. The former is preferred!

b. Solve the problem using LINDO (or equivalent LP solver.) Display the range analysis as well as the optimal tableau.

Solution:

LINDO output:

```
MAX     40 S1 + 60 S2 + 55 S3 + 35 T1 + 40 T2 + 44 T3 - 3 R - 10 I1
       - 10 I2 - 10 J1 - 10 J2
  SUBJECT TO
         2)   3 P1 + 2 Q1 <=   200
         3)   3 P2 + 2 Q2 <=   160
         4)   3 P3 + 2 Q3 <=   190
         5)   2 P1 + 2 Q1 <=   140
         6)   2 P2 + 2 Q2 <=   150
         7)   2 P3 + 2 Q3 <=   110
         8) - S1 - I1 + P1 + I0 =    0
         9) - S2 + I1 - I2 + P2 =    0
        10) - S3 + I2 + P3 =    0
        11) - T1 - J1 + Q1 + J0 =    0
        12) - T2 + J1 - J2 + Q2 =    0
        13) - T3 + J2 + Q3 =    0
        14) - R + 4 P1 + 3 Q1 + 4 P2 + 3 Q2 + 4 P3 + 3 Q3 <=   0
  END
  SUB       S1       50.00000
  SUB       S2       45.00000
  SUB       S3       50.00000
  SUB       T1       43.00000
  SUB       T2       50.00000
  SUB       T3       40.00000
  SUB        R      710.00000
  SUB       I0       10.00000
  SUB       J0        5.00000

        OBJECTIVE FUNCTION VALUE
        1)      7590.000

  VARIABLE        VALUE          REDUCED COST
        S1        40.000000          0.000000
        S2        45.000000        -10.000000
        S3        50.000000         -6.000000
        T1        20.000000          0.000000
        T2        50.000000         -5.000000
        T3         5.000000          0.000000
         R       710.000000         -2.000000
        I1        25.000000          0.000000
        I2         0.000000         11.000000
        J1         0.000000         10.000000
        J2         0.000000          1.000000
        P1        55.000000          0.000000
        Q1        15.000000          0.000000
        P2        20.000000          0.000000
```

Q2	50.000000	0.000000
P3	50.000000	0.000000
Q3	5.000000	0.000000
I0	10.000000	-40.000000
J0	5.000000	-35.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	5.000000	0.000000
3)	0.000000	10.000000
4)	30.000000	0.000000
5)	0.000000	10.000000
6)	10.000000	0.000000
7)	0.000000	14.500000
8)	0.000000	-40.000000
9)	0.000000	-50.000000
10)	0.000000	-49.000000
11)	0.000000	-35.000000
12)	0.000000	-35.000000
13)	0.000000	-44.000000
14)	0.000000	5.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
S1	40.000000	5.000000	1.000000
S2	60.000000	INFINITY	10.000000
S3	55.000000	INFINITY	6.000000
T1	35.000000	2.000000	5.000000
T2	40.000000	INFINITY	5.000000
T3	44.000000	1.000000	29.000000
R	-3.000000	INFINITY	2.000000
I1	-10.000000	1.500000	7.500000
I2	-10.000000	11.000000	INFINITY
J1	-10.000000	10.000000	INFINITY
J2	-10.000000	1.000000	INFINITY
P1	0.000000	6.000000	2.000000
Q1	0.000000	2.000000	5.000000
P2	0.000000	7.500000	1.500000
Q2	0.000000	1.000000	5.000000
P3	0.000000	INFINITY	6.000000
Q3	0.000000	6.000000	29.000000
I0	0.000000	INFINITY	40.000000
J0	0.000000	INFINITY	35.000000

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	200.000000	INFINITY	5.000000
3	160.000000	15.000000	3.750000
4	190.000000	INFINITY	30.000000
5	140.000000	11.500000	6.666667
6	150.000000	INFINITY	10.000000
7	110.000000	15.333333	3.333333
8	0.000000	40.000000	10.000000
9	0.000000	40.000000	10.000000
10	0.000000	5.000000	5.000000
11	0.000000	20.000000	23.000000
12	0.000000	15.000000	10.000000
13	0.000000	5.000000	35.000000
14	0.000000	5.000000	23.000000

THE TABLEAU

ROW	(BASIS)	S1	S2	S3	T1	T2	T3
1	ART	0.000	10.000	6.000	0.000	5.000	0.000
2	SLK 2	0.000	0.000	1.000	0.000	0.333	0.000
3	Q2	0.000	0.000	0.000	0.000	1.000	0.000
4	SLK 4	0.000	0.000	-1.000	0.000	0.000	0.000
5	S1	1.000	-1.000	-1.000	0.000	-1.000	0.000
6	SLK 6	0.000	0.000	0.000	0.000	-0.667	0.000
7	Q3	0.000	0.000	-1.000	0.000	0.000	0.000
8	I1	0.000	1.000	0.000	0.000	0.667	0.000
9	T1	0.000	0.000	1.000	1.000	0.333	0.000
10	P3	0.000	0.000	1.000	0.000	0.000	0.000
11	Q1	0.000	0.000	1.000	0.000	0.333	0.000
12	P1	0.000	0.000	-1.000	0.000	-0.333	0.000
13	T3	0.000	0.000	-1.000	0.000	0.000	1.000
14	P2	0.000	0.000	0.000	0.000	-0.667	0.000

ROW	R	I1	I2	J1	J2	P1	Q1
1	2.000	0.000	11.000	10.000	1.000	0.000	0.000
2	-1.000	0.000	1.000	0.333	-0.333	0.000	0.000
3	0.000	0.000	0.000	1.000	-1.000	0.000	0.000
4	0.000	0.000	-1.000	0.000	0.000	0.000	0.000
5	1.000	0.000	0.000	-1.000	1.000	0.000	0.000
6	0.000	0.000	0.000	-0.667	0.667	0.000	0.000
7	0.000	0.000	-1.000	0.000	0.000	0.000	0.000
8	0.000	1.000	-1.000	0.667	-0.667	0.000	0.000
9	-1.000	0.000	1.000	1.333	-0.333	0.000	0.000
10	0.000	0.000	1.000	0.000	0.000	0.000	0.000
11	-1.000	0.000	1.000	0.333	-0.333	0.000	1.000
12	1.000	0.000	-1.000	-0.333	0.333	1.000	0.000
13	0.000	0.000	-1.000	0.000	-1.000	0.000	0.000
14	0.000	0.000	0.000	-0.667	0.667	0.000	0.000

ROW	P2	Q2	P3	Q3	I0	J0	SLK 2
1	0.000	0.000	0.000	0.000	40.000	35.000	0.000
2	0.000	0.000	0.000	0.000	0.000	0.000	1.000
3	0.000	1.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	1.000	0.000	0.000
6	0.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.000	0.000	0.000	1.000	0.000	0.000	0.000
8	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9	0.000	0.000	0.000	0.000	0.000	1.000	0.000
10	0.000	0.000	1.000	0.000	0.000	0.000	0.000
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000
14	1.000	0.000	0.000	0.000	0.000	0.000	0.000

ROW	SLK 3	SLK 4	SLK 5	SLK 6	SLK 7	SLK 14	
1	10.000	0.000	10.000	0.000	14.500	5.000	7590.000
2	1.333	0.000	0.500	0.000	1.500	-1.000	5.000
3	0.000	0.000	0.000	0.000	0.000	0.000	50.000
4	0.000	1.000	0.000	0.000	-1.000	0.000	30.000
5	-1.000	0.000	-1.500	0.000	-1.500	1.000	40.000
6	-0.667	0.000	0.000	1.000	0.000	0.000	10.000
7	0.000	0.000	0.000	0.000	0.500	0.000	5.000
8	-0.333	0.000	0.000	0.000	0.000	0.000	25.000
9	1.333	0.000	2.000	0.000	1.500	-1.000	20.000

10	0.000	0.000	0.000	0.000	0.000	0.000	50.000
11	1.333	0.000	2.000	0.000	1.500	-1.000	15.000
12	-1.333	0.000	-1.500	0.000	-1.500	1.000	55.000
13	0.000	0.000	0.000	0.000	0.500	0.000	5.000
14	0.333	0.000	0.000	0.000	0.000	0.000	20.000
16	0.500	0.000	0.000	0.000	5.000		

c. Describe the optimal solution in a few sentences (in such a way that the plant manager could easily understand the production plan).

Solution:

In month 1, produce 55 units of PS (in addition to the initial inventory of 10 units) and 15 units of QT (in addition to the initial inventory of 5 units). Sell 40 units of PS, and store 25 units. Sell 20 units of QT, leaving no inventory.

In month 2, produce 20 units of PS and 50 units of QT. Sell 45 units of PS (the 20 units produced in month 2, plus the 25 units from inventory), leaving nothing in inventory. Sell 50 units of QT, leaving no inventory.

In month 3, produce 50 units of PS and 5 units of QT. Sell 50 units of PS (the 50 units produced this month) and 5 units of QT (the 5 units produced this month).

Answer the questions below, using the output above for the original problem, if possible. If not possible, you need not run LINDO again.

d. Find the new optimal solution if it costs $11 to hold a unit of PS in inventory at the end of month 1.

Solution: An increase in storage cost would translate as a decrease in the objective (profit) coefficient of I1. The "allowable decrease" in the objective coefficient of I1 is $1.50, and since the $1 increase in cost is less than $1.50, the current solution remains opitimal, although the objective value would be lowered by ($1/unit of inventory)(25 units of inventory) = $25.

e. Find the company's new optimal solution if 210 hours on line 1 are available during month 1.

Solution: The new value 210 hours is an increase of 10 hours. Checking the right-hand-side range of row 2, we see that the "allowable increase" is INFINITY. This is obvious if we notice that there are 5 slack hours in this constraint, i.e., not all of the currently available time is being used. There will therefore be no change in the optimal solution.

f. Find the company's new profit level if 109 hours are available on line 2 during month 3.

Solution: Row 7 imposes the restriction on hours used on line 2 during month 3. The specified value (109) is a decrease of 1 hour in the currently available time. The "dual price" for row 7 is $14.50/hour, and the "allowable decrease" in row 7 is 3.3333 hours, so that we can say that the profit will be lowered by $14.50 to $7590-14.50 = $7575.50.

g. What is the most Cornco should be willing to pay for an extra hour of line 1 time during month 2?

Solution: Row 3 imposes the restriction on hours used on line 1 during month 2. The "dual price" of this row is $10/hour, i.e. the marginal rate of improvement (increase) in the profit is $10 per hour avaiilable on line 1 during month 2. The "allowable increase" is 15, and so up to 15 hours would each be worth $10.

h. What is the most Cornco should be willing to pay for an extra hour of line 1 time during month 3?

Solution: : Row 4 imposes the restriction on hours used on line 1 during month 3. The "dual price" of this row is $0, since there are 30 unused hours on that line in month 3. Therefore, the company should not be willing to pay for any increase in hours during that month.

i. Find the new optimal solution if PS sells for $50 during month 2.

Solution: The current selling price is $60, and so this would be a decrease of $10 in the profit coefficient of the variable S2. The "allowable decrease" of the objective (profit) coefficient of the variable S2 is $10.00, and so the basis would not change. The value of the objective function (profit) would, however, decrease by $10/unit for each of the 45 units sold, i.e., the profit would decrease by $450.

j. Find the new optimal solution if QT sells for $50 during month 3.

Solution: QT currently sells for \$44 in month 3, so this would be an increase of \$6. The "allowable increase" in the objective coefficient of T3 is 1.00, so the basis would change. (Presumably, the number of units of QT sold in month 3, which is currently 5, would increase due to the increased profit which could be obtained.) The new optimal solution cannot be determined easily without running LINDO again with the revised profit coefficient.

k. Suppose spending \$20 on advertising would increase demand for QT in month 2 by 5 units. Should the advertising be done?

Solution: The method of answering this depends upon whether you used a row or a SUB command to impose the sales limit of 50 units. In the former case, you would consult the "dual price" of the row imposing the sales limit to find the increase in profit per unit of sales; if more than \$4/unit (\$20/5 units) <u>and</u> the allowable increase is at least 5, then the answer is "yes". In the case above, I've used the SUB command to impose the sales limit, and so I must consult the "reduced cost" of the variable T2 (sales of QT in month 2). This value is **-**\$5/unit (= rate of "deterioration" in the profit as T2 is increased). Since a negative deterioration is an improvement, this means that each additional unit which could be sold would increase the profit by \$5 (before accounting for the additional advertising cost). Therefore, the advertising expense cannot be justified.

ROW	P2	Q2	P3	Q3	I0	J0	SLK 2
1	0.000	0.000	0.000	0.000	40.000	35.000	0.000
2	0.000	0.000	0.000	0.000	0.000	0.000	1.000
3	0.000	1.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	1.000	0.000	0.000
6	0.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.000	0.000	0.000	1.000	0.000	0.000	0.000
8	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9	0.000	0.000	0.000	0.000	0.000	1.000	0.000
10	0.000	0.000	1.000	0.000	0.000	0.000	0.000
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000
14	1.000	0.000	0.000	0.000	0.000	0.000	0.000

ROW	P2	Q2	P3	Q3	I0	J0	SLK 2
1	0.000	0.000	0.000	0.000	40.000	35.000	0.000
2	0.000	0.000	0.000	0.000	0.000	0.000	1.000
3	0.000	1.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	1.000	0.000	0.000
6	0.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.000	0.000	0.000	1.000	0.000	0.000	0.000
8	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9	0.000	0.000	0.000	0.000	0.000	1.000	0.000
10	0.000	0.000	1.000	0.000	0.000	0.000	0.000
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000
14	1.000	0.000	0.000	0.000	0.000	0.000	0.000

♦♦♦♦♦♦♦♦♦♦♦♦♦♦♦♦♦♦♦♦♦♦♦♦♦♦♦

Solution #2: Use the values given in the textbook for the hours available on lines 1 & 2 in month 1, namely 1200 & 2140. Assume also that the limit of 710 units of raw material applies to each month, rather than the total period.

Define <u>variables</u>

Pt = # units of product PS produced in month t, t=1,2,3

Qt = # units of product QT produced in month t, t=1,2,3
Rt = # units of raw material purchased in month t, t=1,2,3
St = # units of product PS sold in month t, t=1,2,3
Tt = # units of product QT sold in month t, t=1,2,3
It = # units of product PS in inventory at end of month t, t=0,1,2
Jt = # units of product QT in inventory at end of month t, t=0,1,2

Note: The variables Rt could be eliminated, but that would limit an analysis of the sensitivity of the solution to the price of raw materials!

Objective: Maximize profit =

40S1 + 60S2 + 55S3 *(revenue from sale of PS)*
+35T1 + 40T2 + 44T3 *(revenue from sale of QT)*
- 3R1 - 3R2 - 3R3 *(purchase of raw material)*
- 10I1 - 10I2 *(storage cost of PS)*
- 10J1 - 10 J2 *(storage cost of QT)*
Note: it is assumed that raw material cannot be stored!

Subject to the constraints:

$R_t \leq 710$, t=1,2,3	*(limited availability of raw material)*
$S1 \leq 50$, $S2 \leq 45$, $S3 \leq 50$	*(demand constraints for PS)*
$T1 \leq 43$, $T2 \leq 50$, $T3 \leq 40$	*(demand constraints for QT)*
$3P1 + 2Q1 \leq 1200$	*(hours available on line 1, month 1)*
$3P2 + 2Q2 \leq 160$	*(hours available on line 1, month 2)*
$3P3 + 2Q3 \leq 190$	*(hours available on line 1, month 3)*
$2P1 + 2Q1 \leq 2140$	*(hours available on line 2, month 1)*
$2P2 + 2Q2 \leq 150$	*(hours available on line 2, month 2)*
$2P3 + 2Q3 \leq 110$	*(hours available on line 2, month 3)*
$P1 + I0 = 50 + S1 + I1$	*(material balance of PS, month 1)*
$P2 + I1 = 45 + S2 + I2$	*(material balance of PS, month 2)*
$P3 + I2 = 50 + S3$	*(material balance of PS, month 3)*
$Q1 + J0 = 43 + T1 + J1$	*(material balance of QT, month 1)*
$Q2 + J1 = 50 + T2 + J2$	*(material balance of QT, month 2)*
$Q3 + J2 = 40 + T3$	*(material balance of QT, month 3)*
$4P1 + 3Q1 \leq R1$	*(consumption of raw material, month 1)*
$4P2 + 3Q2 \leq R2$	*(consumption of raw material, month 2)*
$4P3 + 3Q3 \leq R3$	*(consumption of raw material, month 3)*

Note: the upper bounds on Rt, St, etc. could be imposed either by using the "simple upper bound" (SUB) command or by adding a row to the problem. The former is preferred!

b. Solve the problem using LINDO (or equivalent LP solver.) Display the range analysis as well as the optimal tableau.

Solution:

LINDO output:

```
MAX     40 S1 + 60 S2 + 55 S3 + 35 T1 + 40 T2 + 44 T3 - 3 R1 - 3 R2
     - 3 R3 - 10 I1 - 10 I2 - 10 I3 - 10 J1 - 10 J2
SUBJECT TO
       2)   3 P1 + 2 Q1 <=   1200
       3)   3 P2 + 2 Q2 <=   160
       4)   3 P3 + 2 Q3 <=   190
       5)   2 P1 + 2 Q1 <=   2140
       6)   2 P2 + 2 Q2 <=   150
       7)   2 P3 + 2 Q3 <=   110
       8) - S1 - I1 + P1 + I0 =    0
       9) - S2 + I1 - I2 + P2 =    0
      10) - S3 + I2 + P3 =    0
      11) - T1 - J1 + Q1 + J0 =    0
      12) - T2 + J1 - J2 + Q2 =    0
      13) - T3 + J2 + Q3 =    0
```

```
      14) - R1 + 4 P1 + 3 Q1 <=   0
      15) - R2 + 4 P2 + 3 Q2 <=   0
      16) - R3 + 4 P3 + 3 Q3 <=   0
 END
 SUB       S1       50.00000
 SUB       S2       45.00000
 SUB       S3       50.00000
 SUB       T1       43.00000
 SUB       T2       50.00000
 SUB       T3       40.00000
 SUB       R1      710.00000
 SUB       R2      710.00000
 SUB       R3      710.00000
 SUB       I0       10.00000
 SUB       J0        5.00000

      OBJECTIVE FUNCTION VALUE
      1)      9043.000

 VARIABLE        VALUE          REDUCED COST
       S1        50.000000        -28.000000
       S2        45.000000        -38.000000
       S3        50.000000        -23.000000
       T1        43.000000        -26.000000
       T2        50.000000        -21.000000
       T3        40.000000        -15.000000
       R1       474.000000          0.000000
       R2       235.000000          0.000000
       R3       215.000000          0.000000
       I1        35.000000          0.000000
       I2         0.000000          0.000000
       I3         0.000000         10.000000
       J1        20.000000          0.000000
       J2        35.000000          0.000000
       P1        75.000000          0.000000
       Q1        58.000000          0.000000
       P2        10.000000          0.000000
       Q2        65.000000          0.000000
       P3        50.000000          0.000000
       Q3         5.000000          0.000000
       I0        10.000000        -12.000000
       J0         5.000000         -9.000000

      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)       859.000000          0.000000
       3)         0.000000          0.000000
       4)        30.000000          0.000000
       5)      1874.000000          0.000000
       6)         0.000000          5.000000
       7)         0.000000         10.000000
       8)         0.000000        -12.000000
       9)         0.000000        -22.000000
      10)         0.000000        -32.000000
      11)         0.000000         -9.000000
      12)         0.000000        -19.000000
      13)         0.000000        -29.000000
      14)         0.000000          3.000000
      15)         0.000000          3.000000
      16)         0.000000          3.000000

 RANGES IN WHICH THE BASIS IS UNCHANGED:

                            OBJ COEFFICIENT RANGES
```

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
S1	40.000000	INFINITY	28.000000
S2	60.000000	INFINITY	38.000000
S3	55.000000	INFINITY	23.000000
T1	35.000000	INFINITY	26.000000
T2	40.000000	INFINITY	21.000000
T3	44.000000	INFINITY	15.000000
R1	-3.000000	0.000000	5.000000
R2	-3.000000	3.000000	0.000000
R3	-3.000000	3.000000	0.000000
I1	-10.000000	0.000000	5.000000
I2	-10.000000	0.000000	INFINITY
I3	-10.000000	10.000000	INFINITY
J1	-10.000000	3.333333	0.000000
J2	-10.000000	20.000000	0.000000
P1	0.000000	0.000000	5.000000
Q1	0.000000	3.333333	0.000000
P2	0.000000	5.000000	0.000000
Q2	0.000000	0.000000	3.333333
P3	0.000000	INFINITY	0.000000
Q3	0.000000	0.000000	20.000000
I0	0.000000	INFINITY	12.000000
J0	0.000000	INFINITY	9.000000

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	1200.000000	INFINITY	859.000000
3	160.000000	35.000000	10.000000
4	190.000000	INFINITY	30.000000
5	2140.000000	INFINITY	1874.000000
6	150.000000	10.000000	35.000000
7	110.000000	30.000000	10.000000
8	0.000000	59.000000	75.000000
9	0.000000	59.000000	35.000000
10	0.000000	5.000000	20.000000
11	0.000000	78.666664	58.000000
12	0.000000	78.666664	20.000000
13	0.000000	78.666664	20.000000
14	0.000000	474.000000	236.000000
15	0.000000	235.000000	475.000000
16	0.000000	215.000000	495.000000

THE TABLEAU

ROW	(BASIS)	S1	S2	S3	T1	T2	T3
1	ART	28.000	38.000	23.000	26.000	21.000	15.000
2	SLK 2	-3.000	-3.000	-2.000	-2.000	-2.000	-2.000
3	P2	0.000	0.000	0.000	0.000	0.000	0.000
4	SLK 4	0.000	0.000	-1.000	0.000	0.000	0.000
5	SLK 5	-2.000	-2.000	-2.000	-2.000	-2.000	-2.000
6	R2	0.000	0.000	0.000	0.000	0.000	0.000
7	R3	0.000	0.000	1.000	0.000	0.000	0.000
8	R1	4.000	4.000	3.000	3.000	3.000	3.000
9	I1	0.000	1.000	0.000	0.000	0.000	0.000
10	P3	0.000	0.000	1.000	0.000	0.000	0.000
11	P1	1.000	1.000	0.000	0.000	0.000	0.000
12	J1	0.000	0.000	1.000	0.000	1.000	1.000
13	J2	0.000	0.000	1.000	0.000	0.000	1.000
14	Q1	0.000	0.000	1.000	1.000	1.000	1.000
15	Q2	0.000	0.000	0.000	0.000	0.000	0.000
16	Q3	0.000	0.000	-1.000	0.000	0.000	0.000

ROW	R1	R2	R3	I1	I2	I3	J1
1	0.000	0.000	0.000	0.000	0.000	10.000	0.000
2	0.000	0.000	0.000	0.000	1.000	0.000	0.000
3	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.000	-1.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000	0.000
6	0.000	1.000	0.000	0.000	0.000	0.000	0.000
7	0.000	0.000	1.000	0.000	1.000	0.000	0.000
8	1.000	0.000	0.000	0.000	-1.000	0.000	0.000
9	0.000	0.000	0.000	1.000	-1.000	0.000	0.000
10	0.000	0.000	0.000	0.000	1.000	0.000	0.000
11	0.000	0.000	0.000	0.000	-1.000	0.000	0.000
12	0.000	0.000	0.000	0.000	1.000	0.000	1.000
13	0.000	0.000	0.000	0.000	1.000	0.000	0.000
14	0.000	0.000	0.000	0.000	1.000	0.000	0.000
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000
16	0.000	0.000	0.000	0.000	-1.000	0.000	0.000

ROW	J2	P1	Q1	P2	Q2	P3	Q3
1	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3	0.000	0.000	0.000	1.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000	0.000
6	0.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.000	0.000	0.000	0.000	0.000	0.000	0.000
8	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10	0.000	0.000	0.000	0.000	0.000	1.000	0.000
11	0.000	1.000	0.000	0.000	0.000	0.000	0.000
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000
13	1.000	0.000	0.000	0.000	0.000	0.000	0.000
14	0.000	0.000	1.000	0.000	0.000	0.000	0.000
15	0.000	0.000	0.000	0.000	1.000	0.000	0.000
16	0.000	0.000	0.000	0.000	0.000	0.000	1.000

ROW	I0	J0	SLK 2	SLK 3	SLK 4	SLK 5	SLK 6
1	12.000	9.000	0.000	0.000	0.000	0.000	5.000
2	3.000	2.000	1.000	1.000	0.000	0.000	0.000
3	0.000	0.000	0.000	1.000	0.000	0.000	-1.000
4	0.000	0.000	0.000	0.000	1.000	0.000	0.000
5	2.000	2.000	0.000	0.000	0.000	1.000	1.000
6	0.000	0.000	0.000	1.000	0.000	0.000	0.500
7	0.000	0.000	0.000	0.000	0.000	0.000	0.000
8	-4.000	-3.000	0.000	-1.000	0.000	0.000	-0.500
9	0.000	0.000	0.000	-1.000	0.000	0.000	1.000
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000
11	-1.000	0.000	0.000	-1.000	0.000	0.000	1.000
12	0.000	0.000	0.000	1.000	0.000	0.000	-1.500
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000
14	0.000	-1.000	0.000	1.000	0.000	0.000	-1.500
15	0.000	0.000	0.000	-1.000	0.000	0.000	1.500
16	0.000	0.000	0.000	0.000	0.000	0.000	0.000

ROW	SLK 7	SLK 14	SLK 15	SLK 16	
1	10.000	3.000	3.000	3.000	9043.000
2	1.000	0.000	0.000	0.000	859.000
3	0.000	0.000	0.000	0.000	10.000
4	-1.000	0.000	0.000	0.000	30.000
5	1.000	0.000	0.000	0.000	1874.000
6	0.000	0.000	-1.000	0.000	235.000
7	1.500	0.000	0.000	-1.000	215.000

8	-1.500	-1.000	0.000	0.000	474.000
9	0.000	0.000	0.000	0.000	35.000
10	0.000	0.000	0.000	0.000	50.000
11	0.000	0.000	0.000	0.000	75.000
12	-0.500	0.000	0.000	0.000	20.000
13	-0.500	0.000	0.000	0.000	35.000
14	-0.500	0.000	0.000	0.000	58.000
15	0.000	0.000	0.000	0.000	65.000
16	0.500	0.000	0.000	0.000	5.000

c. Describe the optimal solution in a few sentences (in such a way that the plant manager could easily understand the production plan).

Solution:

In month 1, purchase 474 units of raw material, which is used to produce 75 units of PS and 58 units of QT. Sell 50 units of PS, and store 35 units. Sell 43 units of QT and store 20 units.

In month 2, purchase 235 units of raw material, which is used to produce 10 units of PS and 65 units of QT. Sell 45 units of PS (the 10 units produced in month 2, plus the 35 units from inventory), leaving nothing in inventory. Sell 50 units of QT, leaving 15 units to be added to the 20 units already in inventory (total 35).

In month 3, purchase 215 units of raw material, which is used to produce 50 units of PS and 5 units of QT. Sell 50 units of PS (the 50 units produced this month) and 40 units of QT (the 5 units produced this month, plus the 35 units in inventory).

Answer the questions below, using the output above for the original problem, if possible. If not possible, you need not run LINDO again.

d. Find the new optimal solution if it costs $11 to hold a unit of PS in inventory at the end of month 1.

Solution: An increase in storage cost would translate as a decrease in the objective (profit) coefficient of I1. The "allowable decrease" in the objective coefficient of I1 is $5, and since the $1 increase in cost is less than $5, the current solution remains opitimal, although the objective value would be lowered by ($1/unit of inventory)(35 units of inventory) = $35.

e. Find the company's new optimal solution if 210 hours on line 1 are available during month 1.

Solution: A reduction to 210 hours is a reduction of 1200-210=990 hours. Checking the right-hand-side range of row 2, we see that the "allowable decrease" is only 859 (the current "slack" in that constraint). Therefore the basis will change (and hence, the values of the basic variables). Obtaining the new solution would require using LINDO to solve the problem with the new right-hand-side.

f. Find the company's new profit level if 109 hours are available on line 2 during month 3.

Solution: Row 7 imposes the restriction on hours used on line 2 during month 3. The "dual price" for row 7 is $10/hour, and the "allowable decrease" in row 7 is 10 hours, so that we can say that the profit will be lowered by $10 to $9033.

g. What is the most Cornco should be willing to pay for an extra hour of line 1 time during month 2?

Solution: Row 3 imposes the restriction on hours used on line 1 during month 2. The "dual price" of this row is $0/hour (although there is no "slack" in that constraint, i.e. the solution is degenerate). The "allowable increase" is 35, and so the value of one extra hour of time on this line is zero.

h. What is the most Cornco should be willing to pay for an extra hour of line 1 time during month 3?

Solution: : Row 4 imposes the restriction on hours used on line 1 during month 3. The "dual price" of this row is $0, since there are 30 unused hours on that line in month 3. Therefore, the company should not be willing to pay for any increase in hours during that month.

i. Find the new optimal solution if PS sells for $50 during month 2.

Solution: The current selling price is $60, and so this would be a decrease of $10 in the profit coefficient of the variable S2. The "allowable decrease" of the objective (profit) coefficient of the variable S2 is $38, and so the basis would not change, nor would the values of any variables, although, because S2=45, the profit would go down by ($10/unit)(45 units) = $450.

j. Find the new optimal solution if QT sells for $50 during month 3.

Solution: QT currently sells for $44 in month 3, so this would be an increase of $6. The "allowable increase" in the objective coefficient of T3 is INFINITY (since regardless of the increase in profitability of QT, only 40 units can be sold during month 3), so the basis would not change, nor would the values of any basic variables, but because 40 units of QT are sold in month 3, the profit would go up by ($6/unit)(40 units)

k. Suppose spending $20 on advertising would increase demand for QT in month 2 by 5 units. Should the advertising be done?

Solution: The method of answering this depends upon whether you used a row or a SUB command to impose the sales limit of 50 units. In the former case, you would consult the "dual price" of the row imposing the sales limit to find the increase in profit per unit of sales; if more than $4/unit ($20/5 units) and the allowable increase is at least 5, then the answer is "yes". In the case above, I've used the SUB command to impose the sales limit, and so I must consult the "reduced cost" of the variable T2 (sales of QT in month 2). This value is -$38/unit (rate of deterioration as T2 is increased). Since a negative deterioration is an improvement, this means that each additional unit which could be sold would increase the profit by $38. Assuming that T2 could be increase by at least 1 unit without changing the basis, the answer would be "yes", the advertising should be done. (To determine whether the basis must be changed would require performing a minimum ratio test, using the substitution rates found in the tableau.)

Note: I suspect a typographical error in the table of available hours on the two production lines during month 1: probably 1200 for line 1 should be 200 and 2140 for line 2 should be 140!

2. *Production Planning* *for a Shoe Company, Problem 3, page 349::* A shoe company forecasts the following demands during the next 6 months:

Month	demand
October	200
November	260
December	240
January	340
February	190
March	150

It costs $7 to produce a pair of shoes with regular-time labor, and $11 with overtime labor. During each month, regular production is limited to 200 pairs of shoes and overtime production to 100 pairs of shoes. It costs $1 per month to hold a pair of shoes in inventory.

a. Formulate a balanced transportation problem (i.e., provide a transportation tableau, explaining the meaning of "shipments" associated with each cell) to minimize the total cost of meeting the next 6 months' demand on time.

		OCT	NOV	DEC	JAN	FEB	MAR	Unused Capacity	supply
OCT	RT	7	8	9	10	11	12	0	200
	OT	11	12	13	14	15	16	0	100
NOV	RT		7	8	9	10	11	0	200
	OT		11	12	13	14	15	0	100
DEC	RT			7	8	9	10	0	200
	OT			11	12	13	14	0	100
JAN	RT				7	8	9	0	200
	OT				11	12	13	0	100
FEB	RT					7	8	0	200
	OT					11	12	0	100
MAR	RT						7	0	200
	OT						11	0	100
demand		200	260	240	340	190	150	420	1800

b. Apply the "Least-cost" method to find a feasible solution. What is the total cost?
Solution: The solution found by this method is degenerate, having only 17 positive shipments, whereas the required number of basic variables is (m+n-1) = 12 + 7 - 1 = 18. In order to determine which zero shipment should be basic, one should not remove both a row and a column when simultaneously reducing a row supply and a column demand to zero. This will then result in inserting a zero shipment in that row or that column at a later iteration. The total cost of the solution below is $10660.

		OCT	NOV	DEC	JAN	FEB	MAR	Unused Capacity	supply
OCT	RT	200 [7]	**0** [8]	[9]	[10]	[11]	[12]	[0]	200
	OT	[11]	[12]	[13]	[14]	[15]	[16]	100 [0]	100
NOV	RT		200 [7]	[8]	[9]	[10]	[11]	[0]	200
	OT		60 [11]	[12]	[13]	[14]	[15]	40 [0]	100
DEC	RT			200 [7]	[8]	[9]	[10]	[0]	200
	OT			40 [11]	40 [12]	[13]	[14]	20 [0]	100
JAN	RT				200 [7]	[8]	[9]	[0]	200
	OT				100 [11]	[12]	[13]	[0]	100
FEB	RT					190 [7]	[8]	10 [0]	200
	OT					[11]	[12]	100 [0]	100
MAR	RT						150 [7]	50 [0]	200
	OT						[11]	100 [0]	100
demand		200	260	240	340	190	150	420	1800

c. Compute a set of dual variables, and use them to test your solution for optimality. Is there any cell (shipment) with a negative reduced cost?
Solution: Any one of the dual variable may be assigned any arbitrary value-- I have chosen to assign U1 = 0 initially. This then determines exactly all of the remaining dual variables, as shown below. (First V_1 and V_2 are computed, then using the value of V_2, U_3 and U_4 are computed, then using U_4, V_7 is computed, etc.)

U:			OCT	NOV	DEC	JAN	FEB	MAR	Unused Capacity	supply
V:			7	8	8	9	4	4	-3	
0	OCT	RT	200 / 7	0 / 8	9	10	11	12	0	200
3	OCT	OT	11	12	13	14	15	16	100 / 0	100
-1	NOV	RT		200 / 7	8	9	10	11	0	200
3	NOV	OT		60 / 11	12	13	14	15	40 / 0	100
-1	DEC	RT			200 / 7	8	9	10	0	200
3	DEC	OT			40 / 11	40 / 12	13	14	20 / 0	100
-2	JAN	RT				200 / 7	8	9	0	200
2	JAN	OT				100 / 11	12	13	0	100
3	FEB	RT					190 / 7	8	10 / 0	200
3	FEB	OT					11	12	100 / 0	100
3	MAR	RT						150 / 7	50 / 0	200
3	MAR	OT						11	100 / 0	100
demand			200	260	240	340	190	150	420	1800

Using the dual variables, the reduced cost of each nonbasic cell can next be computed, using the formula $C_{ij} - (U_i + V_j)$. For example, the reduced cost of the cell in row 5, column 4 (regular-time production in November) is 8 - (9-1) = 0. This indicates that increasing the shipment in this cell will have no effect on the objective function. After computing all the reduced costs, we find that all of them are either positive or zero, and that the current solution is therefore optimal (although not the only optimal solution-- increasing X54

d. If the above solution is not optimal, perform an iteration of the transportation simplex method to improve the solution. What is the improvement in the cost? *(You need not continue further to find the optimal solution.)*

3. *(Problem 6, page 350)* A bank has two sites at which checks are processed. Site 1 can process 10,000 checks per day and site 2 can process 6000 checks per day. The bank processes three types of checks: vendor checks, salary checks, and personal checks. Each day 5000 checks of each type must be processed. The processing cost per check depends on the site at which the check is processed:

type	Site #1	Site #2
vendor checks	5¢	3¢
salary checks	4¢	4¢
personal checks	2¢	5¢

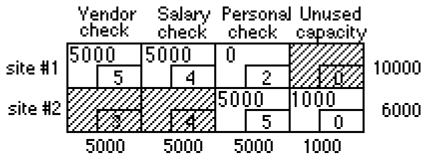
(a.) Formulate a balanced transportation problem to minimize the daily cost of processing checks. (That is, provide the transportation tableau for the problem.) What is the number of basic variables in any basic solution to this problem?

	Vendor check	Salary check	Personal check	Unused capacity	
site #1	5	4	2	0	10000
site #2	3	4	5	0	6000
	5000	5000	5000	1000	

Solution: The number of basic variables must be (m+n-1)=2+4-1=5

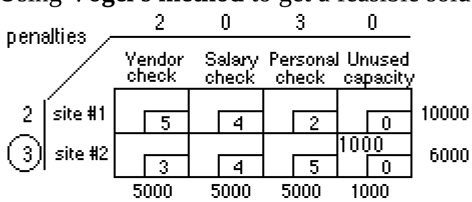
(b.) Use both the Northwest-Corner and Vogel's Approximation Method to find a basic feasible solution to the problem. Compute the total cost for each solution.

Solution: **Northwest-Corner method** yields:

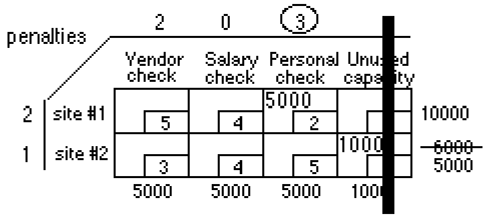


Note that at the second step, when inserting the shipment in row 1, column 2, the shipment of 5000 uses all the remaining supply and fills all the demand in that row & column. This indicates that the resulting solution will be degenerate, i.e., that one (or more) basic variables will equal zero. In this situation, one should remove the row or the column, but not both! In the above case, I removed the second column, leaving row 1, column 3 as the new "northwest" corner, and since the remaining supply in the first row is zero, place a "0" in that cell, and remove row 1. Row 2, column 3 then becomes the new "northwest" corner, etc. I could have removed row 1 instead of column 2 after the second step above, in which case the cell in row 2, column 2 would have become basic (but with a zero shipment). The choice is arbitrary. The cost of this solution is 5(5000) +4(5000) +2(0) +5(5000) +0(1000) = 70000¢, i.e., $700.

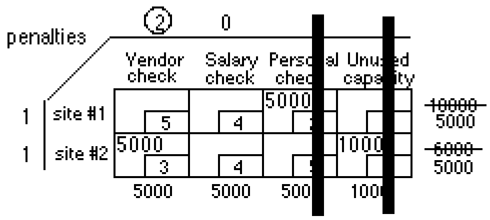
Using **Vogel's method** to get a feasible solution, we first compute the penalties:



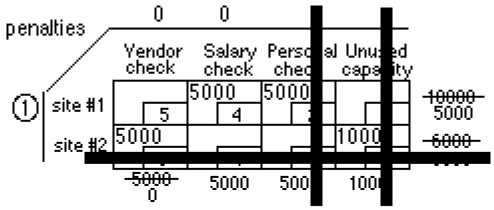
Step (1): The largest penalty is 3 (tie!). Selecting row 2 as indicated, and the smallest cost in that row (0 in column 4), we make a shipment of min{1000, 6000} = 1000. This fills all of the demand for this destination, and so the column is removed and the penalties re-computed:



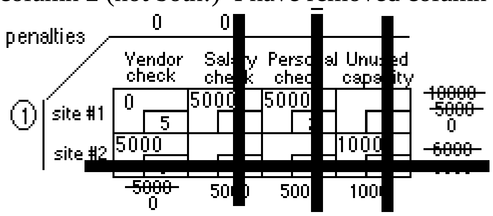
Step (2): The largest penalty is again 3 in column 3, and so we select that column and the smallest cost in that column (2 in row 1). Make a shipment of min{5000, 10000} = 5000, filling all the demand for this destination, so that column 3 is next removed. Re-computing the penalties yields:



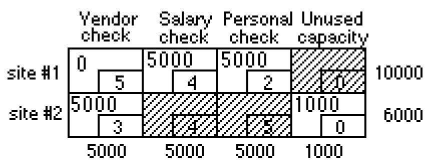
Step (3): The largest penalty is now 2 in column 1. The smallest cost in that column is 3 in row 2, so we make a shipment in that cell of min{5000, 5000} = 5000 (tie!), using all the remaining supply in row 2 and filling all the demand in column 1. This indicates that the resulting solution will be degenerate. We should next remove either row 2 or column 1 *(but not both!)* and re-compute the penalties. Here, I've removed row 2:



Step (4): The largest penalty is now 1 in row 1, and the smallest cost in that row is 4 in column 2. We make a shipment of min{5000,5000} = 5000 as shown, which uses all the remaining supply in that row and fills all the demand in that column. Again, however, I should remove only row 1 or column 2 (not both!) I have removed column 2:

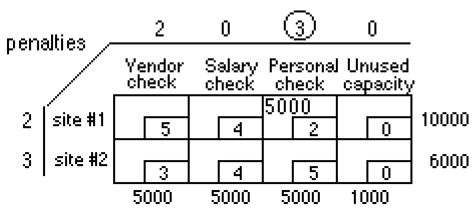


Step (5): The largest penalty is now 1 in row 1, and the smallest cost is 5 in column 1. We then make a shipment of min{0,0} = 0 in this cell, yielding the (degenerate) feasible solution:

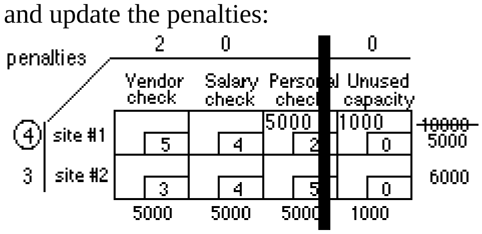


The cost of this solution is 3(5000) + 4(5000) + 2(5000) + 0(1000) = 90000 cents = $900, considerably less than the solution found by the Northwest corner method.

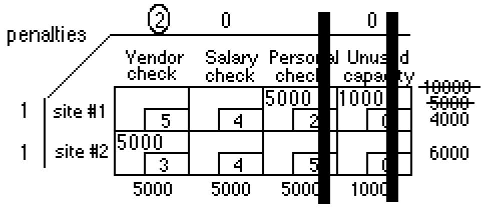
Suppose that I had broken the tie differently in choosing the largest penalty in the first step, by choosing column 3 instead of row 2:



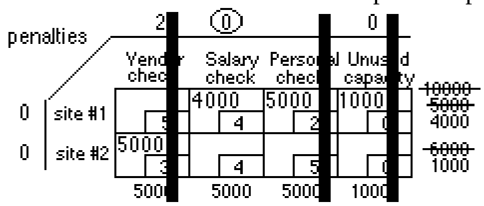
Step (1): We make a shipment of min{5000, 10000} =5000 in row 1, column 3, remove column 3, and update the penalties:



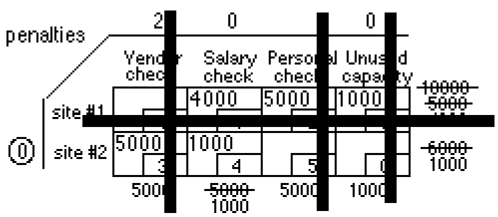
Step (2): Next, select row 1 and column 4, and make a shipment of min{1000, 5000} = 1000. Then remove column 4 and recompute the penalties:



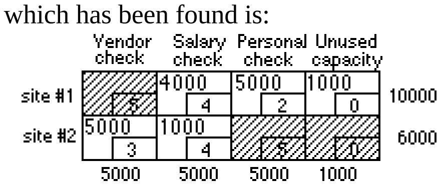
Step (3): Next, select column 1 and row 1, and make a shipment of min{5000, 6000} = 5000 in this cell. Remove column 1 and recompute the penalties:



Step (4): Now we select column 2, and (breaking a tie!) row 1, making a shipment of min{4000, 5000} = 4000. Remove row 1, leaving only a single cell:



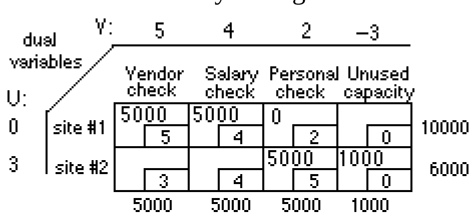
Step (5): The shipment in this cell (row 2, column 2) is 1000. The feasible (non-degenerate) solution which has been found is:



The cost of this solution is 3(5000) +4(4000) +4(1000) +2(5000) + 0(1000) = 45000 cents = $450, half the cost of the earlier solution found by Vogel's method!

(c.) Starting with the Northwest-Corner solution, perform the simplex algorithm to find the optimal solution to this problem. At each iteration, state the values of the dual variables and reduced costs of each nonbasic "shipment".

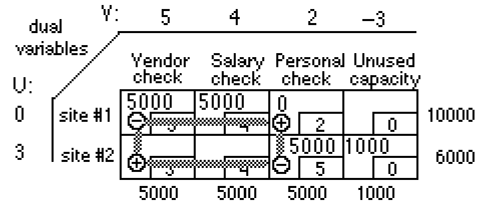
Solution: Arbitrarily setting the dual variable U1 = 0, we get the following set of dual variables:



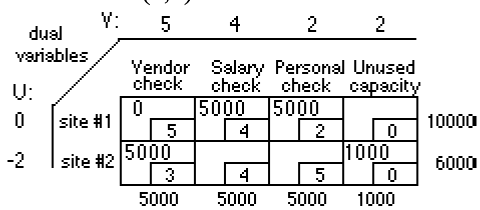
We next compute the reduced costs of the nonbasic cells:

row 1, column 4: 0 - (0-3) = +3
row 2, column 1: 3 - (3+5) = -5
row 2, column 2: 4 - (3+4) = -4

Either X_{21} or X_{22} can enter the basis. Let's somewhat arbitrarily select X_{21}. Increasing the shipment in this cell will require (in order to keep the supplies & demands balanced) decreasing the shipments in cell (1,1) and cell (2,3) and increasing the cell (1,3):



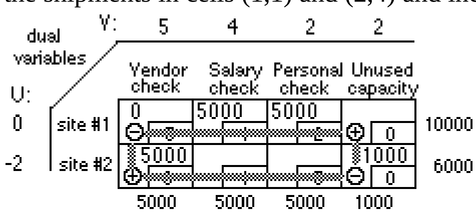
As the shipment in cell (2,1) increases to 5000, the shipments in cells (1,1) and (2,3) both decrease to zero simultaneously. We must select one of these two cells to leave the basis. I've arbitrarily selected cell (1,1) to remain in the basis. The dual variables must now be recomputed:



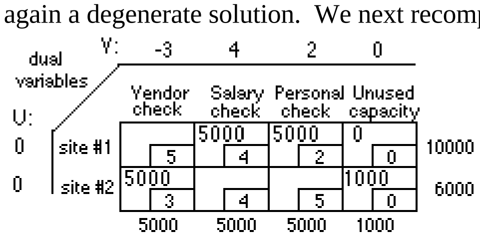
We next compute the new reduced costs of the nonbasic cells:

row 1, column 4: 0 - (0+2) = -2
row 2, column 2: 4 - (4-2) = +2
row 2, column 3: 5 - (2-2) = +5

Therefore, X_{14} in cell (1,4) should enter the basis. Increasing this shipment will require decreasing the shipments in cells (1,1) and (2,4) and increasing the shipment in cell (2,1):



As we try to increase the shipment in cell (1,4), we find that we are immediately "blocked" by the required decrease in cell (1,1), which is already zero. Therefore, we must change the basis by replacing cell (1,1) by cell (1,4), but not changing any of the values of the shipments, obtaining again a degenerate solution. We next recompute the dual variables:



The new reduced costs are:

row 1, column 1: 5 - (0-3) = +2
row 2, column 2: 4 - (0+4) = 0
row 2, column 3: 5 - (0+2) = +3

The reduced costs are nonnegative, and so the current solution is optimal (although the zero reduced cost of cell (2,2) indicates that it could enter the basis without changing the optimality. If this were done, we would get a different (nondegenerate) solution. The cost of these solutions is 45000 cents = $450.

(d.) How far from optimal (as a % of the optimal cost) was the solution found by Vogel's Approximation Method?

Solution: The optimal solution is the same cost as one of the two solutions which would be found by Vogel's method. The other solution found by Vogel's method has a cost which is double the optimal solution!

❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ Homework #6 ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍

(1.) Project Scheduling: Hawkeye Construction Co. has prepared the following table listing the tasks required to complete construction of a house:

Task #	Task Description	Immediate Predecessor(s)	Expected time (days)	Standard Deviation (days)
1	Walls & ceilings	2	5	1
2	Foundation	none	μ	σ
3	Roof timbers	1	2	0
4	Roof sheathing	3	3	0.5
5	Electrical wiring	1	4	1
6	Roof shingles	4	8	2
7	Exterior siding	8	5	1
8	Windows	1	2	0
9	Paint	6,7,10	2	0
10	Inside wallboard	8,5	3	0.5

a. Draw an A-O-A (activity on arrow) network representing this project. Are any "dummy" tasks required?

b. Number the nodes so that if there is an arrow from node i to node j, then i<j.

c. Suppose that the most likely completion time for task #2 is estimated to be 2 days, with the optimistic and pessimistic estimates 1 day and 4 days, respectively. Assuming a Beta distribution, what is the expected time and standard deviation for this task?

In parts (c)-(g), assume that the expected completion times will be the actual completion times:
c. Compute the early times for each node.

d. What is the earliest completion time for the house?

e. Compute the latest times for each node in order to complete the house as early as possible.

f. For each task, compute the

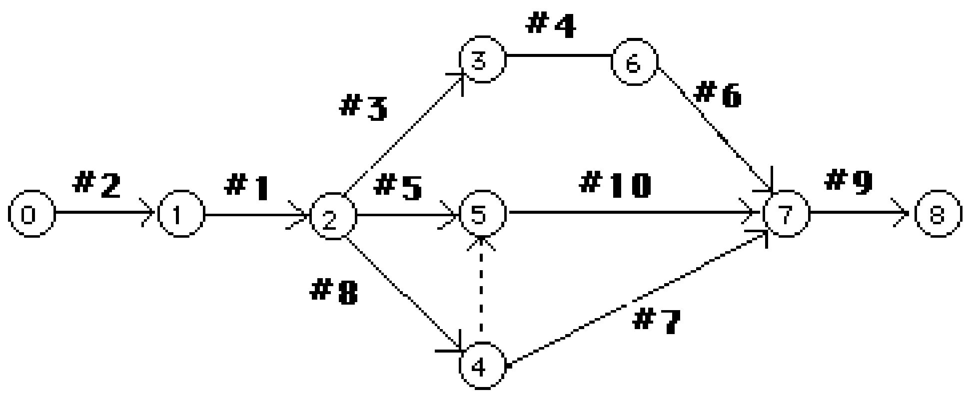
- Early Start Time
- Early Finish Time
- Late Start Time
- Late Finish Time
- Total Float ("slack")
- Free Float ("slack")

g. Which tasks are "critical"?

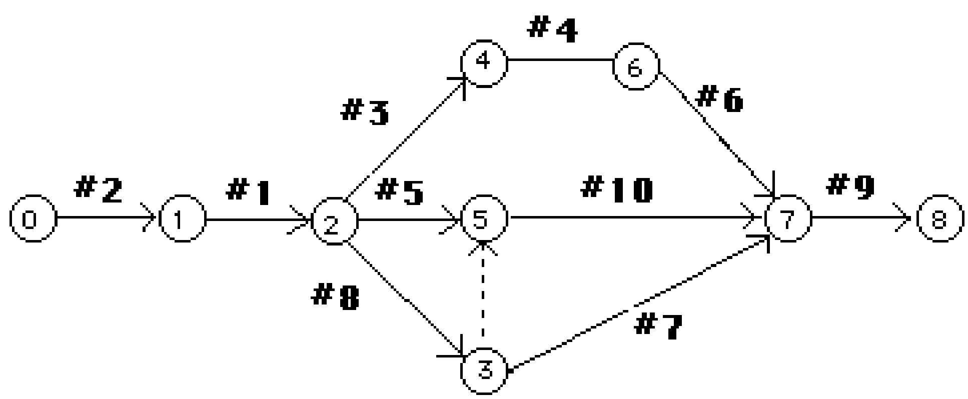
h. What, according to the assumptions of PERT (i.e., the Central Limit Theorem), is the probability distribution of the completion time of the house?

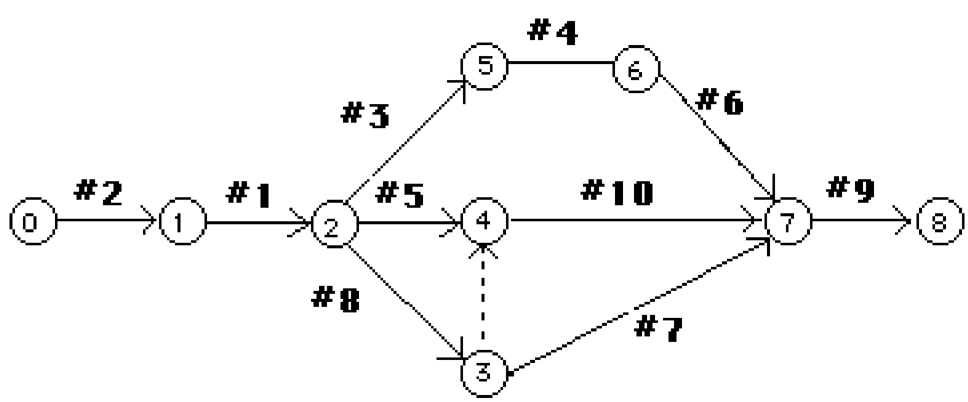
i. What is the probability that the house can be completed within a time which is no greater than your answer to (d) plus one additional day?

j. Draw an A-O-N (activity on node) network representing this same project.
(1.) Project Scheduling
a. One "dummy" activity (shown as the dotted line) is required for the A-O-A diagram:



b. The nodes may be numbered as above, or as in either diagram below:

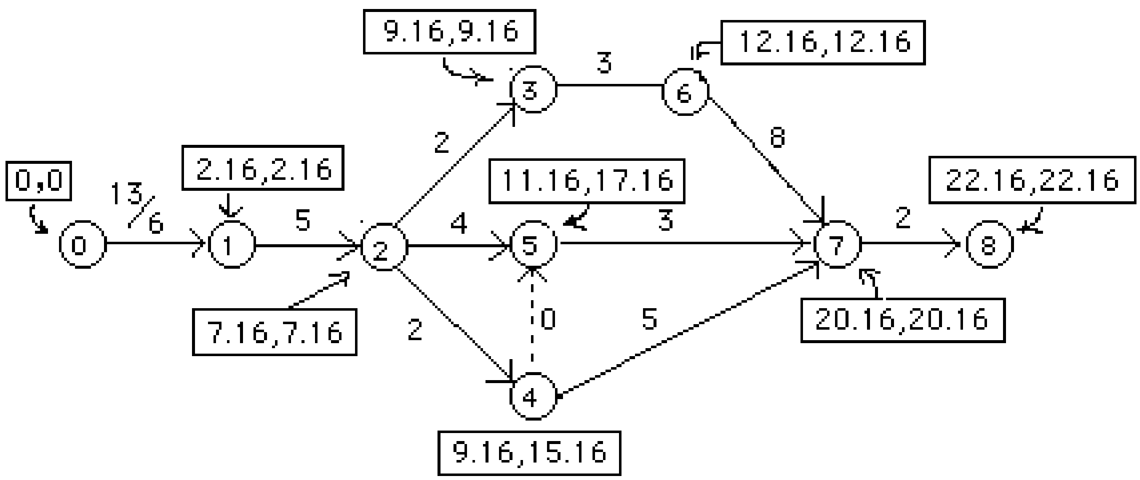




c. Using the parameters a=1, b=4, m=2 (optimistic, pessimistic, and most likely times, respectively), the mean and standard deviation are computed as follows:

$$\mu = \frac{a+4m+b}{6} = \frac{13}{6},\ \sigma = \frac{b-a}{6} = \frac{1}{2}$$

d. & e. The Early Times ET(i), and the Late Times LT(i) of each node i,are shown in the boxes below:

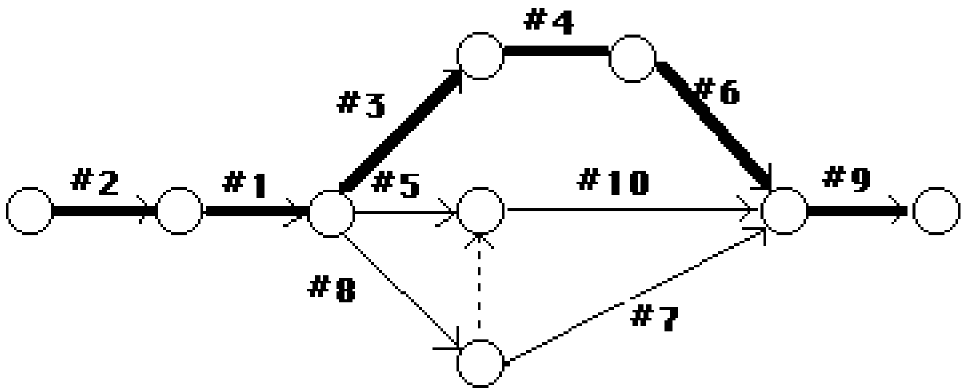


f.

critical	I	D	ES	EF	LS	LF	TS	FS
**	1	5.00	2.16	7.16	2.16	7.16	0.00	0.00
**	2	2.16	0.00	2.16	0.00	2.16	0.00	0.00
**	3	2.00	7.16	9.16	7.16	9.16	0.00	0.00
**	4	3.00	9.16	12.16	9.16	12.16	0.00	0.00
	5	4.00	7.16	11.16	13.16	17.16	6.00	0.00
**	6	8.00	12.16	20.16	12.16	20.16	0.00	0.00
	7	5.00	9.16	14.16	15.16	20.16	6.00	6.00
	8	2.00	7.16	9.16	13.16	15.16	6.00	0.00
**	9	3.00	20.16	23.16	20.16	23.16	0.00	0.00
	10	3.00	11.16	14.16	17.16	20.16	6.00	6.00

I: node #
D: duration
ES: early start time
EF: early finish time
LS: late start time
LF: late finish time
TS: total slack
FS: free slack

g. The critical activities are those marked with the asterisk above, i.e., those with zero total slack (float). These are #1, 2, 3, 4, 6, & 9:



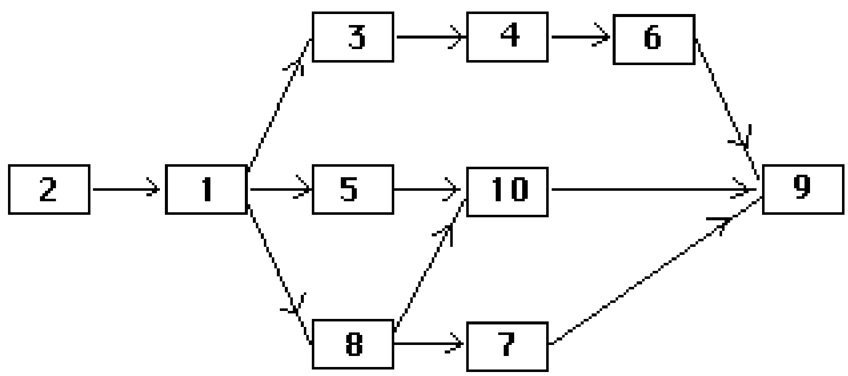
h. According to the assumptions of PERT (i.e., the Central Limit Theorem), the completion time of the house will have a normal distribution, with mean which is the sum of the mean times of the critical activities, and variance which is the sum of the variances of the critical activities. Therefore, the mean is $5+\mu+2+3+8+2=22\ ^1/_6$ (where $\mu=2\ ^1/_6$), and the variance is $1^2 + \sigma^2 + 0 + (0.5)^2 + 2^2 + 0 = 5.5$ (where $\sigma = 0.5$). The standard deviation of the project completion time is therefore approximately 2.345 days.

i. The probability that the house can be completed within a time which is no greater than the expected completion time ($22\ ^1/_6$ days) plus one additional day can be found from a table for the standard normal distribution:

$$P\left\{T \leq 23\tfrac{1}{6}\right\} = P\left\{\frac{T\text{-}22.16}{2.345} \leq \frac{23.16\text{-}22.16}{2.345}\right\} = P\{Z \leq 0.4264\} \approx 0.66$$

where Z has the N(0.1) distribution.

j. An A-O-N (activity on node) network representing this same project is shown below:



2. ***Decision Analysis:*** (Exercise 1, page 731 of textbook) Pizza King and Noble Greek are two competing restaurants. Each must determine simultaneously whether to undertake small, medium, or large advertising campaigns. Pizza King believes that it is equally liely that Noble Greek will undertake a small, medium, or a large advertising campaign. Given the actions chosen by each restaurant, Pizza King's profits are:

Pizza King chooses	Noble Greek Chooses Small	Medium	Large
Small	\$6000	\$5000	\$2000
Medium	\$5000	\$6000	\$1000
Large	\$9000	\$6000	\$0

Determine Pizza King's choice of advertising campaigns for each of the following criteria:

a. Maximize minimum profit
b. Maximize maximum profit
c. Minimize maximum regret

3. ***Decision Trees:*** (2, page 753 of textbook) The Decision Sciences Department is trying to determine which of two copying machines to purchase. Both machines will satisfy the department's needs for the next ten years. Machine 1 costs $2000 and has a maintenance agreement, which, for an annual fee of $150, covers all repairs. machine 2 costs $3000, and its annual maintenance cost is a random variable. At present, the department believes that there is a 40% chance that the annual maintenance cost for machine 2 will be $0, a 40% chance that it will be $100, and a 20% chance that it will be $200.

Before the purchase decision is made, the department can have a trained repairman evaluate the quality of machine 2. If the repairman believes that machine 2 is satisfactory, there is a 60% chance that the annual maintenance cost will be $0 and a 40% chance that it will be $100. If the repairman believes that machine 2 is unsatisfactory, there is a 20% chance that the annual maintenance cost will be $0, a 40% chance that it will be $100, and a 40% chance that it will be $200. If there is a 50% chance that the repairman will give a satisfactory report, what is **EVSI** (expected value of sample information, i.e., the expected value of the repairman's evaluation)? IF the repairman charges $40, what should the Decision Sciences Department do? What is the **EVPI** (expected value of perfect information)?

4. ***Bayes' Rule:*** Acme Manufacturing produces "widgets". Depending upon whether the manufacturing process is "in control" or "out of control", the defective rate will be either 4% (acceptable) or 15% (unacceptable), respectively. Denote these "states of nature" by S_1 and S_2, respectively. Based upon historical data, Acme estimates a 5% chance that a manufactured lot of widgets will be unacceptable. Instead of shipping lots based solely on prior probabilities, a test sample of two items is taken from each lot, which gives rise to three possible outcomes:

O_1: Both items are good
O_2: One item is good
O_3: Both items are defective

a. Compute (assuming the binomial distribution):

$P\{O_1|S_1\}$ = P{both items are good | process is in control}
$P\{O_2|S_1\}$ = P{one item is good | process is in control}
$P\{O_3|S_1\}$ = P{both items are defective | process is in control}
$P\{O_1|S_2\}$ = P{both items are good | process is out of control}
$P\{O_2|S_2\}$ = P{one item is good | process is out of control}
$P\{O_3|S_2\}$ = P{both items are defective | process is out of control}

b. Determine the *posterior* probabilities

$P\{S_1|O_1\}$ = P{process is in control | both items good}
$P\{S_1|O_2\}$ = P{process is in control | one item is good}
$P\{S_1|O_3\}$ = P{process is in control | both items are defective}
$P\{S_2|O_1\}$ = P{process is out of control | both items good}
$P\{S_2|O_2\}$ = P{process is out of control | one item is good}
$P\{S_2|O_3\}$ = P{process is out of control | both items are defective}

❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ Homework #7 ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍

1. The board of directors of General Wheel Corporation is considering 7 large capital investments. These investments differ in the estimated long-run profit (net present value) they will generate, as well as in the amount of capital required, as shown by the following table (in units of millions of dollars):

	Investment Opportunity						
	1	2	3	4	5	6	7
Estimated profit	17	10	15	19	7	13	9
Capital required	43	28	34	48	17	32	23

The total amount of capital available for these investments is 100 million dollars. Investment opportunities 1 and 2 are mutually exclusive, and so are 3 and 4. Furthermore, neither 3 nor 4 can be undertaken unless either 1 or 2 (or both) is undertaken. There are no such restrictions on investment opportunities 5, 6, and 7. The objective is to select the combination of capital investments that will maximize the total estimated long-run profit (net present value).

a. Formulate an integer linear program for this problem, using binary variables.

Solution: This belongs to a class of problems known as "knapsack" problems, but with some additional constraints.

Define X_j = 1 if investment opportunity #j is selected, and
0 otherwise.

Then the objective is

$$\text{Maximize } 17X_1 + 10X_2 + 15X_3 + 19X_4 + 7X_5 + 13X_6 + 9X_7$$

and the budget constraint is

$$43X_1 + 28X_2 + 34X_3 + 48X_4 + 17X_5 + 32X_6 + 23X_7 \leq 100$$

Additional constraints are:

$X_1 + X_2 \leq 1$ (#1 & #2 are mutually exclusive)

$X_3 + X_4 \leq 1$ (#3 & #4 are mutually exclusive)

The final constraint, "neither #3 or #4 may be selected unless either #1 or #2 are selected", may be stated thus:

$$X_3 \leq X_1 + X_2$$

$$X_4 \leq X_1 + X_2$$

or, since #3 and #4 are mutually exclusive, by the single constraint

$$X_3 + X_4 \leq X_1 + X_2$$

b. Using LINDO, find the optimal solution.
(Note that the variables are specified to be binary by the command "INTEGER X1", etc., after "END". LINDO forces each INTEGER variable to be either 0 or 1.)

Solution: The LINDO output follows:

```
MAX     17 X1 + 10 X2 + 15 X3 + 19 X4 + 7 X5 + 13 X6 + 9 X7
SUBJECT TO
       2)   43 X1 + 28 X2 + 34 X3 + 48 X4 + 17 X5 + 32 X6 + 23 X7 <=   100
       3)   X1 + X2 <=   1
       4)   X3 + X4 <=   1
       5) - X1 - X2 + X3 <=   0
       6) - X1 - X2 + X4 <=   0
END
INTE     7

LP OPTIMUM FOUND AT STEP     27
OBJECTIVE VALUE =   41.4375000
FIX ALL VARS.(    4)  WITH RC >   .343750
SET       X6 TO <=         0 AT    1, BND=   39.000000    TWIN=0.10000000E+31
DELETE       X6 AT LEVEL     1
RELEASE FIXED VARIABLES

NEW INTEGER SOLUTION OF   41.00000     AT BRANCH      3 PIVOT     36
```

```
      OBJECTIVE FUNCTION VALUE
      1)    41.00000

  VARIABLE              VALUE             REDUCED COST
    X1                1.000000               .000000
    X2                0.000000               .000000
    X3                1.000000             -.526315
    X4                0.000000               .000000
    X5                 .000000               .157895
    X6                 .000000               .473684
    X7                1.000000               .000000

     ROW            SLACK OR SURPLUS            DUAL PRICES
    2)                 .000000                   .421053
    3)                 .000000                 -1.105263
    4)                 .000000                  -.526315
    5)                0.000000                   .000000
    6)                1.000000                   .000000
    7)                 .000000                   .684211

 NO. ITERATIONS=      36
 BRANCHES=    3 DETERM.=-38.000E  0
 FIX ALL VARS.(    4)  WITH RC >  0.300007E-01
 SET       X3 TO <=         0 AT    1, BND=   33.000000   TWIN=0.10000000E+31
 DELETE       X3 AT LEVEL     1
 ENUMERATION COMPLETE. BRANCHES=     4 PIVOTS=    46

 LAST INTEGER SOLUTION IS THE BEST FOUND
```

That is, the optimal set of investments consists of #1, #3, and #7.

2. **Integer Programming Model Formulation.** Coach Night is trying to choose the starting linerup for the basketball team. The team consists of seven players who have been rated (on a scale of 1=poor to 3=excellent) according to their ball-handling, shooting, rebounding, and defensive abilities. The positions that each player is allowed to play (G=guard, C=center, F=forward) and the player's abilities are:

Player	Position	Ball-handling	Shooting	Rebounding	Defense
1	G	3	3	1	3
2	C	2	1	3	2
3	G-F	2	3	2	2
4	F-C	1	3	3	1
5	G-F	1	3	1	2
6	F-C	3	1	2	3
7	G-F	3	2	2	1

The five-player starting lineup must satisfy the following restrictions:
(i) At least 3 members must be able to play guard,
at least 2 members must be able to play forward,
and at least one member must be able to play center.
(ii) The average ball-handling, shooting, and rebounding level of the starting lineup must each be at least 2.
(iii) If player 3 starts, then player 6 cannot start.
(iv) If player 1 starts, then players 4 and 5 must both start.
(v) Either player 2 or player 3 (or both) must start.
Given these constraints, Coach wants to maximize the total defensive ability of the starting team. Formulate an integer LP that will help him choose his starting team, and use LINDO (or other software) to find the optimal solution.

Solution:
Define ***variables***:

$$X_j = \begin{cases} 1 \text{ if player } i \text{ is selected} \\ 0 \text{ otherwise} \end{cases}$$

Constraints:

The number of players selected must be exactly five:

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 = 5$$

At least 3 members must be able to play guard:

$$X_1 + X_3 + X_5 + X_7 \geq 3$$

At least 2 members must be able to play forward:

$$X_3 + X_5 + X_7 \geq 2$$

At least one member must be able to play center:

$$X_2 + X_4 + X_6 \geq 1$$

The average ball-handling level of the starting lineup must be at least 2:

$$\frac{3X_1 + 2X_2 + 2X_3 + X_4 + X_5 + 3X_6 + 3X_7}{5} \geq 2 \Rightarrow 3X_1 + 2X_2 + 2X_3 + X_4 + X_5 + 3X_6 + 3X_7 \geq 10$$

The average shooting level of the starting lineup must be at least 2:

$$\frac{3X_1 + X_2 + 3X_3 + 3X_4 + 3X_5 + X_6 + 2X_7}{5} \geq 2 \Rightarrow 3X_1 + X_2 + 3X_3 + 3X_4 + 3X_5 + X_6 + 2X_7 \geq 10$$

The average rebounding level of the starting lineup must be at least 2:

$$\frac{X_1 + 3X_2 + 2X_3 + 3X_4 + X_5 + 2X_6 + 2X_7}{5} \geq 2 \Rightarrow X_1 + 3X_2 + 2X_3 + 3X_4 + X_5 + 2X_6 + 2X_7 \geq 10$$

If player 3 starts, then player 6 cannot start:

$$1 - X_3 \geq X_6 \Leftrightarrow X_3 + X_6 \leq 1$$

If player 1 starts, then players 4 and 5 must both start:

$$X_1 \leq X_4 \;\&\; X_1 \leq X_5 \Rightarrow 2X_1 \leq X_4 + X_5$$

Note: the single inequality on the right is equivalent the pair of inequalities on the left if all variables are binary. However, if they are continuous variables restricted to the interval [0,1], the single inequality is implied by the pair on the left, but not vice-versa. In ILP, it is better for the sake of computational efficiency to use the pair of inequalities, which gives a smaller feasible region for the LP obtained by relaxing the integer restrictions.

Either player 2 or player 3 (or both) must start:

$$X_2 + X_3 \geq 1$$

Objective: Maximize the total defensive ability of the team:

$$\text{Maximize } 3X_1 + 2X_2 + 2X_3 + X_4 + 2X_5 + 3X_6 + X_7$$

LINDO output:

```
 MAX     3 X1 + 2 X2 + 2 X3 + X4 + 2 X5 + 3 X6 + X7
  SUBJECT TO
         2)   X1 + X2 + X3 + X4 + X5 + X6 + X7 =    5
         3)   X1 + X3 + X5 + X7 >=   3
         4)   X3 + X5 + X7 >=   2
         5)   X2 + X4 + X6 >=   1
         6)   3 X1 + 2 X2 + 2 X3 + X4 + X5 + 3 X6 + 3 X7 >=   10
         7)   3 X1 + X2 + 3 X3 + 3 X4 + 3 X5 + X6 + 2 X7 >=   10
         8)   X1 + 3 X2 + 2 X3 + 3 X4 + X5 + 2 X6 + 2 X7 >=   10
         9)   X3 + X6 <=   1
```

```
       10)   X1 - X4 <=   0
       11)   X1 - X5 <=   0
       12)   X2 + X3 >=   1
 END
 INTE     7
```

Solution:

```
LP OPTIMUM FOUND AT STEP     17
OBJECTIVE VALUE =   9.71428585

SET       X3 TO <=     0 AT    1, BND=   9.000     TWIN=-0.1000E+31     28

NEW INTEGER SOLUTION OF    9.00000000     AT BRANCH      1 PIVOT      28
BOUND ON OPTIMUM:  9.000000
DELETE       X3 AT LEVEL     1
ENUMERATION COMPLETE. BRANCHES=     1 PIVOTS=      28

LAST INTEGER SOLUTION IS THE BEST FOUND
RE-INSTALLING BEST SOLUTION...

       OBJECTIVE FUNCTION VALUE
       1)      9.000000

 VARIABLE        VALUE          REDUCED COST
       X1         1.000000         -3.000000
       X2         1.000000         -2.000000
       X3         0.000000         -2.000000
       X4         1.000000         -1.000000
       X5         1.000000         -2.000000
       X6         0.000000         -3.000000
       X7         1.000000         -1.000000

      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)         0.000000          0.000000
       3)         0.000000          0.000000
       4)         0.000000          0.000000
       5)         1.000000          0.000000
       6)         0.000000          0.000000
       7)         2.000000          0.000000
       8)         0.000000          0.000000
       9)         1.000000          0.000000
      10)         0.000000          0.000000
      11)         0.000000          0.000000
      12)         0.000000          0.000000

NO. ITERATIONS=      28
BRANCHES=    1 DETERM.=  1.000E    0
```

That is, the starting lineup should consist of players 1, 2, 4, 5, and 7.

❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ Homework #8 ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍

1. Integer LP Modeling. The R&D Division of a company has been developing four possible new product lines. Management must now make a decision as to which of these four products actually will be produced, and at what levels.
A substantial cost is associated with beginning the production of any product, as given in the first row of the following table. The marginal net revenue from each unit produced is given in the second row of the table.

	Product 1	2	3	4
Startup cost ($thousands)	50	40	70	60
Marginal revenue ($)	70	60	90	80

Define continuous decision variables X_1, X_2, X_3, and X_4 to be the production levels of products 1 through 4, respectively. Management has imposed the following policy constraints on these variables:

i) No more than two of the products may be produced.
ii) Either product 3 or 4 can be produced only if either product 1 or 2 is produced.
iii) Depending upon the type of machine installed on the production line, the limited capacity of the production line requires that either

$$5X_1 + 3X_2 + 6X_3 + 4X_4 \leq 6000 \text{ if machine 1 is installed,}$$

or

$$4X_1 + 6X_2 + 3X_3 + 5X_4 \leq 6000 \text{ if machine 2 is installed.}$$

a. Formulate the problem as an integer LP with binary variables.

Solution: Define the continuous variables X1 through X4 as suggested, plus the following additional decision variables:

Y_i = 1 if product i is to be produced, 0 otherwise (i=1,2,3,4)

Z_i = 1 if machine i is installed on the production line, 0 otherwise (i=1,2)

Then the objective will be to maximize profits (revenue minus startup costs):

$$\text{Maximize } 70X_1 + 60X_2 + 90X_3 + 80X_4 - 50000Y_1 - 40000Y_2 - 70000Y_3 - 60000Y_4$$

The constraints will be:

$Y_1 + Y_2 + Y_3 + Y_4 \leq 2$ (at most 2 products may be produced)

$Y_3 \leq Y_1 + Y_2$ (Product 3 cannot be produced unless either #1 or #2 are produced)

$Y_4 \leq Y_1 + Y_2$ (Likewise for product 4)

The two alternative constraints in (iii) are

$5X_1 + 3X_2 + 6X_3 + 4X_4 \leq 6000$ if $Z_1=1$

$4X_1 + 6X_2 + 3X_3 + 5X_4 \leq 6000$ if $Z_2=1$

These can be stated as follows, where "M" is a suitably large number:

$5X_1 + 3X_2 + 6X_3 + 4X_4 \leq 6000 + M(1-Z_1)$

$4X_1 + 6X_2 + 3X_3 + 5X_4 \leq 6000 + M(1-Z_2)$

For example, if Z_1 =1, the right-hand-side of the first constraint of this pair is 6000, while if Z_1 =0, the right-hand-side is a "large" number. How big should "M" be? Probably a value of 10,000 will be large enough, and will be used in LINDO.
We also need constraints

$Z_1 + Z_2 = 1$

Finally, we need constraints which ensure that , for each product i, if Y_i is zero, then X_i must be zero:

$X_i \leq N_i Y_i$, i=1,2,3,4

where N_i is a "large" number, at least as large as the maximum capacity for product i. Therefore, we may use $N_1=N_4=1500$ (=max{6000/5, 6000/4}), and $N_2=N_3=2000$ (=max{6000/3, 6000/6}).

b. Using LINDO, find the optimal solution.

Solution: The LINDO output follows:

```
MAX  -50000Y1 -40000Y2 -70000Y3 -60000Y4 +70X1 +60X2 +90X3 +80X4
SUBJECT TO
     2)      Y1 + Y2 + Y3 + Y4 <=   2
     3)      - Y1 - Y2 + Y3 <=   0
     4)      - Y1 - Y2 + Y4 <=   0
     5)      10000 Z1 + 5 X1 + 3 X2 + 6 X3 + 4 X4 <=   16000
     6)      10000 Z2 + 4 X1 + 6 X2 + 3 X3 + 5 X4 <=   16000
     7)      Z1 + Z2 =    1
     8)      - 1500 Y1 + X1 <=   0
     9)      - 2000 Y2 + X2 <=   0
    10)      - 2000 Y3 + X3 <=   0
    11)      - 1500 Y4 + X4 <=   0
 END
 INTE       Y1
 INTE       Y2
 INTE       Y3
 INTE       Y4
 INTE       Z1
 INTE       Z2

 LP OPTIMUM FOUND AT STEP     64
 OBJECTIVE VALUE =   116111.100
 SET       Z1 TO >=         1 AT    1, BND=   80000.000     TWIN=  82500.000

 NEW INTEGER SOLUTION OF   80000.00     AT BRANCH      5 PIVOT     73
       OBJECTIVE FUNCTION VALUE

        1)    80000.0000

   VARIABLE        VALUE              REDUCED COST
      Y1              .000000          50000.000000
      Y2             1.000000           4000.000000
      Y3              .000000          58000.000000
      Y4              .000000          24000.000000
      Z1             1.000000         140000.000000
      Z2              .000000               .000000
      X1              .000000               .000000
      X2          2000.000000               .000000
      X3              .000000               .000000
      X4              .000000               .000000


      ROW         SLACK OR SURPLUS         DUAL PRICES
       2)            1.000000               .000000
       3)            1.000000               .000000
       4)            1.000000               .000000
       5)             .000000             14.000000
       6)         4000.000000               .000000
       7)             .000000               .000000
       8)             .000000               .000000
       9)             .000000             18.000000
      10)             .000000              6.000000
      11)             .000000             24.000000

 BOUND ON OPTIMUM:  82500.00
 FLIP       Z1 TO <=         0 AT    1 WITH BND=    82500.000
 SET       Y1 TO >=         1 AT    2, BND=   60000.000     TWIN=  70000.000
 DELETE       Y1 AT LEVEL     2
 DELETE       Z1 AT LEVEL     1
 ENUMERATION COMPLETE. BRANCHES=     6 PIVOTS=    81

 LAST INTEGER SOLUTION IS THE BEST FOUND
```

That is, the optimal solution is to install machine #1 on the production line and to produce only product #2. The maximum amount of this product which can be produced is 2000. This will provide a profit of $80000.

56:171 Operations Research
Homework #9 - Due Wednesday, November 11, 1998

1. Let X_n denote the quality of the n^{th} item produced by a production system, with $X_n = 0$ meaning "good" and $X_n = 1$ meaning "defective". The qualities of two successive items produced are not independent; an item is much more likely to be defective if it follows production of a defective item than if the preceding item is non-defective. Suppose that we treat this system as a Markov chain with transition probabilities

$$P = \begin{bmatrix} .98 & .02 \\ .60 & .40 \end{bmatrix}$$

That is, the probability that the next item to be produced is "defective", given that the latest item is "good", is 2 %, while if the latest item is "defective", that probability that the next item is defective is 40%.

(You may use the ***MARKOV*** *workspace to compute the answers to the following questions, or you may do the computations manually.)*

a. Draw a diagram with two nodes, representing this Markov Chain.

b. What is the probability that the fourth item is defective, given that the **first** item is defective? ....given that the **first** item is non-defective?

c. Are we assured that this Markov Chain will have a steady-state distribution? (What property is required? Is this property present here?)

d. Write down the equations which determine the steady-state distribution. If they have a solution, solve for π.

e. What percentage of the items from this production line can we expect to be defective?

f. If the first item produced is "good", what is the probability that the first *defective* item is the **third** item produced. What is the name used for this probability?

g. If the first item produced is "good", what is the expected number of items produced before the *first* defective item is produced?

❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ Homework #10❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍

1. Consider an inventory system in which the number of items on the shelf is checked at the end of each day. The maximum number on the shelf is 8. If 3 or fewer units are on the shelf, the shelf is refilled overnight. The demand distribution is as follows:

x	0	1	2	3	4	5	6
P{D=x}	0.1	0.15	0.25	0.25	0.15	0.05	0.05

(We assume that there is never a demand for more than six units during any day.)

The system is modeled as a Markov chain, with the state defined as the number of units on the shelf at the end of each day. The probability transition matrix is:

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from	1	2	3	4	5	6	7	8	9
1	0	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1
2	0	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1
3	0	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1
4	0	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1
5	0.25	0.25	0.25	0.15	0.1	0	0	0	0
6	0.1	0.15	0.25	0.25	0.15	0.1	0	0	0
7	0.05	0.05	0.15	0.25	0.25	0.15	0.1	0	0
8	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1	0
9	0	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1

a. Explain the derivation of the values P_{19}, P_{35}, P_{51}, P_{83} above. (Note that state 1=inventory level 0, etc.)

Solution: P_{19} = P{shelf is full at end of day tomorrow | shelf is empty at end of day today}
= P{demand = 0} = 10%

P_{35} = P{4 units on shelf at end of day tomorrow | 2 units on shelf at end of day today}
= P{ demand = 4 } = 15% (since shelf will have been replenished and have 8 units)

P_{51} = P{demand ≥ 4} = P{demand = 4 or 5 or 6} = 15% + 5% + 5% = 25%

P_{83} = P{demand = 5} = 5%

The steady-state distribution of the above Markov chain is:

```
Steady State Distribution
-----------------------------

i                  Pi
-            -------------

1            0.06471513457
2            0.07698357218
3            0.1304613771
4            0.1355295351
5            0.16322964
6            0.1698706746
7            0.1384131423
8            0.0754980776
9            0.04529884656
```

b. Write two of the equations which define this steady-state distribution. How many equations must be solved to yield the solution above?

Solution: One equation states that the sum of the probabilities is 1.00, i.e.,

$$\sum_{n=1}^{9} \pi_n = 1$$

The other equations are of the form p = p*P, i.e.,* p_n *= inner product of* p *and column n of P:*

$$\begin{cases} \pi_1 = 0.25\,\pi_5 + 0.1\,\pi_6 + 0.05\,\pi_7 \\ \pi_2 = 0.25\,\pi_5 + 0.15\,\pi_6 + 0.05\,\pi_7 + 0.05\,\pi_8 \\ \pi_3 = 0.05\,\pi_1 + 0.05\,\pi_2 + 0.05\,\pi_3 + 0.05\,\pi_4 + 0.25\,\pi_5 + 0.25\,\pi_6 + 0.15\,\pi_7 + 0.05\,\pi_8 + 0.05\,\pi_9 \\ \qquad \textit{etc.} \end{cases}$$

c. What is the average number on the shelf at the end of each day?

Solution: $\sum_{n=1}^{9} (n-1)\pi_n = 3.968123034$

The mean first passage matrix is:

```
                      Mean First Passage Times
                      ------------------------

f|
r|
o|    1       2       3       4       5       6       7       8       9
m| ------- ------- ------- ------- ------- ------- ------- ------- -------
1| 15.4523 12.9898 7.6651  7.37847 5.69593 5.24463 6.39169 12.6646 22.0756
2| 15.4523 12.9898 7.6651  7.37847 5.69593 5.24463 6.39169 12.6646 22.0756
3| 15.4523 12.9898 7.6651  7.37847 5.69593 5.24463 6.39169 12.6646 22.0756
4| 15.4523 12.9898 7.6651  7.37847 5.69593 5.24463 6.39169 12.6646 22.0756
5| 12.2711 10.4926 6.64702 7.25983 6.12634 6.35574 7.50281 13.7757 23.1867
6| 14.3163 11.5197 6.47734 6.42023 5.85772 5.88683 7.68799 13.9609 23.3719
7| 14.632  12.4406 7.01794 6.24734 5.2518  5.79028 7.22475 14.3004 23.7114
8| 15.2275 12.1857 7.62978 6.77218 5.19576 5.29848 7.10625 13.2454 24.1281
9| 15.4523 12.9898 7.6651  7.37847 5.69593 5.24463 6.39169 12.6646 22.0756
```

d. If the shelf is full Monday morning, what is the expected number of days until the shelf is first emptied ("stockout")?

Solution: m_{91} = 15.4523. *Note that this is the expected number of days, counting from Sunday night, which is when the system would have been observed.*

e. What is the expected time between stockouts?
Solution: $m_{11} = 1/\pi_1$ = 15.4523

f. How frequently will the shelf be restocked? (i.e. what is the average number of days between restocking?)
Solution: The steadystate probability that the shelf is restocked is $\sum_{n=1}^{4} \pi_n$ = 0.407689619

The frequency that the system visits the set of states {1,2,3,4} is therefore the reciprocal of this probability, i.e., $\frac{1}{0.407689619}$ = 2.452846365

2. Consider a manufacturing process in which raw parts (blanks) are machined on three machines, and inspected after each machining operation. The relevant data is as follows:

Manufacturing System Parameters: 56:171 O.R. HW#10

Station	Machine Operation			Inspection			
i	T	C	S	T	C	S	R
1	0.5	20	10	0.1	15	10	5
2	0.75	20	5	0.2	15	10	3
3	0.25	20	2	0.25	15	5	2

Pack & Ship: 0.1 hrs at 10 $/hr
Cost per blank: $50; Scrap Value: $10

T = time (hrs) per operation
C = cost ($/hr) of operation
S = scrap rate (%)
R = rework rate (%)

For example, machine #1 requires 0.5 hrs, at $20/hr., and has a 10% scrap rate. Those parts completing this operation are inspected, requiring 0.1 hr. at $15/hr. The inspector scraps 10%, and sends 5% back to machine #1 for rework (after which it is again inspected, etc.)
The Markov chain model of a part moving through this system has transition probability matrix:

56:171 O.R. HW#10: Transition Probabilities

from	1	2	3	4	5	6	7	8
1	0	0.9	0	0	0	0	0	0.1
2	0.05	0	0.85	0	0	0	0	0.1
3	0	0	0	0.95	0	0	0	0.05
4	0	0	0.03	0	0.87	0	0	0.1
5	0	0	0	0	0	0.98	0	0.02
6	0	0	0	0	0.02	0	0.93	0.05
7	0	0	0	0	0	0	1	0
8	0	0	0	0	0	0	0	1

a. Draw the diagram for this Markov chain and describe each state.
b. Which states are transient? which are absorbing?
Solution: States 7 and 8 are absorbing, since $\pi_{77} = \pi_{88} = 1.00$

The absorption probabilities are:

A = Absorption Probability Matrix

OK	Scrap
0.6335	0.3665
0.7039	0.2961
0.7909	0.2091
0.8325	0.1675
0.9296	0.07038
0.9486	0.05141

The matrix E is as follows:

E = Expected No. Visits

1	2	3	4	5	6
1.047	0.9424	0.8245	0.7833	0.6951	0.6812
0.05236	1.047	0.9162	0.8704	0.7723	0.7569
0	0	1.029	0.9779	0.8678	0.8504
0	0	0.03088	1.029	0.9134	0.8952
0	0	0	0	1.02	0.9996
0	0	0	0	0.0204	1.02

c. Explain how E was computed. Explain how A was computed, given E.
Solution: $E = \left(I - Q\right)^{-1}$ where Q is the submatrix of P, consisting of probabilities of transitions between transient states, i.e.,

$$Q = \begin{bmatrix} 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0.05 & 0 & 0.85 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.95 & 0 & 0 \\ 0 & 0 & 0.03 & 0 & 0.87 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.98 \\ 0 & 0 & 0 & 0 & 0.02 & 0 \end{bmatrix}$$

Therefore,

$$E = \begin{bmatrix} 1 & -0.9 & 0 & 0 & 0 & 0 \\ -0.05 & 1 & -0.85 & 0 & 0 & 0 \\ 0 & 0 & 1 & -0.95 & 0 & 0 \\ 0 & 0 & -0.03 & 1 & -0.87 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.98 \\ 0 & 0 & 0 & 0 & -0.02 & 1 \end{bmatrix}^{-1}$$

d. What percent of the parts which are started are successfully completed?
Solution: the absorption probability $a_{17} = 63.37\%$

e. What is the expected number of blanks which should be required to fill an order for 100 completed parts?
Solution: Since one entering part yields, on average, 0.6337 completed parts,

the expected number of entering parts per completed part is the reciprocal of 0.6337, or

1.57803377, i.e., $\dfrac{1 \textit{ entering part}}{0.6337 \textit{ completed part}} = \dfrac{1.57803377 \textit{ entering parts}}{1 \textit{ completed part}}$

f. What percent of the parts arriving at machine #2 will be successfully completed?
Solution: $a_{37} = 0.7909$, *i.e.*, 79.09%.

g. What is the expected total number of inspections which entering parts will undergo?
Solution: Since inspections are done at states 2 & 4, we add the expected number of visits to these

states, starting at the state 1: $e_{12} + e_{14} = 0.9424 + 0.7833 = 1.7257$

h. Explain the meaning of the number appearing in row 3, column 2 of the A matrix.
Solution: Note that the third row of matrix A (absorption probabilities) corresponds to the third

transient state, while the second column of matrix A corresponds to the second absorbing state.

Thus, $a_{38} = 0.2091$ *is the probability that a part which reaches state 3, i.e., the second machine, will*

eventually be scrapped.

i. Explain the meaning of the number appearing in row 3, column 3 of the E matrix.
Solution: the rows and columns of E (expected number of visits) correspond to the transient states.

Thus, $e_{33} = 1.029$ *is the expected number of times that a part which reaches the second machine is*

processed on that machine.

j. To fill the order for 100 completed parts, what is the expected man-hour requirement for each machine? for each inspection station?
Solution: We multiply the man-hours per transient state times the expected number of visits to

obtain the expected man-hour requirements at each state:

$$\sum_{n=1}^{6} T_n e_{1n} = 0.5235 + 0.09424 + 0.625875 + 0.15666 + 0.173775 + 0.1703 = 1.74435$$

k. What are the expected direct costs (row materials + operating costs - scrap value of rejected parts) per completed part?
Solution: We multiply the cost of labor for each state time the expected man-hour requirements at

that state, and sum:

$$\sum_{n=1}^{6} C_n T_n e_{1n} = 10.47 + 1.4136 + 12.5175 + 2.3499 + 3.4755 + 2.5545 = 32.781 \textit{ dollars}$$

3. The Minnesota State University admissions office has modeled the path of a student through the university as a Markov Chain:

	Freshman	Sophomore	Junior	Senior	Quits	Graduates
Freshman	0.10	0.80	0	0	0.10	0
Sophomore	0	0.10	0.85	0	0.05	0
Junior	0	0	0.15	0.80	0.08	0
Senior	0	0	0	0.10	0.05	0.85
Quits	0	0	0	0	1.00	0
Graduates	0	0	0	0	0	1.00

Each student's state is observed at the beginning of each fall semester. For example, if a student is a junior at the beginning of the current fall semester, there is an 80% chance that he will be a senior at the beginning of the next fall semester, a 15% chance that he will still be a junior, and a 5% chance that he will have quit. (We will assume that once a student quits, he never re-enrolls.)

a. If a student enters Minnesota State U. as a freshman, how many years can he expect to spend as a student there? $e_{11} + e_{12} + e_{13} + e_{14}$

$= 1.111111 + 0.98765432 + 0.98765432 + 0.87791495 = 3.96433459$ *years*

Note that this does <u>not</u> mean that the expected time until graduation is less than 4 years, since this expected number includes time spent at the university by students who drop out!

b. If he survives until his junior year, what is the probability that he will graduate?

Solution: $a_{36} = 0.888888$

c. What fraction of entering freshman will graduate?

Solution: $a_{16} = 0.74622771$

You may use the following computational results to answer the questions above:

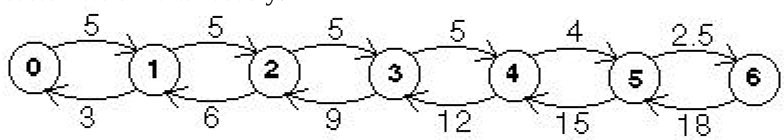
```
A = Absorption Probabilities
----------------------------
  |
 f|
 r|
 o|      5             6
 m| -----------   ----------
 1| 0.25377229    0.74622771
 2| 0.16049383    0.83950617
 3| 0.11111111    0.88888889
 4| 0.055555556   0.94444444
```

```
  E = Expected No. Visits to Transient States
  -------------------------------------------
f|
r|
o|     1          2          3          4
m| --------- ---------- ---------- ----------
1| 1.1111111 0.98765432 0.98765432 0.87791495
2| 0         1.1111111  1.1111111  0.98765432
3| 0         0          1.1764706  1.0457516
4| 0         0          0          1.1111111
```

❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ Homework #11 ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍

1. Birth-Death Process. Ed's Diner has room to seat only 6 persons. The customers arrive at the entrance in a random fashion at the rate of 5 customers per hour, and they stay on the average of 20 minutes (the total time in the diner, including waiting for the meal to be served, eating the meal, etc.) having an exponential distribution). However, if the seats in the diner are completely filled, an arriving customer will not come into the diner, but will go next door to Burger Master. Furthermore, when only 1 seat is left, the probability of an arriving customer coming into the diner is only 0.5, and when only 2 seats are left, the probability of a customer coming in is 0.8. Otherwise, all arriving customers will enter.

(a.) Draw a flow diagram for a **birth-death model** of this system. How many "servers" does this "queueing system" have (i.e., how many customers can be served simultaneously)?
Solution: This is considered to be a queueing system with 6 servers, since all six tables can serve customers simultaneously.



(b.) Find the steady-state distribution of the number of customers in the diner.
Solution:

$$\frac{1}{\pi_0} = 1 + \frac{5}{3} + \frac{5}{3}\times\frac{5}{6} + \frac{5}{3}\times\frac{5}{6}\times\frac{5}{9} + \frac{5}{3}\times\frac{5}{6}\times\frac{5}{9}\times\frac{5}{12} + \frac{5}{3}\times\frac{5}{6}\times\frac{5}{9}\times\frac{5}{12}\times\frac{4}{15} + \frac{5}{3}\times\frac{5}{6}\times\frac{5}{9}\times\frac{5}{12}\times\frac{4}{15}\times\frac{2.5}{18}$$

= 1 + 1.66667 + 1.38889 + 0.771605 + 0.321502 + 0.0857339 + 0.0119075 = 5.2463
and so π_0 = 1/5.2463 = 0.19061.
The other probabilities are found by multiplying π_0 by each term above:

$\pi_1 = 1.66667\pi_0 = 0.317683$,
$\pi_2 = 1.38889\pi_0 = 0.264736$,
$\pi_3 = 0.771605\pi_0 = 0.147076$,
$\pi_4 = 0.321502\pi_0 = 0.0612815$,
$\pi_5 = 0.0857339\pi_0 = 0.0163417$,
$\pi_6 = 0.0119075\pi_0 = 0.00226969$.

(c.) How many seats of the restaurant will be occupied on the average?
Solution: $L = \sum_{n=0}^{6} n\pi_n = 1.62884$

(d.) What is the average arrival rate? *Solution*: $\bar{\lambda} = \sum_{n=0}^{6} \lambda_n\pi_n = 4.02492$ / hour

(e.) During an 8-hour day, how many customers will Ed be expected to serve?
Solution: Since customers arrive at the average rate of 4.02492/hour, during an 8-hour day, 8hr (4.02492/hr.) = 32.1994 customers will arrive.

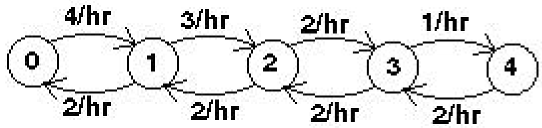
(f.) What fraction of the potential customers is Ed losing to his competition (including those who are discouraged from entering when there is only one or two seats remaining)?
Solution: In state 4 he is losing 1 potential customer per hour, in state 5 he is losing 2.5 customers/hour, while in state 6 he is losing 5 customer/hour.
Thus he is losing customers at the average rate of $1\pi_4 + 2.5\pi_5 + 5\pi_6 = 0.113484$/hour. Therefore the fraction which he is losing due to crowded conditions is (0.113484/hour)/(5/hour) = 2.27%.

2. A job shop has four numerically controlled machines that are capable of operating on their own (i.e., without a human operator) once they have been set up with the proper cutting tools and all adjustments are made. Each setup requires the skills of an experienced machine operater, and the time need to complete a setup is exponentially distributed with a mean of 30 minutes. When the setup is complete, the machine operater pushes a button, and the machine requires no further attention until it has finished its job, when it is ready for another setup. The job times are exponentially distributed with a mean of one hour. The question is, "how many machine operaters should there be to tend the machines?" At opposite extremes, there could be one operater tending all four machines, or there could be four operators. The optimal number of machine operaters obviously depends on

a trade-off between the cost of operators and the cost of idle machines. Of course, machinists are paid the same regardless of how much work they do, but each machine incurs idle-time costs only when it is idle. (If more than one operator is used, any idle operator will tend any machine that completes its job, rather than each operator being assigned to a certain machine or set of machines.)

Assume that the cost of a machinist (including fringe benefits, etc.) is \$20 per hour, and that the cost of an idle machine (including lost revenues, etc.) is \$60 per hour of idleness. For each alternative (i.e., 1, 2, 3, or 4 machinists) answer (a) through (f):
Solution:
Alternative: 1 machinist

a. Sketch the transition diagram and set up the transition rate matrix.



b. Compute the steady-state distribution.
Solution:
$\frac{1}{\pi_0} = 1 + \frac{4}{2} + \frac{4}{2}\times\frac{3}{2} + \frac{4}{2}\times\frac{3}{2}\times\frac{2}{2} + \frac{4}{2}\times\frac{3}{2}\times\frac{2}{2}\times\frac{1}{2} = 10.5$, so $\pi_0 = 0.0952381$.
$\pi_1 = 2\pi_0 = 0.190476$, $\pi_2 = 3\pi_0 = 0.285714$, $\pi_3 = 3\pi_0 = 0.285714$, and $\pi_4 = 3\pi_0 = 0.142857$.

c. What is the percent of the time that each machinist is busy?
Solution: $1 - \pi_0 = 90.48\%$.

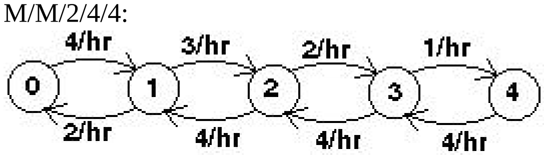
d. What is the average number of machines in operation?
Solution: Denote this quantity by $\underline{N} = \sum_{n=0}^{4} (4 - n)\pi_n = 1.8095$ machines.

e. What is the utilization of each machine, i.e., the percent of the time that each machine is busy?
Solution: 1.8095/4 = 45.24%

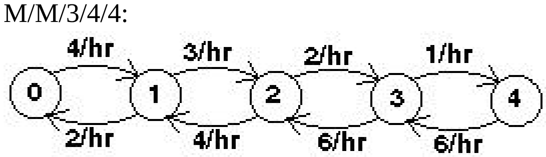
f. What is the total cost of the alternative?
Solution: 1(\$20/hr.) + (1 - 0.4524)(\$60/hr.) = \$52.86/hr.

The diagrams for the other alternatives are:

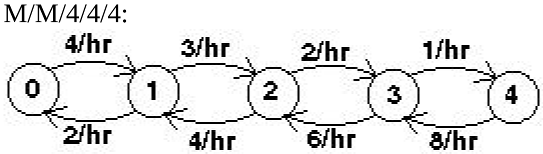
M/M/2/4/4:



M/M/3/4/4:



M/M/4/4/4:



In similar fashion to the case above, we evaluate each of the other alternatives:

Alternative:	1 machinist	2 machinists	3 machinists	4 machinists
π_0 =	0.095238	0.183908	0.196721	0.197531
π_1 =	0.190476	0.367816	0.393443	0.395062
π_2 =	0.285714	0.275862	0.295082	0.296296
π_3 =	0.285714	0.137931	0.098361	0.0987654
π_4 =	0.285714	0.0344828	0.016393	0.0123457
$\underline{N}$ =	1.8095	2.52874	2.65574	2.66667

Utilization=	45.24%	63.2185%	66.3935%	66.6667%
Cost of idle machine:	$32.86/hr.	$22.07/hr.	$20.16/hr.	$20/hr.
Labor cost:	$20/hr.	$40/hr.	$60/hr.	$80.00/hr.
Total cost=	$52.86/hr	$62.07/hr.	$80.16/hr.	$100.00/hr.

g. What is the optimal number of machinists?
Solution: The least cost alternative is to use a single machinist, which would have a cost of $52.86/hr.

3. (From text by Winston, page 1179, #6). The manager of a large group of employees must decide if she needs another photocopying machine. The cost of a machine is $40 per 8-hour day whether or not the machine is in use. An average of 4 people per hour need to use the copying machine. Each person uses the copier for an average of 10 minutes. Interarrival times and copying times are assumed to be exponentially distributed. Employees are paid $8 per hour, and we assume that a waiting cost (equal to the salary of the worker) is incurred when a worker is waiting in line or is using the copying machine. How many copying machines should be rented?

Solution: If *c* is the number of copy machines rented, then the queue will be of the form M/M/c.
For each value of c, we must compute L (the average number of customers in the line). The cost of lost labor will then be ($8/hr)L.

Alternative:	1 machine	2 machines
π_0 =	0.333333	0.5
π_1 =	0.222222	0.333333
π_2 =	0.148148	0.111111
π_3 =	0.0987654	0.037037
π_4 =	0.0658436	0.0123457
L =	2	0.75
Cost of lost labor:	$16/hr.	$6
Rental cost:	$40/hr.	$80/hr.
Total cost=	$56/hr.	$86/hr.

The optimal number of copy machines to be rented is 1.

❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ Homework #12 ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍

1. Game of Matches. Consider the game discussed in class, but with 25 matches initially on the table, and, at each turn, from one to ***four*** matches may be removed. As before, the person removing the last match is the loser.

a. If you are given the option of having the first turn or allowing you opponent to be first, what would be your choice?

b. Describe a strategy which would then allow you to win the game.

2. Power Plant Capacity Planning. A power company is doing long-range planning and has forecast additional demand for electricity which would require addition of the following number of generators for each of the next six years:

Year	1999	2000	2001	2002	2003	2004
Needed:	1	2	3	5	6	7
Cost/generator	5.4	5.6	5.8	5.7	5.5	5.2 ($millions)

That is, by the end of 1999 one additional generator must be on-line, by the end of 2000 two additional generators must be on-line (including the one required in 1999), and at the end of the year 2004 a total of seven additional generators must be on-line.

At most three generators may be added during any year, with the cost per generator as given above. In addition, a cost of 2 million dollars is incurred during any year in which a generator is added to cover costs which do not depend upon the number of generators installed. The power company wishes to know the schedule for adding the generators such that the requirements are met and total cost is minimized.

A dynamic programming model has been defined, in which the stage is the number of years remaining during the planning period, e.g., the year 1999 is stage 6, 2000 is stage 5, etc. The state of the system, S_n , is the number of generators which have been installed prior to stage n. The function $f_n(S_n)$ is defined to be the minimum total cost of meeting the requirements during stages n, n-1, ... 1 if at the beginning of stage n, S_n generators have been installed. Therefore, the power company wishes to determine $f_6(0)$, the minimum total cost for stages 6, 5, 4, 3, 2, and 1, given that S_6=0, i.e., no generators have as yet been added.

Below is shown the computation performed to solve the problem by dynamic programming, with several values blanked out:

```
                    Powerplant Capacity Planning

Recursion type: backward


                    ---Stage 1---

   s \ x:  0       1       2       3
   ------------------------------------
   0 | 9999.00 9999.99 9999.99 9999.99
   1 | 9999.00 9999.99 9999.99 9999.99
   2 | 9999.00 9999.99 9999.99 9999.99
   3 | 9999.00 9999.99 9999.99 9999.99
   4 | 9999.00 9999.99 9999.99   17.60
   5 | 9999.00 9999.99   12.40 9999.99
   6 | 9999.00    7.20 9999.99 9999.99
   7 |    0.00 9999.99 9999.99 9999.99
```

---Stage 2---

s \ x:	0	1	2	3
0	9999.00	9999.99	9999.99	9999.99
1	9999.00	9999.99	9999.99	9999.99
2	9999.00	9999.99	9999.99	9999.99
3	9999.00	9999.99	9999.99	25.70
4	9999.00	9999.99	20.20	18.50
5	9999.00	14.70	13.00	9999.99
6	7.20	7.50	9999.99	9999.99
7	0.00	9999.99	9999.99	9999.99

---Stage 3---

s \ x:	0	1	2	3
0	9999.00	9999.99	9999.99	9999.99
1	9999.00	9999.99	9999.99	9999.99
2	9999.00	9999.99	9999.99	32.10
3	9999.00	9999.99	26.40	26.30
4	9999.00	20.70	20.60	19.10
5	13.00	14.90	13.40	9999.99
6	7.20	7.70	9999.99	9999.99
7	0.00	9999.99	9999.99	9999.99

---Stage 4---

s \ x:	0	1	2	3
0	9999.00	9999.99	9999.99	45.70
1	9999.00	9999.99	39.90	38.50
2	9999.00	34.10	32.70	32.40
3	26.30	26.90	26.60	26.60
4	19.10	20.80	20.80	19.40
5	13.00	15.00	13.60	9999.99
6	7.20	7.80	9999.99	9999.99
7	0.00	9999.99	9999.99	9999.99

---Stage 5---

s \ x:	0	1	2	3
0	9999.00	9999.99	45.60	45.10
1	9999.00	40.00	39.50	37.90
2	32.40	33.90	32.30	31.80
3	26.30	26.70	26.20	26.00
4	19.10	20.60	20.40	18.80

```
   5 |   13.00   14.80   13.20 9999.99
   6 |    7.20    7.60 9999.99 9999.99
   7 |    0.00 9999.99 9999.99 9999.99
```

```
                  ---Stage 6---

   s \ x:  0       1       2       3
_____________________________________

   0 | 9999.00   45.30   44.60   44.20
   1 |   37.90   39.20   38.80   37.00
   2 |   31.80   33.40   31.60   31.20
   3 |   26.00   26.20   25.80   25.40
   4 |   18.80   20.40   20.00   18.20
   5 |   13.00   14.60   12.80 9999.99
   6 |    7.20    7.40 9999.99 9999.99
   7 |    0.00 9999.99 9999.99 9999.99
```

```
      Display_Deterministic_Tables
VALUE ERROR
      Display_Deterministic_Tables
      ^
      Display_Tables
```

```
                             ____________________
                             |                   |
                             |      Optimal      |
                             |Returns & Decisions|
                             |___________________|

------------------------------------------------------------
Stage 6:
               Optimal        Optimal    Resulting
 State      Values        Decisions     State
   0   44.20   3         3
   1   37.00   3         4
   2   31.20   3         5
   3   25.40   3         6
   4   18.20   3         7
   5   12.80   2         7
   6    7.20   0         6
   7    0.00   0         7

------------------------------------------------------------
Stage 5:
               Optimal        Optimal    Resulting
 State      Values        Decisions     State
   0   45.10   3         3
   1   37.90   3         4
   2   31.80   3         5
   3   26.00   3         6
   4   18.80   3         7
   5   13.00   0         5
   6    7.20   0         6
   7    0.00   0         7

------------------------------------------------------------
Stage 4:
```

State	Optimal Values	Optimal Decisions	Resulting State
0	45.70	3	3
1	38.50	3	4
2	32.40	3	5
3	26.30	0	3
4	19.10	0	4
5	13.00	0	5
6	7.20	0	6
7	0.00	0	7

Stage 3:

State	Optimal Values	Optimal Decisions	Resulting State
0	9999.00	0	0
1	9999.00	0	0
2	32.10	3	5
3	26.30	3	6
4	19.10	3	7
5	13.00	0	5
6	7.20	0	6
7	0.00	0	7

Stage 2:

State	Optimal Values	Optimal Decisions	Resulting State
0	9999.00	0	0
1	9999.00	0	0
2	9999.00	0	0
3	25.70	3	6
4	18.50	3	7
5	13.00	2	7
6	7.20	0	6
7	0.00	0	7

Stage 1:

State	Optimal Values	Optimal Decisions	Resulting State
0	9999.00	0	0
1	9999.00	0	0
2	9999.00	0	0
3	9999.00	0	0
4	17.60	3	7
5	12.40	2	7
6	7.20	1	7
7	0.00	0	7

Powerplant Capacity Planning

*** Optimal value is 44.2 ***

STAGE	STATE	DECISION
0	0	3
1	3	3
2	6	0
3	6	0
4	6	0
5	6	1
6	7	

3. ***Dynamic Programming Algorithm for Knapsack Problem.*** Consider the knapsack problem in which 5 items are available to fill a knapsack with capacity 13 pounds. The weight and value of each item are shown below:

```
Number of items: 5
Capacity of Knapsack: 13
Maximum units of any item to be included is 1
```

i	W	V
1	6	11
2	5	9
3	3	5
4	2	3
5	2	2

```
W = 'weight' of item
V = value of item
```

Dynamic programming was used to find the contents with maximum value, subject to the weight limitation. Below are the computations for the various stages:

---STAGE 1---

s \ x:	0	1
0	0.00	¯99999.00
1	0.00	¯99999.00
2	0.00	¯99999.00
3	0.00	¯99999.00
4	0.00	¯99999.00
5	0.00	¯99999.00
6	0.00	11.00
7	0.00	11.00
8	0.00	11.00
9	0.00	11.00
10	0.00	11.00
11	0.00	11.00
12	0.00	11.00
13	0.00	11.00

---STAGE 2---

s \ x:	0	1
0	0.00	¯99999.00
1	0.00	¯99999.00
2	0.00	¯99999.00
3	0.00	¯99999.00
4	0.00	¯99999.00
5	0.00	9.00
6	11.00	9.00
7	11.00	9.00
8	11.00	9.00
9	11.00	9.00
10	11.00	$\boxed{}$
11	11.00	20.00
12	11.00	20.00
13	11.00	20.00

---STAGE 3---

s \ x:	0	1
0	0.00	¯99999.00
1	0.00	¯99999.00
2	0.00	¯99999.00
3	0.00	5.00
4	0.00	5.00
5	9.00	5.00
6	11.00	5.00
7	11.00	5.00
8	11.00	
9	11.00	16.00
10	11.00	16.00
11	20.00	16.00
12	20.00	16.00
13	20.00	16.00

---STAGE 4---

s \ x:	0	1
0	0.00	¯99999.00
1	0.00	¯99999.00
2	0.00	3.00
3	5.00	3.00
4	5.00	3.00
5	9.00	8.00
6	11.00	8.00
7	11.00	12.00
8	14.00	14.00
9	16.00	14.00
10		17.00
11	20.00	19.00
12	20.00	19.00
13	20.00	23.00

---STAGE 5---

s \ x:	0	1
0	0.00	¯99999.00
1	0.00	¯99999.00
2	3.00	2.00
3	5.00	2.00
4	5.00	5.00
5	9.00	7.00
6	11.00	7.00
7	12.00	11.00
8	14.00	13.00
9	16.00	14.00
10	17.00	16.00
11	20.00	18.00
12	20.00	19.00
13	23.00	22.00

a. Three of the values in the tables above have been blanked out. Compute these values (and explain the computation!)

b. Using the tables above, prepare tables for each stage showing the optimal value and the optimal decision for each state value.

c. Explain how the optimal contents of the knapsack may be found using the tables in (b).

❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ Homework #13 ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍

1. *(Exercise 6, page 1083-1084 of text by Winston)* The following is the original statement of the problem: *Bectol, Inc. is building a dam. A total of 10,000,000 cu ft of dirt is needed to construct the dam. A bulldozer is used to collect dirt for the dam. Then the dirt is moved via dumpers to the dam site. Only one bulldozer is available, and it rents for $100 per hour. Bectol can rent, at $40 per hour, as many dumpers as desired. Each dumper can hold 1000 cu ft of dirt. It takes an average of 12 minutes for the bulldozer to load a dumper with dirt, and it takes each dumper an average of five minutes to deliver the dirt to the dam and return to the bulldozer. Making appropriate assumptions about exponentiality so as to obtain a birth/death model, determine the optimal number of dumpers and the minimum total expected cost of moving the dirt needed to build the dam.*

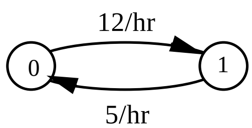
Solution to this original statement of the problem:

We have to use 10,000,000/1000=10,000 times of dumper to deliver all the dirt.

Case 1 : One dumper :

Define state 0 : no dumper in the system,

state 1 : one dumper in the system.



Steady-state Distribution

i	Pi	CDF
0	0.294118	0.294118
1	0.705882	1.000000

The average departure rate of dumper is $(1-\pi_0)5=0.705882(5)=3.52941$(times/hr)

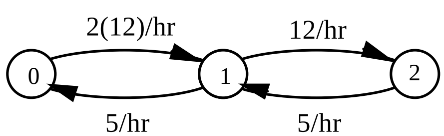
The total cost = (10,000/3.52941)($100+$40)=396667.

Case 2 : Two dumpers.

Define state 0 : no dumper in the system,

state 1 : one dumper in the system,

state 2 : two dumpers in the system, one is being served and another is waiting.



Steady-state Distribution

i	Pi	CDF
0	0.057737	0.057737
1	0.277136	0.334873
2	0.665127	1.000000

The total cost={10,000/[$(1-\pi_0)$(5)]}{$100+2($40)}=382059.

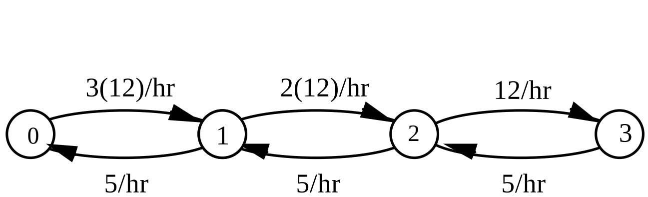
Case 3 : Three dumpers :

Define state 0 : no dumper in the system,

state 1 : one dumper in the system,

state 2 : two dumpers in the system, one is being served and another is waiting.

state 3 : three dumpers in the system, one is being served, and the other two are waiting.



Steady-state Distribution

i	Pi	CDF
0	0.007955	0.007955
1	0.057277	0.065233
2	0.659836	0.340164
3	0.659836	1.000000

The total cost = {10.000/[$(1-\pi_0)$(5)]}{$100+3($40)}=443528

Thus, the optimal number of dumpers is 2.

Suppose that improved methods have now been implemented so that the time to load a dumper has been decreased from 12 to 10 minutes, but that the loading site has changed so that it now requires 6 minutes instead of 5 minutes for a dumper to unload and return for its next load. Recompute the optimal number of dumpers under these new conditions.

2. **Optimization of System Reliability:** A system consists of 3 devices, each subject to possible failure, all of which must function in order for the system to function. In order to increase the reliability of the system, redundant units may be included, so that the system continues to function if at least one of the redundant units remains functional. The data are:

Device	Reliability (%)	Weight (kg.)
1	75	1
2	80	2
3	90	3

If we include a single unit of each device, then the system reliability will be the product of the device reliabilities, i.e., (0.75)(0.80)(0.90) = 53.55%. However, by including redunant units of one or more devices, we can substantially increase the reliability. Thus, for example, if 2 redundant units of device #1 were included, the reliability of device #1 will be increased from 75% to $1 - (0.25)^2 = 93.75\%$. That is, the probability that both units fail, assuming independent failures, is 0.25x0.25 = .0625. Suppose that the system may weigh no more than 10 kg. (Since at least one of each device must be included, a total of 6 kg, this leaves 4 kg available for redundant units.) Assume that no more than 3 units of any type need be considered. We wish to compute the number of units of each device type to be installed in order to maximize the system reliability, subject to the maximum weight restriction.

Assume that the devices are considered in the order: #3, #2, and finally, #1. The optimal value function is defined to be:

$F_n(S)$ = maximum reliability which can be achieved for devices #n, n-1, ... 1, given that the weight used by these devices cannot exceed S (the state variable)

The optimal value for the problem is therefore given by $F_3(10)$. The computation is done in the backward order, i.e., first the optimal value function $F_1(S)$ is computed for each value of the available weight S, then $F_2(S)$, until finally $F_3(10)$ has been computed.

The reliability of each device as a function of the number x of redundant units is $1 - (1-R_i)^x$ where R_i is the reliability of a single unit of device i:

Reliability (%) vs # redundant units

i	1	2	3
1	70	91	97.3
2	85	97.75	99.6625
3	90	99	99.9

The following output is produced during the solution of the problem:

Stage 1

s \ x	1	2	3
1	0.7000		
2	0.7000	0.9100	
3	0.7000	0.9100	0.9730
4	0.7000	0.9100	0.9730
5	0.7000	0.9100	0.9730
6	0.7000	0.9100	0.9730
7	0.7000	0.9100	0.9730
8	0.7000	0.9100	0.9730
9	0.7000	0.9100	0.9730
10	0.7000	0.9100	0.9730

Stage 2

s \ x	1	2	3
3	0.5950		
4	0.7735		
5		0.6843	
6	0.8270	0.8895	
7	0.8270	0.9511	0.6976
8	0.8270	0.9511	0.9069
9	0.8270	0.9511	0.9697
10	0.8270	0.9511	0.9697

Stage 3

s \ x	1	2
6	0.5355	
7	0.6961	
8	0.7443	
9	0.8006	0.5891
10	0.8560	

a. Fill in the two blanks in the tables above. That is,

→ if there is an available capacity of 5 kg. when considering device #2, and only 1 unit of this device is selected, what is the maximum reliability that can be achieved for the subsystem consisting of devices 1 & 2?

→ if, when considering device #3 there is 10 kg available capacity and 2 units of this device are included, what is the maximum reliability that can be achieved for the system consisting of devices 1, 2, & 3?

The tables showing the values of f_3, f_2, and f_1 are:

Stage 1

State	f_1	Optimal Decisions	Resulting State
1	0.7000	1	0
2	0.9100	2	0
3	0.9730	3	0
4	0.9730	3	1
5	0.9730	3	2
6	0.9730	3	3
7	0.9730	3	4
8	0.9730	3	5
9	0.9730	3	6
10	0.9730	3	7

Stage 2

State	f_2	Optimal Decisions	Resulting State
3	0.5950	1	1
4	0.7735	1	2
5	0.8270	1	3
6			
7	0.9511	2	3
8	0.9511	2	4
9	0.9697	3	3
10	0.9697	3	4

Stage 3

State	f_3	Optimal Decisions	Resulting State
6	0.5355	1	3
7	0.6961	1	4
8	0.7443	1	5
9	0.8006	1	6
10	0.8560	1	7

b. Fill in the three blanks in the table above for stage #2. *That is, when considering device #2, if six kilograms of capacity is available, what is the maximum reliability that can be achieved for the subsystem consisting of devices #1&2? How many units of device #2 should be selected? After including this number of units of device #2, how many kg of capacity are available when considering the next item (#1 at stage 1)?*

c. What is the optimal system reliability if 10 kg. is available for the devices ?

d. What is the optimal number of units of each device if 10 kg. is available?

e. What is the optimal system reliability if only 9 kg. were available for the devices? If only 9 kg. were available, how many units of each device should be included in the system? *(This question can be answered without recomputing the tables above!)*