# 56:171
# Operations Research
# Fall 2001

# Homework

© D.L.Bricker
Dept of Mechanical & Industrial Engineering
University of Iowa

| 56:171 Operations Research<br>Homework #1 - Due Wednesday, September 12, 2001 |
|---|

*In each case below, you must formulate a linear programming model that will solve the problem. Be sure to define the meaning of your variables! Then use LINDO (or other appropriate software) to find the optimal solution. State the optimal objective value, and describe in "layman's terms" the optimal decisions.*

**1.** A company makes two products in a single plant. It runs this plant for 100 hours each week. Each unit of product A that the company produces consumes two hours of plant capacity, earns the company a profit of $1000, and causes, as an undesirable side effect, the emission of 4 ounces of particulates. Each unit of product B that the company produces consumes one hour of capacity, earns the company a profit of $2000, and causes the emission of 3 ounces of particulates and 1 ounce of chemicals. The EPA (environmental Protection Agency) requires the company to limit particulate emission to at most 240 ounces per week and chemical emission to at most 60 ounces per week.

a. Write down the linear programming model for maximizing the company's profits subject to the restrictions on production capacity and emissions.
b. What is the optimal solution of the LP?

**2.** Cattle feed can be mixed from oats, corn, alfalfa, and peanut hulls. The following table shows the current cost per ton (in dollars) of each of these ingredients, together with the percentage of recommended daily allowances for protein, fat, and fiber that a serving of it fulfills.

| | Oats | Corn | Alfalfa | Peanut hulls |
|---|---|---|---|---|
| % protein | 60 | 80 | 55 | 40 |
| % fat | 50 | 70 | 40 | 100 |
| % fiber | 90 | 30 | 60 | 80 |
| Cost $/ton | 200 | 150 | 100 | 75 |

We want to find a minimum cost way to produce feed that satisfies at least 60% of the daily allowance for protein and fiber while not exceeding 60% of the fat allowance.

**3.** "Mama's Kitchen" serves from 5:30 a.m. each morning until 1:30 p.m. in the afternoon. Tables are set and cleared by busers working 4-hour shifts beginning on the hour from 5:00 a.m. through 10:00 a.m. Most are college students who hate to get up in the morning, so Mama's pays $9 per hour for the 5:00, 6:00, and 7:00 a.m. shifts, and $7.50 per hour for the others. (That is, a person works a shift consisting of 4 consecutive hours, with the wages equal to 4×$9 for the three early shifts, and 4×$7.50 for the 3 later shifts.) The manager seeks a minimum cost staffing plan that will have at least the number of busers on duty each hour as specified below:

| | 5 am | 6 am | 7 am | 8 am | 9 am | 10am | 11am | Noon | 1 pm |
|---|---|---|---|---|---|---|---|---|---|
| #reqd | 2 | 3 | 5 | 5 | 3 | 2 | 4 | 6 | 3 |

4. a. Draw the feasible region of the following LP:

$$\begin{array}{ll} \text{Maximize} & 3X_1 + 2X_2 \\ \text{subject to} & 4X_1 + 7X_2 \leq 28 \\ & X_1 + X_2 \leq 6 \\ & 3X_1 + X_2 \leq 9 \\ & X_1 \geq 0,\ X_2 \geq 0 \end{array}$$

b. Use the simplex algorithm to find the optimal solution of the above LP. *(Show the initial and each succeeding tableau.)*

c. On the sketch of the feasible region in (a), indicate the initial basic solution and the basic solution at each succeeding iteration.

5a. What is INFORMS?

5b. Find (on the INFORMS website at http://www.informs.org) a definition of "Operations Research".

56:171 Operations Research
Homework #2 -- Due Wednesday, Sept. 19

**The Diet Problem**. "The goal of the diet problem is to find the cheapest combination of foods that will satisfy all the daily nutritional requirements of a person." Go to the URL:

http://www-fp.mcs.anl.gov/otc/Guide/CaseStudies/diet/index.html

and click on *"Give it a try."* Then on the next page select *"Edit the constraints"* and click on *"Go on"*.

a. What are the restrictions on calories in the default set of requirements?

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| ☐ | Frozen Broccoli | $0.16 | 10 Oz Pkg | ☐ | Carrots,Raw | $0.07 | 1/2 Cup Shredded |
| ☐ | Celery, Raw | $0.04 | 1 Stalk | ☐ | Frozen Corn | $0.18 | 1/2 Cup |
| ☐ | Lettuce,Iceberg,Raw | $0.02 | 1 Leaf | ☐ | Peppers, Sweet, Raw | $0.53 | 1 Pepper |
| ☐ | Potatoes, Baked | $0.06 | 1/2 Cup | ☐ | Tofu | $0.31 | 1/4 block |
| ☐ | Roasted Chicken | $0.84 | 1 lb chicken | ☐ | Spaghetti W/ Sauce | $0.78 | 1 1/2 Cup |
| ☐ | Tomato,Red,Ripe,Raw | $0.27 | 1 Tomato, 2-3/5 In | ☐ | Apple,Raw,W/Skin | $0.24 | 1 Fruit,3/Lb,Wo/Rf |
| ☐ | Banana | $0.15 | 1 Fruit,Wo/Skn&Seeds | ☐ | Grapes | $0.32 | 10 Fruits,Wo/Rf |
| ☐ | Kiwifruit,Raw,Fresh | $0.49 | 1 Med Frt,Wo/Skin | ☐ | Oranges | $0.15 | 1 Frt,2-5/8 Diam |
| ☐ | Bagels | $0.16 | 1 Oz | ☐ | Wheat Bread | $0.05 | 1 Sl |
| ☐ | White Bread | $0.06 | 1 Sl | ☐ | Oatmeal Cookies | $0.09 | 1 Cookie |
| ☐ | Apple Pie | $0.16 | 1 Oz | ☐ | Chocolate Chip Cookies | $0.03 | 1 Cookie |
| ☐ | Butter,Regular | $0.05 | 1 Pat | ☐ | Cheddar Cheese | $0.25 | 1 Oz |
| ☐ | 3.3% Fat,Whole Milk | $0.16 | 1 C | ☐ | 2% Lowfat Milk | $0.23 | 1 C |
| ☐ | Skim Milk | $0.13 | 1 C | ☐ | Poached Eggs | $0.08 | Lrg Egg |
| ☐ | Scrambled Eggs | $0.11 | 1 Egg | ☐ | Bologna,Turkey | $0.15 | 1 Oz |
| ☐ | Frankfurter, Beef | $0.27 | 1 Frankfurter | ☐ | Ham,Sliced,Extralean | $0.33 | 1 Sl,6-1/4x4x1/16 In |
| ☐ | Kielbasa,Prk | $0.15 | 1 Sl,6x3-3/4x1/16 In | ☐ | Cap'N Crunch | $0.31 | 1 Oz |
| ☐ | Cheerios | $0.28 | 1 Oz | ☐ | Corn Flks, Kellogg'S | $0.28 | 1 Oz |
| ☐ | Raisin Brn, Kellg'S | $0.34 | 1.3 Oz | ☐ | Rice Krispies | $0.32 | 1 Oz |
| ☐ | Special K | $0.38 | 1 Oz | ☐ | Oatmeal | $0.82 | 1 C |
| ☐ | Malt-O-Meal,Choc | $0.52 | 1 C | ☐ | Pizza W/Pepperoni | $0.44 | 1 Slice |
| ☐ | Taco | $0.59 | 1 Small Taco | ☐ | Hamburger W/Toppings | $0.83 | 1 Burger |
| ☐ | Hotdog, Plain | $0.31 | 1 Hotdog | ☐ | Couscous | $0.39 | 1/2 Cup |
| ☐ | White Rice | $0.08 | 1/2 Cup | ☐ | Macaroni,Ckd | $0.17 | 1/2 Cup |
| ☐ | Peanut Butter | $0.07 | 2 Tbsp | ☐ | Pork | $0.81 | 4 Oz |
| ☐ | Sardines in Oil | $0.45 | 2 Sardines | ☐ | White Tuna in Water | $0.69 | 3 Oz |
| ☐ | Popcorn,Air-Popped | $0.04 | 1 Oz | ☐ | Potato Chips,Bbqflvr | $0.22 | 1 Oz |
| ☐ | Pretzels | $0.12 | 1 Oz | ☐ | Tortilla Chip | $0.19 | 1 Oz |
| ☐ | Chicknoodl Soup | $0.39 | 1 C (8 Fl Oz) | ☐ | Splt Pea&Hamsoup | $0.67 | 1 C (8 Fl Oz) |
| ☐ | Vegetbeef Soup | $0.71 | 1 C (8 Fl Oz) | ☐ | Neweng Clamchwd | $0.75 | 1 C (8 Fl Oz) |
| ☐ | Tomato Soup | $0.39 | 1 C (8 Fl Oz) | ☐ | New E Clamchwd,W/Mlk | $0.99 | 1 C (8 Fl Oz) |
| ☐ | Crm Mshrm Soup,W/Mlk | $0.65 | 1 C (8 Fl Oz) | ☐ | Beanbacn Soup,W/Watr | $0.67 | 1 C (8 Fl Oz) |

Go back to the previous page, where approximately 100 foods are listed for your selection. Choose "Default requirements", and select 15 foods which you think would provide an economical menu meeting the requirements. Then click on "Go on" again.

b. What is the minimum-cost menu meeting the nutritional requirements using the foods you indicated? Indicate the solution in the left 2 columns of the table below.

Change the default upper limit on calories to 1500/day and solve the problem again. (Be sure that the lower bound ≤ upper bound!)

c. What is the minimum-cost menu meeting the nutritional requirements using the foods you indicated? Indicate the solution in the right 2 columns of the table below.

| Quantity (# servings) | Cost | Food (& serving size) | Quantity (# servings) | Cost |
|---|---|---|---|---|
| | | 1. | | |
| | | 2. | | |
| | | 3. | | |
| | | 4. | | |
| | | 5. | | |
| | | 6. | | |
| | | 7. | | |
| | | 8. | | |
| | | 9. | | |
| | | 10. | | |
| | | 11. | | |
| | | 12. | | |
| | | 13. | | |
| | | 14. | | |
| | | 15. | | |
| Total Cost: | $ | <><><><><><><> | Total Cost: | $ |

2. Below are several simplex tableaus. Assume that the objective in each case is to be **minimized**. Classify each tableau by writing to the right of the tableau a letter **A** through **G**, according to the descriptions below. *Also answer the question accompanying each classification, if any.*

**(A)** Nonoptimal, nondegenerate tableau with bounded solution. *Circle a pivot element which would improve the objective.*

**(B)** Nonoptimal, degenerate tableau with bounded solution. *Circle an appropriate pivot element. Would the objective improve with this pivot?*

**(C)** Unique nondegenerate optimum.

**(D)** Optimal tableau, with alternate optimum. *State the values of the basic variables. Circle a pivot element which would lead to another optimal basic solution. Which variable will enter the basis, and at what value?*

**(E)** Objective unbounded (below). *Specify a variable which, when going to infinity, will make the objective arbitrarily low.*

**(F)** Tableau with infeasible primal

**Warning:** Some of these classifications might be used for more than one tableau, while others might not be used at all!

| (i) -z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -3 | 0 | 1 | 1 | 0 | 0 | 2 | 3 | -45 | |
| 0 | 0 | 0 | -4 | 0 | 0 | 1 | 0 | 0 | 9 | ______ |
| 0 | -6 | 0 | 3 | -2 | 1 | 0 | 2 | 3 | 5 | |
| 0 | 4 | 1 | 2 | -5 | 0 | 0 | 1 | 1 | 8 | |

| (ii) -z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | -1 | 3 | 0 | 0 | 2 | -2 | -45 | |
| 0 | 0 | 0 | -4 | 0 | 0 | 1 | 3 | 0 | 9 | ________ |
| 0 | -4 | 1 | 2 | -5 | 0 | 0 | -2 | 1 | 0 | |
| 0 | -6 | 0 | 3 | -2 | 1 | 0 | -4 | 3 | 5 | |

| (iii) -z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 1 | 1 | 0 | 0 | 3 | 5 | -45 | |
| 0 | 0 | 0 | -4 | 0 | 0 | 1 | 3 | 0 | 3 | ________ |
| 0 | 4 | 1 | 2 | -5 | 0 | 0 | 2 | 1 | 7 | |
| 0 | -6 | 0 | 3 | -2 | 1 | 0 | -4 | 3 | 15 | |

| (iv) -z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 1 | -3 | 0 | 0 | 2 | 0 | -45 | |
| 0 | 0 | 0 | -1 | 0 | 0 | 1 | 3 | 0 | 9 | ________ |
| 0 | 4 | 1 | -4 | -5 | 0 | 0 | 2 | 1 | 3 | |
| 0 | -6 | 0 | 3 | -2 | 1 | 0 | -4 | 3 | 5 | |

| (v) -z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 0 | 1 | 0 | 0 | 0 | 12 | -45 | |
| 0 | 0 | 0 | -4 | 0 | 0 | 1 | 3 | 0 | 9 | ______ |
| 0 | 4 | 1 | 2 | -5 | 0 | 0 | 2 | 1 | 8 | |
| 0 | -6 | 0 | 3 | -2 | 1 | 0 | -4 | 3 | 5 | |

| (vi) -z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 1 | 3 | 0 | 0 | 2 | 0 | -45 | |
| 0 | 0 | 0 | -4 | 0 | 0 | 1 | 3 | 0 | 9 | |
| 0 | -6 | 0 | 3 | -2 | 1 | 0 | -4 | 3 | 5 | ________ |
| 0 | 4 | 1 | 2 | -5 | 0 | 0 | 2 | 1 | 8 | |

| (vii)-z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 1 | 1 | 0 | 0 | -2 | 0 | -45 | |
| 0 | 4 | 1 | 2 | -5 | 0 | 0 | 2 | 1 | 5 | ________ |
| 0 | -6 | 0 | 3 | 2 | 1 | 0 | -4 | 3 | 0 | |
| 0 | 0 | 0 | -4 | 0 | 0 | 1 | 3 | 0 | 9 | |

| (viii)-z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 0 | -1 | 3 | 0 | 0 | 2 | 0 | -45 | |
| 0 | 0 | 0 | -4 | 0 | 0 | 1 | 3 | 0 | 9 | |
| 0 | 6 | 0 | 3 | -2 | 1 | 0 | -4 | 3 | 5 | ________ |
| 0 | 4 | 1 | 2 | -5 | 0 | 0 | 2 | 1 | 8 | |

| (ix) -z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 1 | 4 | 0 | 0 | -2 | 2 | -45 | |
| 0 | 0 | 0 | -4 | 0 | 0 | 1 | -3 | 0 | 3 | ________ |
| 0 | 4 | 1 | 2 | -5 | 0 | 0 | 2 | 1 | -8 | |
| 0 | -6 | 0 | 3 | -2 | 1 | 0 | -4 | 3 | 15 | |

56:171 Operations Research
Homework #3 -- Due Wednesday, 26 September 2001

1. ***Revised Simplex Algorithm:*** Consider the LP:

$$\text{Minimize } z = 3x_1 + 2x_2 + 6x_3$$

$$\text{subject to } \begin{cases} 4x_1 + 8x_2 - x_3 \le 5 \\ 7x_1 - 2x_2 + 2x_3 \ge 4 \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{cases}$$

*Note: In the computation that follows, you need not use more than 3 significant digits.*

By introducing slack and surplus variables, the problem is rewritten as Min cx subject to Ax=b, x≥0 where C = [3, 2, 6, 0, 0], b = [5, 4] and A = $\begin{bmatrix} 4 & 8 & -1 & 1 & 0 \\ 7 & -2 & 2 & 0 & -1 \end{bmatrix}$.

Suppose that "Phase I" has found the initial basis B ={1,2} for the constraints, i.e., basic variables $x_1$ and $x_2$.

a. Then using the revised simplex method requires computation of:

$$c_B = [\_ \quad \_], A^B = \begin{bmatrix} - & - \\ \_ & \_ \end{bmatrix}, \left(A^B\right)^{-1} = \begin{bmatrix} - & - \\ \_ & \_ \end{bmatrix},$$

$$x_B = \left(A^B\right)^{-1} b = [\_ \quad \_], \pi = c_B \left(A^B\right)^{-1} = [\_ \quad \_]$$

b. Use the simplex multiplier vector π to compute the reduced cost of $x_3$:

$$\bar{c}_3 = c_3 - \pi A^3 = \_\_\_\_\_\_$$

c. Will entering $x_3$ into the basis improve the solution? _____

d. Use the simplex multiplier vector π to compute the reduced cost of $x_4$:

$$\bar{c}_4 = c_4 - \pi A^4 = \_\_\_\_\_\_$$

e. Will entering $x_4$ into the basis improve the solution? _____

f. Select either $x_3$ or $x_4$ to enter the basis, and compute the substitution rates (where j=3 or 4):

$$\alpha = \left(A^B\right)^{-1} A^j = \begin{bmatrix} \text{—} \\ \_\_ \end{bmatrix}$$

g. Perform the minimum ratio test to determine which variable leaves the basis. The new basis is B={ ___, ___ }.

h. Compute, for this new basis,

$$c_B = [\_ \quad \_], A^B = \begin{bmatrix} - & - \\ \_ & \_ \end{bmatrix}, \left(A^B\right)^{-1} = \begin{bmatrix} - & - \\ \_ & \_ \end{bmatrix},$$

$$x_B = \left(A^B\right)^{-1} b = [0.571, \quad \_], \pi = c_B \left(A^B\right)^{-1} = [0, \quad \_]$$

i. Find a nonbasic variable, if any, which would improve the solution if entered into the basis, and determine which variable would be replaced in the basis.

*You need not continue computations after two iterations even if the solution at that point is not optimal!*

2. ***LP Duality:*** Write the dual of the following LP:

$$Min\ 3x_1 + 2x_2 - 4x_3$$

$$\text{subject to} \begin{cases} 5x_1 - 7x_2 + x_3 \geq 12 \\ x_1 - x_2 + 2x_3 = 18 \\ 2x_1 - x_3 \leq 6 \\ x_2 + 2x_3 \geq 10 \\ x_j \geq 0,\ \text{j=1,2,3} \end{cases}$$

3. Consider the following primal LP problem:

$$Max\ x_1 + 2x_2 - 9x_3 + 8x_4 - 36x_5$$

$$\text{subject to} \begin{cases} 2x_2 - x_3 + x_4 - 3x_5 \leq 40 \\ x_1 - x_2 + 2x_4 - 2x_5 \leq 10 \\ x_j \geq 0,\ \text{j=1,2,3,4,5} \end{cases}$$

a. Write the dual LP problem
b. Sketch the feasible region of the dual LP in 2 dimensions, and use it to find the optimal solution.
c. Using complementary slackness conditions,
   - write equations which must be satisfied by the optimal primal solution x*
   - which primal variables must be zero?
d. Using the information in (c), determine the optimal solution x*.
e. Compare the optimal objective values of the primal and dual solutions.

4. ***LP Sensitivity Analysis:*** Cornco produces two products: PS and QT. The sales price for each product and the maximum quantity of each that can be sold during each of the next three months are:

| | Month 1 | | Month 2 | | Month 3 | |
|---|---|---|---|---|---|---|
| Product | Price | Demand | Price | Demand | Price | Demand |
| PS | \$40 | 50 | \$60 | 45 | \$55 | 50 |
| QT | \$35 | 43 | \$40 | 50 | \$44 | 40 |

Each product must be processed through two assembly lines: 1 & 2. The number of hours required by each product on each assembly line are:

| Product | Line 1 | Line 2 |
|---|---|---|
| PS | 3 hours | 2 hours |
| QT | 2 hours | 2 hours |

The number of hours available on each assembly line during each month are:

| Line | Month 1 | Month 2 | Month 3 |
|---|---|---|---|
| 1 | 200 | 160 | 190 |
| 2 | 140 | 150 | 110 |

Each unit of PS requires 4 pounds of raw material while each unit of QT requires 3 pounds. A total of 710 units of raw material can be purchased during the three-month interval at \$3 per pound. At the beginning of month 1, 10 units of PS and 5 units of QT are available. It costs \$10 to hold a unit of a unit of either product in inventory for a month.

***Solution:***

Define variables

Pt = # units of product PS produced in month t, t=1,2,3
Qt = # units of product QT produced in month t, t=1,2,3
R = (total) # units of raw material purchased
St = # units of product PS sold in month t, t=1,2,3
Tt = # units of product QT sold in month t, t=1,2,3
It = # units of product PS in inventory at end of month t, t=0,1,2
Jt = # units of product QT in inventory at end of month t, t=0,1,2

Objective: Maximize profit =

$40S1 + 60S2 + 55S3$ *(revenue from sale of PS)*
$+35T1 + 40T2 + 44T3$ *(revenue from sale of QT)*
$- 3R$ *(purchase of raw material)*
$- 10I1 - 10I2$ *(storage cost of PS)*
$- 10J1 - 10 J2$ *(storage cost of QT)*

Subject to the constraints:

| | |
|---|---|
| $R \le 710$ | *(limited availability of raw material)* |
| $S1 \le 50$, $S2 \le 45$, $S3 \le 50$ | *(demand constraints for PS)* |
| $T1 \le 43$, $T2 \le 50$, $T3 \le 40$ | *(demand constraints for QT)* |
| $3P1 + 2Q1 \le 200$ | *(hours available on line 1, month 1)* |
| $3P2 + 2Q2 \le 160$ | *(hours available on line 1, month 2)* |
| $3P3 + 2Q3 \le 190$ | *(hours available on line 1, month 3)* |
| $2P1 + 2Q1 \le 140$ | *(hours available on line 2, month 1)* |
| $2P2 + 2Q2 \le 150$ | *(hours available on line 2, month 2)* |
| $2P3 + 2Q3 \le 110$ | *(hours available on line 2, month 3)* |
| $P1 + I0 = 50 + S1 + I1$ | *(material balance of PS, month 1)* |
| $P2 + I1 = 45 + S2 + I2$ | *(material balance of PS, month 2)* |
| $P3 + I2 = 50 + S3$ | *(material balance of PS, month 3)* |
| $Q1 + J0 = 43 + T1 + J1$ | *(material balance of QT, month 1)* |
| $Q2 + J1 = 50 + T2 + J2$ | *(material balance of QT, month 2)* |
| $Q3 + J2 = 40 + T3$ | *(material balance of QT, month 3)* |
| $4P1+3Q1+4P2+3Q2+4P3+3Q3 \le R$ | *(consumption of raw material)* |

Note: the upper bounds on R, St, Tt, etc. could be imposed either by using the "simple upper bound" (SUB) command or by adding a row to the problem. The former is preferred!

LINDO output:

```
MAX     40 S1 + 60 S2 + 55 S3 + 35 T1 + 40 T2 + 44 T3 - 3 R - 10 I1
      - 10 I2 - 10 J1 - 10 J2
  SUBJECT TO
         2)   3 P1 + 2 Q1 <=   200
         3)   3 P2 + 2 Q2 <=   160
         4)   3 P3 + 2 Q3 <=   190
         5)   2 P1 + 2 Q1 <=   140
         6)   2 P2 + 2 Q2 <=   150
         7)   2 P3 + 2 Q3 <=   110
         8) - S1 - I1 + P1 + I0 =    0
         9) - S2 + I1 - I2 + P2 =    0
        10) - S3 + I2 + P3 =    0
        11) - T1 - J1 + Q1 + J0 =    0
        12) - T2 + J1 - J2 + Q2 =    0
        13) - T3 + J2 + Q3 =    0
        14) - R + 4 P1 + 3 Q1 + 4 P2 + 3 Q2 + 4 P3 + 3 Q3 <=   0
  END
  SUB       S1       50.00000
  SUB       S2       45.00000
  SUB       S3       50.00000
  SUB       T1       43.00000
  SUB       T2       50.00000
  SUB       T3       40.00000
  SUB        R      710.00000
  SUB       I0       10.00000
  SUB       J0        5.00000

        OBJECTIVE FUNCTION VALUE
        1)      7590.000

  VARIABLE        VALUE          REDUCED COST
        S1       40.000000          0.000000
        S2       45.000000        -10.000000
```

| | | |
|---|---|---|
| S3 | 50.000000 | -6.000000 |
| T1 | 20.000000 | 0.000000 |
| T2 | 50.000000 | -5.000000 |
| T3 | 5.000000 | 0.000000 |
| R | 710.000000 | -2.000000 |
| I1 | 25.000000 | 0.000000 |
| I2 | 0.000000 | 11.000000 |
| J1 | 0.000000 | 10.000000 |
| J2 | 0.000000 | 1.000000 |
| P1 | 55.000000 | 0.000000 |
| Q1 | 15.000000 | 0.000000 |
| P2 | 20.000000 | 0.000000 |
| Q2 | 50.000000 | 0.000000 |
| P3 | 50.000000 | 0.000000 |
| Q3 | 5.000000 | 0.000000 |
| I0 | 10.000000 | -40.000000 |
| J0 | 5.000000 | -35.000000 |

| ROW | SLACK OR SURPLUS | DUAL PRICES |
|---|---|---|
| 2) | 5.000000 | 0.000000 |
| 3) | 0.000000 | 10.000000 |
| 4) | 30.000000 | 0.000000 |
| 5) | 0.000000 | 10.000000 |
| 6) | 10.000000 | 0.000000 |
| 7) | 0.000000 | 14.500000 |
| 8) | 0.000000 | -40.000000 |
| 9) | 0.000000 | -50.000000 |
| 10) | 0.000000 | -49.000000 |
| 11) | 0.000000 | -35.000000 |
| 12) | 0.000000 | -35.000000 |
| 13) | 0.000000 | -44.000000 |
| 14) | 0.000000 | 5.000000 |

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

| VARIABLE | CURRENT COEF | ALLOWABLE INCREASE | ALLOWABLE DECREASE |
|---|---|---|---|
| S1 | 40.000000 | 5.000000 | 1.000000 |
| S2 | 60.000000 | INFINITY | 10.000000 |
| S3 | 55.000000 | INFINITY | 6.000000 |
| T1 | 35.000000 | 2.000000 | 5.000000 |
| T2 | 40.000000 | INFINITY | 5.000000 |
| T3 | 44.000000 | 1.000000 | 29.000000 |
| R | -3.000000 | INFINITY | 2.000000 |
| I1 | -10.000000 | 1.500000 | 7.500000 |
| I2 | -10.000000 | 11.000000 | INFINITY |
| J1 | -10.000000 | 10.000000 | INFINITY |
| J2 | -10.000000 | 1.000000 | INFINITY |
| P1 | 0.000000 | 6.000000 | 2.000000 |
| Q1 | 0.000000 | 2.000000 | 5.000000 |
| P2 | 0.000000 | 7.500000 | 1.500000 |
| Q2 | 0.000000 | 1.000000 | 5.000000 |
| P3 | 0.000000 | INFINITY | 6.000000 |
| Q3 | 0.000000 | 6.000000 | 29.000000 |
| I0 | 0.000000 | INFINITY | 40.000000 |
| J0 | 0.000000 | INFINITY | 35.000000 |

RIGHTHAND SIDE RANGES

| ROW | CURRENT RHS | ALLOWABLE INCREASE | ALLOWABLE DECREASE |
|---|---|---|---|
| 2 | 200.000000 | INFINITY | 5.000000 |
| 3 | 160.000000 | 15.000000 | 3.750000 |
| 4 | 190.000000 | INFINITY | 30.000000 |

| | | | |
|---|---|---|---|
| 5 | 140.000000 | 11.500000 | 6.666667 |
| 6 | 150.000000 | INFINITY | 10.000000 |
| 7 | 110.000000 | 15.333333 | 3.333333 |
| 8 | 0.000000 | 40.000000 | 10.000000 |
| 9 | 0.000000 | 40.000000 | 10.000000 |
| 10 | 0.000000 | 5.000000 | 5.000000 |
| 11 | 0.000000 | 20.000000 | 23.000000 |
| 12 | 0.000000 | 15.000000 | 10.000000 |
| 13 | 0.000000 | 5.000000 | 35.000000 |
| 14 | 0.000000 | 5.000000 | 23.000000 |

THE TABLEAU

| ROW | (BASIS) | S1 | S2 | S3 | T1 | T2 | T3 |
|---|---|---|---|---|---|---|---|
| 1 | ART | 0.000 | 10.000 | 6.000 | 0.000 | 5.000 | 0.000 |
| 2 | SLK 2 | 0.000 | 0.000 | 1.000 | 0.000 | 0.333 | 0.000 |
| 3 | Q2 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 |
| 4 | SLK 4 | 0.000 | 0.000 | -1.000 | 0.000 | 0.000 | 0.000 |
| 5 | S1 | 1.000 | -1.000 | -1.000 | 0.000 | -1.000 | 0.000 |
| 6 | SLK 6 | 0.000 | 0.000 | 0.000 | 0.000 | -0.667 | 0.000 |
| 7 | Q3 | 0.000 | 0.000 | -1.000 | 0.000 | 0.000 | 0.000 |
| 8 | I1 | 0.000 | 1.000 | 0.000 | 0.000 | 0.667 | 0.000 |
| 9 | T1 | 0.000 | 0.000 | 1.000 | 1.000 | 0.333 | 0.000 |
| 10 | P3 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 |
| 11 | Q1 | 0.000 | 0.000 | 1.000 | 0.000 | 0.333 | 0.000 |
| 12 | P1 | 0.000 | 0.000 | -1.000 | 0.000 | -0.333 | 0.000 |
| 13 | T3 | 0.000 | 0.000 | -1.000 | 0.000 | 0.000 | 1.000 |
| 14 | P2 | 0.000 | 0.000 | 0.000 | 0.000 | -0.667 | 0.000 |

| ROW | R | I1 | I2 | J1 | J2 | P1 | Q1 |
|---|---|---|---|---|---|---|---|
| 1 | 2.000 | 0.000 | 11.000 | 10.000 | 1.000 | 0.000 | 0.000 |
| 2 | -1.000 | 0.000 | 1.000 | 0.333 | -0.333 | 0.000 | 0.000 |
| 3 | 0.000 | 0.000 | 0.000 | 1.000 | -1.000 | 0.000 | 0.000 |
| 4 | 0.000 | 0.000 | -1.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 5 | 1.000 | 0.000 | 0.000 | -1.000 | 1.000 | 0.000 | 0.000 |
| 6 | 0.000 | 0.000 | 0.000 | -0.667 | 0.667 | 0.000 | 0.000 |
| 7 | 0.000 | 0.000 | -1.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 8 | 0.000 | 1.000 | -1.000 | 0.667 | -0.667 | 0.000 | 0.000 |
| 9 | -1.000 | 0.000 | 1.000 | 1.333 | -0.333 | 0.000 | 0.000 |
| 10 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 11 | -1.000 | 0.000 | 1.000 | 0.333 | -0.333 | 0.000 | 1.000 |
| 12 | 1.000 | 0.000 | -1.000 | -0.333 | 0.333 | 1.000 | 0.000 |
| 13 | 0.000 | 0.000 | -1.000 | 0.000 | -1.000 | 0.000 | 0.000 |
| 14 | 0.000 | 0.000 | 0.000 | -0.667 | 0.667 | 0.000 | 0.000 |

| ROW | P2 | Q2 | P3 | Q3 | I0 | J0 | SLK 2 |
|---|---|---|---|---|---|---|---|
| 1 | 0.000 | 0.000 | 0.000 | 0.000 | 40.000 | 35.000 | 0.000 |
| 2 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 |
| 3 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 4 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 5 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 |
| 6 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 7 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 |
| 8 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 9 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 |
| 10 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 11 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 12 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 13 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 14 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| ROW | SLK 3 | SLK 4 | SLK 5 | SLK 6 | SLK 7 | SLK 14 | |
|---|---|---|---|---|---|---|---|
| 1 | 10.000 | 0.000 | 10.000 | 0.000 | 14.500 | 5.000 | 7590.000 |
| 2 | 1.333 | 0.000 | 0.500 | 0.000 | 1.500 | -1.000 | 5.000 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 50.000 |
| 4 | 0.000 | 1.000 | 0.000 | 0.000 | -1.000 | 0.000 | 30.000 |
| 5 | -1.000 | 0.000 | -1.500 | 0.000 | -1.500 | 1.000 | 40.000 |
| 6 | -0.667 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 10.000 |
| 7 | 0.000 | 0.000 | 0.000 | 0.000 | 0.500 | 0.000 | 5.000 |
| 8 | -0.333 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 25.000 |
| 9 | 1.333 | 0.000 | 2.000 | 0.000 | 1.500 | -1.000 | 20.000 |
| 10 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 50.000 |
| 11 | 1.333 | 0.000 | 2.000 | 0.000 | 1.500 | -1.000 | 15.000 |
| 12 | -1.333 | 0.000 | -1.500 | 0.000 | -1.500 | 1.000 | 55.000 |
| 13 | 0.000 | 0.000 | 0.000 | 0.000 | 0.500 | 0.000 | 5.000 |
| 14 | 0.333 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 20.000 |
| 16 | 0.500 | 0.000 | 0.000 | 0.000 | 5.000 | | |

*Answer the questions below, using the output above for the original problem, if possible. If not possible, you need not run LINDO again—simply state that it is necessary to run LINDO again.*

*Note (i) the "new optimal solution" may be identical to the current optimal solution!*

*(ii) even if the optimal basis is unchanged, the values of the basic variables in the optimal solution may change!*

a. Find the new optimal solution if it costs \$11 to hold a unit of PS in inventory at the end of month 1.

b. Find the company's new optimal solution if 210 hours on line 1 are available during month 1.

c. Find the company's new profit level if 109 hours are available on line 2 during month 3.

d. What is the most Cornco should be willing to pay for an extra hour of line 1 time during month 2?

e. What is the most Cornco should be willing to pay for an extra hour of line 1 time during month 3?

f. Find the new optimal solution if PS sells for \$50 during month 2.

g. Find the new optimal solution if QT sells for \$50 during month 3.

h. Suppose spending \$20 on advertising would increase demand for QT in month 2 by 5 units. Should the advertising be done?

| 56:171 Operations Research |
|---|
| Homework #4 – due 3 October 2001 |

***1. Linear Programming sensitivity.*** A paper recycling plant processes box board, tissue paper, newsprint, and book paper into pulp that can be used to produce three grades of recycled paper (grades 1, 2, and 3). The prices per ton and the pulp contents of the four inputs are:

| Input type | Cost $/ton | Pulp content |
|---|---|---|
| Box board | 5 | 15% |
| Tissue paper | 6 | 20% |
| Newsprint | 8 | 30% |
| Book paper | 10 | 40% |

Two methods, de-inking and asphalt dispersion, can be used to process the four inputs into pulp. It costs $20 to de-ink a ton of any input. The process of de-inking removes 10% of the input's pulp. It costs $15 to apply asphalt dispersion to a ton of material. The asphalt dispersion process removes 20% of the input's pulp. At most 3000 tons of input can be run through the asphalt dispersion process or the de-inking process. Grade 1 paper can only be produced with newsprint or book paper pulp; grade 2 paper, only with book paper, tissue paper, or box board pulp; and grade 3 paper, only with newsprint, tissue paper, or box board pulp. To meet its current demands, the company needs 500 tons of pulp for grade 1 paper, 500 tons of pulp for grade 2 paper, and 600 tons of pulp for grade 3 paper. The LP model below was formulated to minimize the cost of meeting the demands for pulp.

*Define the variables*

BOX = tons of purchased boxboard
TISS = tons of purchased tissue
NEWS = tons of purchased newsprint
BOOK = tons of purchased book paper
BOX1 = tons of boxboard sent through de-inking
TISS1 = tons of tissue sent through de-inking
NEWS1 = tons of newsprint sent through de-inking
BOOK1 = tons of book paper sent through de-inking
BOX2 = tons of boxboard sent through asphalt dispersion
TISS2 = tons of tissue sent through asphalt dispersion
NEWS2 = tons of newsprint sent through asphalt dispersion
BOOK2 = tons of book paper sent through asphalt dispersion
PBOX = tons of pulp recovered from boxboard
PTISS = tons of pulp recovered from tissue
PNEWS= tons of pulp recovered from newsprint
PBOOK = tons of pulp recovered from book paper
PBOX1 = tons of boxboard pulp used for grade 1 paper,
PBOX2 = tons of boxboard pulp used for grade 2 paper, etc.
...
PBOOK3 = tons of book paper pulp used for grade 3 paper.

The LP model using these variables is:

```
MIN  5 BOX +6 TISS +8 NEWS +10 BOOK +20 BOX1 +20 TISS1 +20 NEWS1
          +20 BOOK1 +15 BOX2 +15 TISS2 +15 NEWS2 +15 BOOK2
 SUBJECT TO
        2) - BOX + BOX1 + BOX2 <=   0
        3) - TISS + TISS1 + TISS2 <=   0
        4) - NEWS + NEWS1 + NEWS2 <=   0
        5) - BOOK + BOOK1 + BOOK2 <=   0
        6)   0.135 BOX1 + 0.12 BOX2 - PBOX =    0
        7)   0.18 TISS1 + 0.16 TISS2 - PTISS =    0
        8)   0.27 NEWS1 + 0.24 NEWS2 - PNEWS =    0
        9)   0.36 BOOK1 + 0.32 BOOK2 - PBOOK =    0
       10) - PBOX + PBOX2 + PBOX3 <=   0
       11) - PTISS + PTISS2 + PTISS3 <=   0
       12) - PNEWS + PNEWS1 + PNEWS3 <=   0
       13) - PBOOK + PBOOK1 + PBOOK2 <=   0
       14)   PNEWS1 + PBOOK1 >=   500
       15)   PBOX2 + PTISS2 + PBOOK2 >=   500
```

```
      16)   PBOX3 + PTISS3 + PNEWS3 >=   600
      17)   BOX1 + TISS1 + NEWS1 + BOOK1 <=   3000
      18)   BOX2 + TISS2 + NEWS2 + BOOK2 <=   3000
END
```

- Rows 2-5 state that only the supply of each input material which is purchased can be processed, either by de-inking or asphalt dispersion.
- Row 6 states that the recovered pulp in the boxboard is 90% of that in the boxboard which is processed by de-inking, i.e., (0.90)(0.15)BOX1, since boxboard is 15% pulp, plus 80% of that in the boxboard which is processed by asphalt dispersion, i.e., (0.80)(0.15)BOX2.
- Rows 7-9 are similar to row 6, but for different input materials.
- Rows 10-13 state that no more than the pulp which is recovered from each input may be used in making paper (grades 1, 2, &/or 3). Note that some variables are omitted, e.g., PBOX1, since boxboard cannot be used in Grade 1 paper.
- Rows 14-16 state that the demand for each grade of paper must be satisfied.
- Rows 17-18 state that each process (de-inking & asphalt dispersion) has a maximum throughput of 3000 tons. (The problem statement is unclear here, and might be interpreted as stating the total throughput of both processes cannot exceed 3000 tons, in which case rows 17&18 would be replaced by

```
17)   BOX1 + TISS1 + NEWS1 + BOOK1
    + BOX2 + TISS2 + NEWS2 + BOOK2 <=   3000
```

The solution found by LINDO is as follows:

```
 LP OPTIMUM FOUND AT STEP     25
       OBJECTIVE FUNCTION VALUE

       1)      140000.0

 VARIABLE        VALUE          REDUCED COST
      BOX         0.000000          0.000000
     TISS         0.000000          6.000000
     NEWS      2500.000000          0.000000
     BOOK      2833.333252          0.000000
     BOX1         0.000000         11.124999
    TISS1         0.000000          1.499999
    NEWS1         0.000000          0.249999
    BOOK1      2333.333252          0.000000
     BOX2         0.000000          9.333334
    TISS2         0.000000          0.222223
    NEWS2      2500.000000          0.000000
    BOOK2       500.000000          0.000000
     PBOX         0.000000          0.000000
    PTISS         0.000000          0.000000
    PNEWS       600.000000          0.000000
    PBOOK      1000.000000          0.000000
    PBOX2         0.000000         19.444445
    PBOX3         0.000000          0.000000
   PTISS2         0.000000         19.444445
   PTISS3         0.000000          0.000000
   PNEWS1         0.000000         19.444445
   PNEWS3       600.000000          0.000000
   PBOOK1       500.000000          0.000000
   PBOOK2       500.000000          0.000000

      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)         0.000000          5.000000
       3)         0.000000          0.000000
       4)         0.000000          8.000000
       5)         0.000000         10.000000
       6)         0.000000       -102.777779
       7)         0.000000       -102.777779
       8)         0.000000       -102.777779
       9)         0.000000        -83.333336
      10)         0.000000        102.777779
      11)         0.000000        102.777779
      12)         0.000000        102.777779
      13)         0.000000         83.333336
      14)         0.000000        -83.333336
      15)         0.000000        -83.333336
```

```
     16)        0.000000       -102.777779
     17)      666.666687          0.000000
     18)        0.000000          1.666667

RANGES IN WHICH THE BASIS IS UNCHANGED:
                          OBJ COEFFICIENT RANGES
VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
     BOX        5.000000         INFINITY         5.000000
    TISS        6.000000         INFINITY         6.000000
    NEWS        8.000000         0.333334         4.666667
    BOOK       10.000000         6.000000         1.999989
    BOX1       20.000000         INFINITY        11.124999
   TISS1       20.000000         INFINITY         1.499999
   NEWS1       20.000000         INFINITY         0.249999
   BOOK1       20.000000         0.249999         0.750001
    BOX2       15.000000         INFINITY         9.333333
   TISS2       15.000000         INFINITY         0.222222
   NEWS2       15.000000         0.222221         4.666667
   BOOK2       15.000000         0.666667         0.222221
    PBOX        0.000000         INFINITY        77.777779
   PTISS        0.000000         INFINITY         1.388890
   PNEWS        0.000000         1.388890        19.444443
   PBOOK        0.000000        19.444443        83.333336
   PBOX2        0.000000         INFINITY        19.444443
   PBOX3        0.000000        19.444443        77.777779
  PTISS2        0.000000         INFINITY        19.444443
  PTISS3        0.000000        19.444443         1.388890
  PNEWS1        0.000000         INFINITY        19.444443
  PNEWS3        0.000000         1.388890        19.444443
  PBOOK1        0.000000        19.444443        83.333336
  PBOOK2        0.000000        19.444443        83.333336

                          RIGHTHAND SIDE RANGES
     ROW         CURRENT        ALLOWABLE        ALLOWABLE
                   RHS           INCREASE         DECREASE
       2        0.000000         0.000000         INFINITY
       3        0.000000         INFINITY         0.000000
       4        0.000000      2500.000000         INFINITY
       5        0.000000      2833.333252         INFINITY
       6        0.000000         0.000000       600.000000
       7        0.000000         0.000000       600.000000
       8        0.000000       120.000008       600.000000
       9        0.000000       240.000015       840.000000
      10        0.000000       600.000000         0.000000
      11        0.000000       600.000000         0.000000
      12        0.000000       600.000000       120.000008
      13        0.000000       840.000000       240.000015
      14      500.000000       240.000015       500.000000
      15      500.000000       240.000015       500.000000
      16      600.000000       120.000008       600.000000
      17     3000.000000         INFINITY       666.666687
      18     3000.000000      2625.000000       500.000000

THE TABLEAU

     ROW  (BASIS)        BOX        TISS        NEWS        BOOK        BOX1       TISS1
       1 ART           0.000       6.000       0.000       0.000      11.125       1.500
       2     BOOK      0.000       0.000       0.000       1.000      -0.062      -0.083
       3 SLK    3      0.000      -1.000       0.000       0.000       0.000       1.000
       4 SLK   17      0.000       0.000       0.000       0.000       0.500       0.333
       5    BOOK1      0.000       0.000       0.000       0.000       0.500       0.667
       6     PBOX      0.000       0.000       0.000       0.000      -0.135       0.000
       7    PTISS      0.000       0.000       0.000       0.000       0.000      -0.180
       8    PNEWS      0.000       0.000       0.000       0.000       0.135       0.180
       9    PBOOK      0.000       0.000       0.000       0.000       0.000       0.000
      10    PBOX3      0.000       0.000       0.000       0.000      -0.135       0.000
      11   PTISS3      0.000       0.000       0.000       0.000       0.000      -0.180
      12   PNEWS3      0.000       0.000       0.000       0.000       0.135       0.180
```

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 13 | PBOOK2 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 14 | PBOOK1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 15 | NEWS2 | 0.000 | 0.000 | 0.000 | 0.000 | 0.562 | 0.750 |
| 16 | NEWS | 0.000 | 0.000 | 1.000 | 0.000 | 0.562 | 0.750 |
| 17 | BOX | 1.000 | 0.000 | 0.000 | 0.000 | -1.000 | 0.000 |
| 18 | BOOK2 | 0.000 | 0.000 | 0.000 | 0.000 | -0.562 | -0.750 |

| ROW | NEWS1 | BOOK1 | BOX2 | TISS2 | NEWS2 | BOOK2 | PBOX |
|---|---|---|---|---|---|---|---|
| 1 | 0.250 | 0.000 | 9.333 | 0.222 | 0.000 | 0.000 | 0.000 |
| 2 | -0.125 | 0.000 | 0.056 | 0.037 | 0.000 | 0.000 | 0.000 |
| 3 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 |
| 4 | 0.000 | 0.000 | 0.444 | 0.296 | 0.000 | 0.000 | 0.000 |
| 5 | 1.000 | 1.000 | -0.444 | -0.296 | 0.000 | 0.000 | 0.000 |
| 6 | 0.000 | 0.000 | -0.120 | 0.000 | 0.000 | 0.000 | 1.000 |
| 7 | 0.000 | 0.000 | 0.000 | -0.160 | 0.000 | 0.000 | 0.000 |
| 8 | 0.000 | 0.000 | 0.120 | 0.160 | 0.000 | 0.000 | 0.000 |
| 9 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 10 | 0.000 | 0.000 | -0.120 | 0.000 | 0.000 | 0.000 | 0.000 |
| 11 | 0.000 | 0.000 | 0.000 | -0.160 | 0.000 | 0.000 | 0.000 |
| 12 | 0.000 | 0.000 | 0.120 | 0.160 | 0.000 | 0.000 | 0.000 |
| 13 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 14 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 15 | 1.125 | 0.000 | 0.500 | 0.667 | 1.000 | 0.000 | 0.000 |
| 16 | 0.125 | 0.000 | 0.500 | 0.667 | 0.000 | 0.000 | 0.000 |
| 17 | 0.000 | 0.000 | -1.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 18 | -1.125 | 0.000 | 0.500 | 0.333 | 0.000 | 1.000 | 0.000 |

| ROW | PTISS | PNEWS | PBOOK | PBOX2 | PBOX3 | PTISS2 | PTISS3 |
|---|---|---|---|---|---|---|---|
| 1 | 0.000 | 0.000 | 0.000 | 19.444 | 0.000 | 19.444 | 0.000 |
| 2 | 0.000 | 0.000 | 0.000 | 3.241 | 0.000 | 3.241 | 0.000 |
| 3 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 4 | 0.000 | 0.000 | 0.000 | 0.926 | 0.000 | 0.926 | 0.000 |
| 5 | 0.000 | 0.000 | 0.000 | -0.926 | 0.000 | -0.926 | 0.000 |
| 6 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 7 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 8 | 0.000 | 1.000 | 0.000 | -1.000 | 0.000 | -1.000 | 0.000 |
| 9 | 0.000 | 0.000 | 1.000 | 1.000 | 0.000 | 1.000 | 0.000 |
| 10 | 0.000 | 0.000 | 0.000 | 1.000 | 1.000 | 0.000 | 0.000 |
| 11 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 1.000 |
| 12 | 0.000 | 0.000 | 0.000 | -1.000 | 0.000 | -1.000 | 0.000 |
| 13 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 1.000 | 0.000 |
| 14 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 15 | 0.000 | 0.000 | 0.000 | -4.167 | 0.000 | -4.167 | 0.000 |
| 16 | 0.000 | 0.000 | 0.000 | -4.167 | 0.000 | -4.167 | 0.000 |
| 17 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 18 | 0.000 | 0.000 | 0.000 | 4.167 | 0.000 | 4.167 | 0.000 |

| ROW | PNEWS1 | PNEWS3 | PBOOK1 | PBOOK2 | SLK 2 | SLK 3 | SLK 4 |
|---|---|---|---|---|---|---|---|
| 1 | 19.444 | 0.000 | 0.000 | 0.000 | 5.000 | 0.000 | 8.000 |
| 2 | 3.241 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 3 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 |
| 4 | 0.926 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 5 | -0.926 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 6 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 7 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 8 | -1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 9 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 10 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 11 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 12 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 13 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 |
| 14 | 1.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 15 | -4.167 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 16 | -4.167 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | -1.000 |
| 17 | 0.000 | 0.000 | 0.000 | 0.000 | -1.000 | 0.000 | 0.000 |
| 18 | 4.167 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| ROW | SLK 5 | SLK 10 | SLK 11 | SLK 12 | SLK 13 | SLK 14 | SLK 15 |
|---|---|---|---|---|---|---|---|
| 1 | 10.000 | 102.778 | 102.778 | 102.778 | 83.333 | 83.333 | 83.333 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | -1.000 | 0.463 | 0.463 | 0.463 | -2.778 | -2.778 | -2.778 |
| 3 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 4 | 0.000 | 3.704 | 3.704 | 3.704 | 2.778 | 2.778 | 2.778 |
| 5 | 0.000 | -3.704 | -3.704 | -3.704 | -2.778 | -2.778 | -2.778 |
| 6 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 7 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 8 | 0.000 | -1.000 | -1.000 | -1.000 | 0.000 | 0.000 | 0.000 |
| 9 | 0.000 | 0.000 | 0.000 | 0.000 | -1.000 | -1.000 | -1.000 |
| 10 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 11 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 12 | 0.000 | -1.000 | -1.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 13 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | -1.000 |
| 14 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | -1.000 | 0.000 |
| 15 | 0.000 | -4.167 | -4.167 | -4.167 | 0.000 | 0.000 | 0.000 |
| 16 | 0.000 | -4.167 | -4.167 | -4.167 | 0.000 | 0.000 | 0.000 |
| 17 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 18 | 0.000 | 4.167 | 4.167 | 4.167 | 0.000 | 0.000 | 0.000 |

| ROW | SLK 16 | SLK 17 | SLK 18 | RHS |
|---|---|---|---|---|
| 1 | 0.10E+03 | 0.00E+00 | 1.7 | -0.14E+06 |
| 2 | 0.463 | 0.000 | 0.111 | 2833.333 |
| 3 | 0.000 | 0.000 | 0.000 | 0.000 |
| 4 | 3.704 | 1.000 | 0.889 | 666.667 |
| 5 | -3.704 | 0.000 | -0.889 | 2333.333 |
| 6 | 0.000 | 0.000 | 0.000 | 0.000 |
| 7 | 0.000 | 0.000 | 0.000 | 0.000 |
| 8 | -1.000 | 0.000 | 0.000 | 600.000 |
| 9 | 0.000 | 0.000 | 0.000 | 1000.000 |
| 10 | 0.000 | 0.000 | 0.000 | 0.000 |
| 11 | 0.000 | 0.000 | 0.000 | 0.000 |
| 12 | -1.000 | 0.000 | 0.000 | 600.000 |
| 13 | 0.000 | 0.000 | 0.000 | 500.000 |
| 14 | 0.000 | 0.000 | 0.000 | 500.000 |
| 15 | -4.167 | 0.000 | 0.000 | 2500.000 |
| 16 | -4.167 | 0.000 | 0.000 | 2500.000 |
| 17 | 0.000 | 0.000 | 0.000 | 0.000 |
| 18 | 4.167 | 0.000 | 1.000 | 500.000 |

*a.* *Complete the following statements:* the optimal solution is to purchase only newsprint and book paper, process ______ tons of the book paper and _____ tons of the newsprint by asphalt dispersion, and the remaining book paper by de-inking. This yields ______ tons of pulp from the newsprint and ______ tons of pulp from the book paper. One-half of the pulp from book paper is used in each of grades 1 & 2 paper, and the newsprint is used in grade 3 paper. This plan will use ______% of the de-inking capacity and ______% of the asphalt dispersion capacity. Note that BOX is a basic variable, but because it has a value of zero, this solution is categorized as ______________.

*b.* How much must newsprint increase in price in order that less would be used? ________

*c.* In the optimal solution, no newsprint is processed by the de-inking. Suppose that 5 tons of newsprint were to be de-inked. How should the solution best be modified to compensate? In particular, what should be the adjusted values of:

| Quantity | Current value | Adjusted value |
|---|---|---|
| BOX = tons of purchased boxboard | _________ | _________ |
| TISS = tons of purchased tissue | _________ | _________ |
| NEWS = tons of purchased newsprint | _________ | _________ |
| BOOK = tons of purchased book paper | _________ | _________ |
| BOX1 = tons of boxboard sent through de-inking | _________ | _________ |
| TISS1 = tons of tissue sent through de-inking | _________ | _________ |
| NEWS1 = tons of newsprint sent through de-inking | _________ | _________ |
| BOOK1 = tons of book paper sent through de-inking | _________ | _________ |
| PNEWS= tons of pulp recovered from newsprint | _________ | _________ |

*d.* Suppose that ten additional tons of pulp for grade 3 paper were required. Is this within the range of requirements for which the current basis is optimal? ________
What would be the effect on the cost? _______
How would the quantities of the four raw materials change?

| Raw material | Current value | Adjusted value |
|---|---|---|
| BOX = tons of purchased boxboard | _________ | _________ |

TISS = tons of purchased tissue __________ __________
NEWS = tons of purchased newsprint __________ __________
BOOK = tons of purchased book paper __________ __________

*(Hint: if the surplus variable for the row stating the requirement were to increase, what effect would it have on the basic variables BOX, NEWS, and BOOK?)*

2. (A modification of Exercise 3, page 317, of Operations Research, by W. Winston). You have been assigned to evaluate the efficiency of the Port Charles Police Department. Eight precincts are to be evaluated. The inputs and outputs for each precinct are as follows:

Inputs:
- Number of policemen
- Number of vehicles used

Outputs:
- Number of patrol units responding to service requests (thousands/year)
- Number of convictions obtained each year (hundreds/year)

The following data has been collected:

| Precinct | # policemen | # vehicles | # responses | #convictions |
|---|---|---|---|---|
| A | 200 | 60 | 6 | 8 |
| B | 250 | 65 | 5.5 | 9 |
| C | 300 | 90 | 8 | 9.5 |
| D | 400 | 120 | 10 | 11 |
| E | 350 | 100 | 9.5 | 9 |
| F | 300 | 80 | 5 | 7.5 |
| G | 275 | 85 | 9 | 8 |
| H | 325 | 75 | 4.5 | 10 |

The city wishes to use this information to determine which precincts, if any, are inefficient.

a. Write the LP model which can be used to compute the efficiency of precinct C.

b. What is the total number of LP problems which need to be solved in order to compute the efficiencies of the eight precincts? _____

One might use LINDO to do the computation, or any of several other software packages for data envelopment analysis—see, for example, the website
http://www.wiso.uni-dortmund.de/lsfg/or/scheel/doordea.htm )
The output below was computed by the APL workspace "DEA" which can be downloaded from the website at URL:
http://asrl.ecn.uiowa.edu/dbricker/APL_software.html

| i | ID | Efficiency | Freq | R |
|---|---|---|---|---|
| 1 | A | 1 | 4 | 1 |
| 2 | B | 1 | 2 | 3 |
| 3 | C | 0.8727 | 0 | 6 |
| 4 | D | 0.809 | 0 | 7 |
| 5 | E | 0.9042 | 0 | 5 |
| 6 | F | 0.6875 | 0 | 8 |
| 7 | G | 1 | 3 | 2 |
| 8 | H | 0.963 | 0 | 4 |

Freq = frequency of appearance in reference sets of inefficient DMUs
R = rank based upon (Efficiency + Freq)

Slack inputs/outputs

| i/o | C | D | E | F | H |
|---|---|---|---|---|---|
| responses | 0 | 0 | 0 | 0 | 1.61111 |
| convictions | 0 | 0 | 0 | 0 | 0 |
| policemen | 2.43056 | 4.86111 | 22.7083 | 6.25 | 35.1852 |
| vehicles | 0 | 0 | 0 | 0 | 0 |

Prices

| i | ID | responses | convictions | policemen | vehicles |
|---|---|---|---|---|---|
| 1 | A | 0.125 | 0.03125 | 0.005 | 0 |
| 2 | B | 0 | 0.111111 | 0.00111111 | 0.0111111 |
| 3 | C | 0.0925926 | 0.0138889 | 0 | 0.0111111 |
| 4 | D | 0.0694444 | 0.0104167 | 0 | 0.00833333 |
| 5 | E | 0.0833333 | 0.0125 | 0 | 0.01 |
| 6 | F | 0.025 | 0.075 | 0 | 0.0125 |
| 7 | G | 0.0909091 | 0.0227273 | 0.00363636 | 0 |
| 8 | H | 0 | 0.0962963 | 0 | 0.0133333 |

Reference Sets

For each DMU, the DMUs in its reference set are listed:

| 3 C | 4 D | 5 E | 6 F | 8 H |
|---|---|---|---|---|
| 1 A | 1 A | 7 G | 2 B | 2 B |
| 7 G | 7 G | 1 A | 1 A | |

c. Using the prices given in the table above for precinct C (e.g. 0.0925926 for *responses*), compute the ratio of the total value of the output variables *responses* and *convictions* to the total value of input variables *policemen* and *vehicles*. _______________

d. By how much should precinct C cut its number of policemen in order to become "efficient" (assuming that they could maintain their current output levels)? _________

3. The ZapCon Company is considering investing in three projects. If it fully invests in a project, the realized cash flows (in millions of dollars) will be as listed in the table below.

| *Time (years)* | *Cash flow project 1* | *Cash flow project 2* | *Cash flow project 3* |
|---|---|---|---|
| 0 | –3 | –2 | –2.0 |
| 0.5 | –1 | –0.5 | –2.0 |
| 1 | –1.8 | 1.5 | –1.8 |
| 1.5 | 0.4 | 1.5 | 1 |
| 2 | 1.8 | 1.5 | 1 |
| 2.5 | 1.8 | 0.2 | 1 |
| 3 | 5.5 | –1.0 | 6 |

For example, project 1 requires an initial cash outflow of $3 million, smaller outlays six months and one year from now, begins paying a small return 1.5 years from now, and a final payback of $5.5 million 3 years from now. Today ZapCon has $2 million in cash. At each time point (0, 0.5, 1, 1.5 2, and 2.5 years from today) the com[any can, if desired, borrow up to $2 million at 3.5% (per 6 months) interest. Leftover cash earns 3% (per six months) interest. For example, if after borrowing and investing at time 0, ZapCon has $1 million, it would receive $30,000 in interest at time 0.5 year. the company's goal is to maximize cash on hand after cash flows 3 years from now are accounted for. What investment and borrowing strategy should it use? Assume that the company can invest in a fraction of a project. For example, if it invests in 0.5 of project 3, it has, for example, cash outflows of -$1 million at times 0 and 0.5. No more than 100% investment in a project is possible, however.

a. Formulate a linear programming model to optimize the investment plan.

b. Use LINDO ( or other LP solver) to find the optimal solution.

56:171 Operations Research

# Homework #5 – due 10 October 2001

**1.** ***Transportation Problem*** Consider the following "balanced" transportation problem with three sources and four destinations, where the transportation cost/unit shipped, supplies available, and amounts required are shown in the table:

| Plant \ Warehouse | 1 | 2 | 3 | 4 | **Supply** |
|---|---|---|---|---|---|
| A | 1 | 4 | 8 | 6 | **7** |
| B | 1 | 10 | 1 | 7 | **10** |
| C | 8 | 5 | 6 | 9 | **3** |
| **Demand** | **6** | **6** | **3** | **5** | |

a. A linear programming model of this problem will have ____ equality constraints (not counting the objective) and ______ basic variables.

b. Find an initial feasible basic solution, using the "Northwest Corner Rule":

| | | | |
|---|---|---|---|
| 1 | 4 | 8 | 6 |
| 1 | 10 | 1 | 7 |
| 8 | 5 | 6 | 9 |

c. The shipping cost of this solution is ______ .

d. Compute the reduced cost of the variable $X_{A4}$ by identifying the "cycle" of adjustments that would be required in the NW-corner solution if $X_{A4}$ were to be increased by one unit.

e. Entering $X_{A4}$ into the basis will _______ (increase/decrease) the objective function by _____ per unit shipped from A to 4.

f. Compute a set of "dual variables" corresponding to the initial NW-corner solution, and use them to compute the reduced cost of $X_{A4}$ :
Corresponding to supply constraints: $U_A$=_____, $U_B$=______, $U_C$=_______
Corresponding to demand constraints: $V_1$=______, $V_2$=______, $V_3$=_______, $V_4$=_______

Reduced cost of $X_{A4}$ is $C_{A4}-(U_A+V_4)$ = ________ – _________ = _________

g. The reduced cost of $X_{B1}$ is is $C_{B1}-(U_B+V_1)$ = _________. Entering $X_{B1}$ into the basis would cause the variable _____ to leave the basis, resulting in the basic solution:

| | | | |
|---|---|---|---|
| 1 | 4 | 8 | 6 |
| 1 | 10 | 1 | 7 |
| 8 | 5 | 6 | 9 |

The increase in $X_{B1}$ (____ units) times the reduced cost ( ______ ) is ________, so that the cost of the new solution is _________ .

h. Continue changing the basis until you have found the optimal solution:

| | | | |
|---|---|---|---|
| 1 | 4 | 8 | 6 |
| 1 | 10 | 1 | 7 |
| 8 | 5 | 6 | 9 |

i. The optimal cost is _______ .

2. *Powerhouse* produces capacitors at three locations: Los Angeles, Chicago, and New York. Capacitors are shipped from these locations to public utilities in five regions of the country: northeast (NE), northwest (NW), midwest (MW), southeast (SE), and southwest (SW). The cost of producing and shipping a capacitor from each plant to each region of the country is given in the table below.

| From \ to | NE | NW | MW | SE | SW |
|---|---|---|---|---|---|
| **LA** | $27.86 | $4.00 | $20.54 | $21.52 | $13.87 |
| **Chicago** | $8.02 | $20.54 | $2.00 | $6.74 | $10.67 |
| **NY** | $2.00 | $27.86 | $8.02 | $8.41 | $15.20 |

Each plant has an annual production capacity of 100,000 capacitors. Each year, each region of the country must receive the following number of capacitors: NE: 55,000; NW: 50,000; MW: 60,000; SE: 60,000; SW: 45,000. *Powerhouse* feels shipping costs are too high, and the company is therefore considering building one or two more production plants. Possible sites are Atlanta and Houston. The costs of producing a capacitor and shipping it to each region of the country are:

| From \ to | NE | NW | MW | SE | SW |
|---|---|---|---|---|---|
| **Atlanta** | $8.41 | $21.52 | $6.74 | $3.00 | $7.89 |
| **Houston** | $15.20 | $13.87 | $10.67 | $7.89 | $3.00 |

It costs $3 million (in current dollars) to build a new plant, and operating each plant incurs a fixed cost (in addition to variable shipping and production costs) of $50,000 per year. A plant at Atlanta or Houston will have the capacity to produce 100,000 capacitors per year.

Assume that future demand patterns and production costs will remain unchanged. If costs are discounted at a rate of $11\,^{1}/_{9}$ % per year, how can *Powerhouse* minimize the present value of all costs associated with meeting current and future demands?

3. The coach of a swim team needs to assign four swimmers to a 400-meter medley relay team. The "best times" (in seconds for 100 meters) achieved by his seven swimmers in each of the strokes are given below. Which swimmer should the coach assign to each of the four strokes? Which swimmers will *not* be assigned to the relay team? Are there more than one optimal solution?

| Stroke | Alan | Ben | Carl | Don | Ed | Fred | George |
|---|---|---|---|---|---|---|---|
| **Backstroke** | 66 | 67 | 66 | 64 | 70 | 68 | 64 |
| **Breaststroke** | 71 | 72 | 70 | 69 | 72 | 72 | 73 |
| **Butterfly** | 65 | 67 | 71 | 74 | 65 | 64 | 64 |
| **Freestyle** | 59 | 59 | 55 | 59 | 54 | 54 | 56 |

56:171 Operations Research

# Homework #6 – due 24 October 2001

**1.** ***Integer LP Model*** A court decision has stated that the enrollment of each high school in Metropolis be at least 20% black. The numbers of black and white high school students in each of the city's five school districts are:

| District | Whites | Blacks |
|---|---|---|
| 1 | 80 | 30 |
| 2 | 70 | 5 |
| 3 | 90 | 10 |
| 4 | 50 | 40 |
| 5 | 60 | 30 |

The distance (in miles) that a student in each district must travel to each high school is:

| District | HS#1 | HS#2 |
|---|---|---|
| 1 | 1.0 | 2.0 |
| 2 | 0.5 | 1.7 |
| 3 | 0.8 | 0.8 |
| 4 | 1.3 | 0.4 |
| 5 | 1.5 | 0.6 |

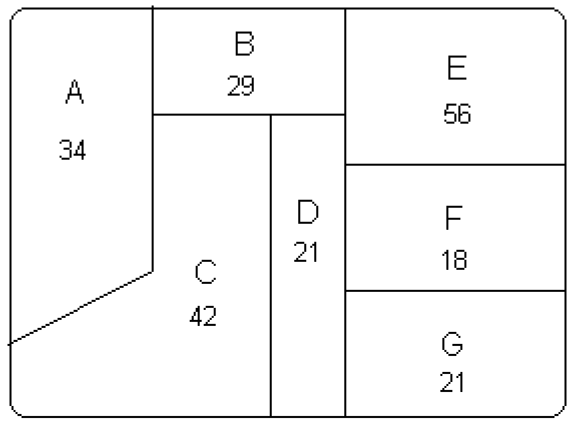
School board policy requires that all students in a given district must attend the same school, and that each school must have an enrollment of at least 150 students. Formulate an integer LP to determine how to minimize the total distance that Metropolis students must travel to high school, and use LINDO (or other ILP solver) to compute the optimal solution.

**2.** ***Integer LP Model*** A company sells seven types of boxes, ranging in volume from 17 to 33 cubic feet. The demand and size of each box are given below.

| Product#: | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Size | 33 | 30 | 26 | 24 | 19 | 18 | 17 |
| Demand | 400 | 300 | 500 | 700 | 200 | 400 | 200 |

The variable cost (in dollars) of producing each box is equal to the box's volume. A fixed cost of $1000 is incurred to produce any of a particular box. If the company desires, demand for a box may be satisfied by a box of larger size. Formulate an integer LP model to minimize the cost of meeting the demand for boxes, and solve, using LINDO (or another ILP solver).

**3.** ***Integer LP Model*** WSP Publishing sells textbooks to college students. WSP has two sales representatives available to assign to the seven-state area (states A through G):




The number of college students (in thousands) in each area is indicated in the figure above. Each sales representative must be assigned to two adjacent states. For example, a sales rep could be assigned to A & B, but not A&D. WSP's goal is to maximize the number of total students in the states assigned to the sales reps. Formulate an integer LP whose solution will tell WSP where to assign the sales reps. Use LINDO (or another ILP solver) to compute the optimal assignment.

| 56:171 Operations Research |
|---|
| Homework #7 – due 31 October 2001 |

**1.** A Markov chain has the transition probability matrix

$$P = \begin{bmatrix} 0 & 0.3 & 0.7 \\ 0.9 & 0.1 & 0 \\ 0.2 & 0 & 0.8 \end{bmatrix}$$

a. Draw the transition diagram, with probabilities indicated.
b. Find the probability distributions of the state for the first five steps, given that it begins in state 3.
c. Find the expected first passage time from state 3 to state 1.
d. What property does this Markov chain have that guarantees the existence of a steady state probability distribution?
e. Write the equations which must be solved in order to compute the steady state distribution.
f. What is the steady state probability distribution?

**2.** An office has two printers, which are very unreliable. It has been observed that when both are working in the morning, there is a 30% chance that one will fail by evening, and a 10% chance that both will fail. If it happens that only one printer is working in the morning, there is a 20% chance that it will fail by evening. . Any printers that fail during the day are picked up by a repairman the next morning, and returned the following morning. (Assume that he can work on more than one printer at a time.)

Model this situation as a Markov chain with the state being the number of failed printers observed in the morning after the repairman has returned any printers but before any failures have occurred. The states then, are 0, 1, & 2.

a. Draw the transition diagram, with probabilities indicated.
b. Write the transition probability matrix.
c. What is the probability distribution of the number of failed printers on Wednesday evening if both printers are working on Monday morning?
d. What property does this Markov chain have that guarantees the existence of a steady state probability distribution?
e. Write the equations which must be solved in order to compute the steady state distribution.
f. What is the steady state probability distribution?

3**.** ***(s,S) Model of Inventory System*** A periodic inventory replenishment system with reorder point **s**=2 and order-up to level **S**=7 is modeled. At the end of each period of demand (day), the inventory is tallied, and if the level is less than or equal to the reorder point (s), enough is ordered (& immediately received) so as to bring the inventory level up to S. The probability distribution is discrete and Poisson, with expected demand 2/day.

The state of the system is the inventory position: if no backorders are permitted, as in this case, this is the stock-on-hand. (Otherwise it is the stock-on-hand if nonnegative, and the number of unfilled orders if negative.)

The following output was obtained using the MARKOV workspace (APL code) which is available from the URL: http://asrl.ecn.uiowa.edu/dbricker/APL_software.html

a. Over a long period of time, what is the percent of the days in which you would expect there to be a stockout (zero inventory)? _____________

b. What will be the average end-of-day inventory level? _____________

c. How often (i.e. once every how many days?) will the inventory be full at the end of the day? _____________

d. How often will the inventory be restocked? _____________

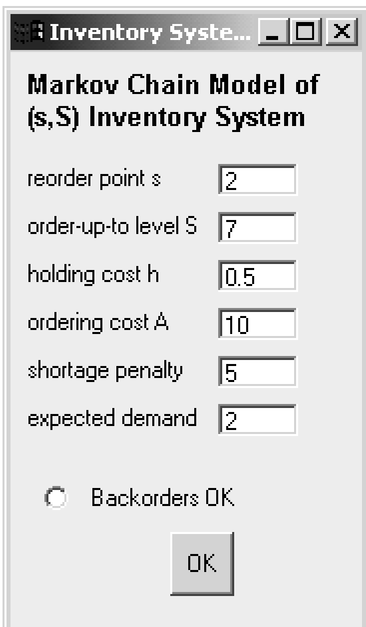
e. If the shelf is full Monday morning, what is the probability that a replenishment occurs Friday evening? _____

f. If the shelf is full Monday morning, what is the probability that the first stockout occurs Friday evening? _____________

g. What is the expected number of days, starting with a full inventory, until a stockout occurs? _____________

h. Starting with a full inventory, what is the expected number of stockouts during the first 30 days? What is the expected number of times that the inventory is restocked? _____________

i. What is the average daily cost of this inventory system--including holding cost of $0.50/unit, replenishment cost of $10 per replenishment, and shortage penalties of $5 per stockout (regardless of the unsatisfied demand)? _____________




**Transition Probability Matrix**

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.004534 | 0.01203 | 0.03609 | 0.09022 | 0.1804 | 0.2707 | 0.2707 | 0.1353 |
| 2 | 0.004534 | 0.01203 | 0.03609 | 0.09022 | 0.1804 | 0.2707 | 0.2707 | 0.1353 |
| 3 | 0.004534 | 0.01203 | 0.03609 | 0.09022 | 0.1804 | 0.2707 | 0.2707 | 0.1353 |
| 4 | 0.3233 | 0.2707 | 0.2707 | 0.1353 | 0 | 0 | 0 | 0 |
| 5 | 0.1429 | 0.1804 | 0.2707 | 0.2707 | 0.1353 | 0 | 0 | 0 |
| 6 | 0.05265 | 0.09022 | 0.1804 | 0.2707 | 0.2707 | 0.1353 | 0 | 0 |
| 7 | 0.01656 | 0.03609 | 0.09022 | 0.1804 | 0.2707 | 0.2707 | 0.1353 | 0 |
| 8 | 0.004534 | 0.01203 | 0.03609 | 0.09022 | 0.1804 | 0.2707 | 0.2707 | 0.1353 |

**Cost Vector**

| i | State | Cost |
|---|---|---|
| 1 | SOH=zero | 10.0 |
| 2 | SOH=one | 10.5 |
| 3 | SOH=two | 11.0 |
| 4 | SOH=three | 1.5 |
| 5 | SOH=four | 2.0 |
| 6 | SOH=five | 2.5 |
| 7 | SOH=six | 3.0 |
| 8 | SOH=seven | 3.5 |

**5-th Power**

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.08798 | 0.09716 | 0.1428 | 0.1681 | 0.1703 | 0.1606 | 0.1212 | 0.05186 |
| 2 | 0.08798 | 0.09716 | 0.1428 | 0.1681 | 0.1703 | 0.1606 | 0.1212 | 0.05186 |
| 3 | 0.08798 | 0.09716 | 0.1428 | 0.1681 | 0.1703 | 0.1606 | 0.1212 | 0.05186 |
| 4 | 0.09102 | 0.09849 | 0.1418 | 0.163 | 0.164 | 0.1598 | 0.126 | 0.05585 |
| 5 | 0.09283 | 0.1009 | 0.1456 | 0.1667 | 0.1649 | 0.156 | 0.1203 | 0.0527 |
| 6 | 0.09241 | 0.1013 | 0.1473 | 0.1699 | 0.1674 | 0.155 | 0.1166 | 0.05016 |
| 7 | 0.08993 | 0.09931 | 0.1457 | 0.1702 | 0.1699 | 0.1575 | 0.1174 | 0.04999 |
| 8 | 0.08798 | 0.09716 | 0.1428 | 0.1681 | 0.1703 | 0.1606 | 0.1212 | 0.05186 |

**Steady State Distribution**

| i | state | P{i} |
|---|---|---|
| 1 | SOH=zero | 0.09024 |
| 2 | SOH=one | 0.09892 |
| 3 | SOH=two | 0.1442 |
| 4 | SOH=three | 0.1675 |
| 5 | SOH=four | 0.1678 |
| 6 | SOH=five | 0.1585 |
| 7 | SOH=six | 0.1207 |
| 8 | SOH=seven | 0.05217 |

## Expected no. of visits during first 30 stages

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2.619 | 2.886 | 4.236 | 4.982 | 5.073 | 4.854 | 3.73 | 1.621 |
| 2 | 2.619 | 2.886 | 4.236 | 4.982 | 5.073 | 4.854 | 3.73 | 1.621 |
| 3 | 2.619 | 2.886 | 4.236 | 4.982 | 5.073 | 4.854 | 3.73 | 1.621 |
| 4 | 2.888 | 3.085 | 4.382 | 4.945 | 4.879 | 4.671 | 3.59 | 1.56 |
| 5 | 2.764 | 3.043 | 4.428 | 5.089 | 4.974 | 4.613 | 3.547 | 1.542 |
| 6 | 2.705 | 2.987 | 4.384 | 5.123 | 5.1 | 4.695 | 3.489 | 1.517 |
| 7 | 2.662 | 2.936 | 4.31 | 5.067 | 5.129 | 4.82 | 3.585 | 1.49 |
| 8 | 2.619 | 2.886 | 4.236 | 4.982 | 5.073 | 4.854 | 3.73 | 1.621 |

**First Visit Probabilities to State 1 from State 8**

| n | P |
|---|---|
| 1 | 0.004534 |
| 2 | 0.07452 |
| 3 | 0.1048 |
| 4 | 0.08298 |
| 5 | 0.07153 |
| 6 | 0.06623 |
| 7 | 0.05974 |
| 8 | 0.05352 |
| 9 | 0.04818 |
| 10 | 0.0434 |
| 11 | 0.03905 |
| 12 | 0.03514 |
| 13 | 0.03163 |
| 14 | 0.02847 |
| 15 | 0.02562 |

**Mean First Passage Time Matrix**

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 11.08 | 10.11 | 6.936 | 5.746 | 5.373 | 5.305 | 7.086 | 19.17 |
| 2 | 11.08 | 10.11 | 6.936 | 5.746 | 5.373 | 5.305 | 7.086 | 19.17 |
| 3 | 11.08 | 10.11 | 6.936 | 5.746 | 5.373 | 5.305 | 7.086 | 19.17 |
| 4 | 8.094 | 8.101 | 5.922 | 5.969 | 6.529 | 6.462 | 8.243 | 20.32 |
| 5 | 9.472 | 8.527 | 5.604 | 5.104 | 5.959 | 6.824 | 8.605 | 20.69 |
| 6 | 10.12 | 9.087 | 5.911 | 4.903 | 5.209 | 6.311 | 9.08 | 21.16 |
| 7 | 10.6 | 9.609 | 6.419 | 5.239 | 5.038 | 5.518 | 8.287 | 21.66 |
| 8 | 11.08 | 10.11 | 6.936 | 5.746 | 5.373 | 5.305 | 7.086 | 19.17 |

56:171 Operations Research

# Homework #8 – due 7 November 2001

1. A factory has a buffer with a capacity of 4 $m^3$ for temporarily storing waste produced by the factory. Each week the factory produces k $m^3$ waste with a probability of $p_k$, where $p_0 = {}^1/_8$, $p_1 = ½$, $p_2 = ¼$, and $p_3 = {}^1/_8$. If the amount of waste produced in one week exceeds the remaining capacity of the buffer, the excess is specially removed at a cost of $100 per $m^3$. At the end of each week, there is a regular opportunity to remove waste from the storage buffer at a fixed cost of $50 and a variable cost of $10 per $m^3$. The following policy is used. If at the end of the week the storage buffer contains more than 2 $m^3$ the buffer is emptied; otherwise no waste is removed. Determine
   a) the frequency of overflows
   b) the frequency that the buffer is emptied
   c) the long-run average cost per week

✷ ✷ ✷ ✷ ✷ ✷ ✷ ✷ ✷

2. For simplicity, suppose that fresh blood obtained by a hospital will spoil if it is not transfused within five days. The hospital receives 100 pints of fresh blood daily from a local blood bank. Two policies are possible for determining the order in which blood is transfused. The following table gives the probabilities of transfusion for blood of various ages under each policy:

| | 0 day old | 1 day old | 2 days old | 3 days old | 4 days old |
|---|---|---|---|---|---|
| Policy 1 | 10% | 20% | 30% | 40% | 50% |
| Policy 2 | 50% | 40% | 30% | 20% | 10% |

For example, under policy 1, blood has a 10% chance of being transfused during its first day at the hospital. Under policy 2, four-day-old blood has a 10% chance of being transfused.

a. A FIFO (first in, first out) blood-issuing policy issues "old" blood first, whereas a LIFO (last in, first out) policy issues "young" blood first. Which policy above represents a LIFO policy, and which represents a FIFO policy?

b. For each policy, determine the probability that a new pint of blood will eventually spoil.

c. For each policy, determine the average number of pints of blood in inventory.

d. For each policy, determine the average age of transfused blood.

e. Which policy do you recommend, based upon these three criteria (stated in b, c, & d)?

✷ ✷ ✷ ✷ ✷ ✷ ✷ ✷ ✷

56:171 Operations Research

Homework #9 – due 14 November 2001

**1. Discrete-time Markov Chain.** *(Based upon Exercise 4, §19.5, page 982 of text by W. Winston)*

At the beginning of each year, my car is in good, fair, or broken-down condition. A good car will be good at the beginning of next year with probability 90%, fair with probability 5%, or broken-down with probability 5%. A fair car will be fair at the beginning of the next year with probability 70%, or broken-down with probability 30%. It costs $12000 to purchase a good car; a fair car can be traded in for $5000; and a broken-down car has no trade-in value and must immediately be replaced by a good car. It costs $1000 per year to operate a good car and $2000 to operate a fair car. Should I replace my car as soon as it becomes a fair car, or should I drive my car until it breaks down? Assume that the cost of operating a car during a year depends on the type of car on hand at the *beginning* of the year (after a new car, if any, arrives).

Define a Markov chain model with three states (Good, Fair, & Broken-down). Assume, as implied by the problem statement, that break-down occurs only at the *end* of a year, and then (at the beginning of the next year) the broken-down car "must immediately be replaced". For each of the two replacement policies mentioned, answer the following:

a. Draw a diagram of the Markov chain and write down the transition probability matrix.

b. Write down the equations which could be solved to obtain the steadystate probabilities.

c. Solve the equations, either manually or using appropriate computer software.

d. Compute the average cost per year for the replacement policy.

e. What is the expected time between break-downs?

f. What replacement policy do you recommend?

*Note: assume that **state 1= Good, state 2= Fair, and state 3= Broken-down**.*


Copyright (c) 2000 by Thaves. Distributed from www.thecomics.com.

2. **Continuous-time Markov Chain**. In exercise 1, the Markov chain model assumes that break-down occurs only at the *end* of a year, and then (at the beginning of the next year) the broken-down car "must immediately be replaced" with a car in good condition. In fact, of course, the change in condition can occur at any time during the year, and a continuous-time Markov chain model would be a closer representation of reality. Let's assume that when my car breaks down, it takes me an average of 0.02 years (about 1 week) to find and purchase a replacement car (and that this delay has exponential distribution.) Again define a Markov chain model with three states (Good, Fair, & Brokendown).

a. What should be the transition rates, so that the probability of a change of condition during a one-year period is in agreement with the probabilities given in exercise 1?

*Hint: The cdf of the exponential distribution is*

$$F(t) = P\{\text{time to next event } \leq t\} = 1 - e^{-\lambda t}$$

*If in state 1 (good car) there is a 10% probability that the system has changed states during the next year, the transition rate $\lambda_1$ should therefore satisfy*

$$F(1) = 1 - e^{-\lambda} = 0.10$$

*The value of $\lambda_{12}$ should be equal to the value of $\lambda_{13}$ (since the transition probabilities $p_{12}$ and $p_{13}$ were each 5%), and $\lambda_1 = \lambda_{12} + \lambda_{13}$, so $\lambda_{12} = \lambda_{13} = 0.5\ \lambda_1$. To get the transition rate $\lambda_{31}$, observe that the expected value of the length of time required to replace my broken-down car is $1/\lambda_{31} = 0.02$ years.*

b. Write the matrix of transition rates.

c. Write the set of equations that must be solved for a steadystate distribution.

d. Find the steadystate distribution.

e. What does this model predict will be my average operating cost/year (not including replacement costs)?

*To compute the average replacement costs per year is not quite so simple. (We must multiply the replacement costs by the expected number of replacements/year, not by $\pi_3$ (the fraction of the year spent in state 3). Let T = average time between replacements. Then*

$$\pi_3 = \frac{\text{average time from breakdown to replacement}}{\text{average length of time between replacements}}$$

$$= \frac{0.02\,yr}{T}$$

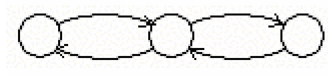
*What then is T? The number of replacements per year should then be 1/T.*

f. What average replacement cost per year is predicted by this model?

✷ ✷ ✷ ✷ ✷ ✷ ✷ ✷ ✷

# Homework #10 – due 28 November 2001

**1. Birth-death process** (exercise 2, section 22.3 of text of Winston. *Numerical values modified*)
My home uses two light bulbs. On average, a light bulb lasts for 30 days (exponentially distributed). When a light bulb burns out, it takes me an average of 5 days (exponentially distributed) before I replace the bulb (one at a time!)



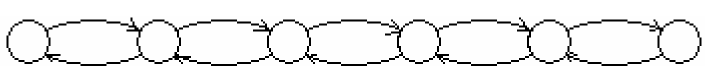
a. Formulate a three-state birth-death model of this situation.

b. Determine the fraction of the time that both light bulbs are working.

c. Determine the fraction of the time that neither light bulb is working.

d. Suppose that, when both bulbs are burned out and I replace a bulb, I replace both bulbs simultaneously. Why is this no longer a birth-death process?


2. **Birth-death process** A local takeout Chinese restaurant has space to accommodate at most five customers. During the frigid Iowa winter, it is noticed that when customers arrive and the restaurant is full, virtually no one waits outside in the subfreezing weather, but instead goes next door to Luigi's Pizza Palace. Customers arrive at the restaurant at the average rate of 10 per hour, according to a Poisson process. The restaurant serves customers one at a time, first-come, first-served, in an average of 3 minutes each (the actual time being exponentially distributed.)



(a.) What is the steady-state distribution of the number of customers in the restaurant?

(b.) What is the average number of customers in the Chinese restaurant at any time?

(c.) What is the *average* arrival rate, considering that when there are 5 customers in the restaurant, the arrival rate is zero?

(d.) According to Little's Law, what is the expected amount of time that a customer spends in the restaurant?

(e.) What is the fraction of potential customers who are lost to the pizza establishment? What is the number of customers per hour lost to the pizza establishment?

3. **Deterministic Dynamic Programming** (A variation of the example presented in class) A utility company must plan expansion of its generating capacity over the next eight years. A forecast has been prepared, specifying the number of additional power plants $R_t$ required at the end of each year t. Each year, at most three plants may be added. The cost of adding a power plant in year t is $C_t$ per plant, plus a fixed cost of $F_t$ (unless no plants are added).

| *Year* $t$ | *Reqd* $R_t$ | *Fixed cost* $F_t$ | *Marginal cost* $C_t$ |
|---|---|---|---|
| 1 | 1 | 2.4 | 3.4 |
| 2 | 2 | 2.4 | 3.5 |
| 3 | 3 | 2.5 | 3.5 |
| 4 | 5 | 2.5 | 3.5 |
| 5 | 7 | 2.6 | 3.4 |
| 6 | 8 | 2.6 | 3.4 |

A dynamic programming model with forward recursion is developed, so that stage 1 = first year (now), stage 6 = final year of planning period. Time value of money is to be considered, with a discount factor = 0.83333. The computations at each stage are shown below in order to minimize the present value of the cost of adding the generating capacity. *Note*: A value "9999.9999" in the table indicates an *infeasible* combination of state & decision.

---Stage 6 (*final year of planning period*) ---

| s \ x: | 0 | 1 | Minimum |
|---|---|---|---|
| 7 | 9999.9999 | 6.0000 | 6.0000 |
| 8 | 0.0000 | 9999.9999 | 0.0000 |

---Stage 5---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| 5 | 9999.9999 | 9999.9999 | 14.4000 | 12.8000 | 12.8000 |
| 6 | 9999.9999 | 11.0000 | 9.4000 | 9999.9999 | 9.4000 |
| 7 | 5.0000 | 6.0000 | 9999.9999 | 9999.9999 | 5.0000 |
| 8 | 0.0000 | 9999.9999 | 9999.9999 | 9999.9999 | 0.0000 |

---Stage 4---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| 4 | 9999.9999 | 16.6667 | 17.3333 | 17.1667 | 16.6667 |
| 5 | 10.6667 | 13.8333 | 13.6667 | 13.0000 | 10.6667 |
| 6 | 7.8333 | 10.1667 | 9.5000 | 9999.9999 | 7.8333 |
| 7 | 4.1667 | 6.0000 | 9999.9999 | 9999.9999 | 4.1667 |
| 8 | 0.0000 | 9999.9999 | 9999.9999 | 9999.9999 | 0.0000 |

---Stage 3---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| 2 | 9999.9999 | 9999.9999 | 23.3889 | 21.8889 | 21.8889 |
| 3 | 9999.9999 | $\boxed{\phantom{0000000}}$ | 18.3889 | 19.5278 | 18.3889 |
| 4 | 13.8889 | 14.8889 | 16.0278 | 16.4722 | 13.8889 |
| 5 | 8.8889 | 12.5278 | 12.9722 | 13.0000 | 8.8889 |
| 6 | 6.5278 | 9.4722 | 9.5000 | 9999.9999 | 6.5278 |
| 7 | 3.4722 | 6.0000 | 9999.9999 | 9999.9999 | 3.4722 |
| 8 | 0.0000 | 9999.9999 | 9999.9999 | 9999.9999 | 0.0000 |

---Stage 2---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| 1 | 9999.9999 | 24.1407 | 24.7241 | 24.4741 | 24.1407 |
| 2 | 18.2407 | 21.2241 | 20.9741 | 20.3074 | 18.2407 |
| 3 | 15.3241 | 17.4741 | 16.8074 | 18.3398 | 15.3241 |
| 4 | 11.5741 | 13.3074 | 14.8398 | 15.7935 | 11.5741 |
| 5 | 7.4074 | 11.3398 | 12.2935 | 12.9000 | 7.4074 |
| 6 | 5.4398 | 8.7935 | 9.4000 | 9999.9999 | 5.4398 |
| 7 | 2.8935 | 5.9000 | 9999.9999 | 9999.9999 | 2.8935 |
| 8 | 0.0000 | 9999.9999 | 9999.9999 | 9999.9999 | 0.0000 |

```
                    ---Stage 1 (first year of planning period )---
  s   \ x:    0         1         2         3   | Minimum
    0  | 9999.9999   25.9173   24.4006   25.3701|   24.4006
```

a. What annual percent return on investment is implied by the discount factor $\beta = 0.833333$? _________

b. One value is missing in the table, i.e., the total cost of years 3, 4, 5, and 6 if, at the beginning of year 3, the company has already added 3 plants and decides to add 1 additional plant. What is this value?

Construction cost in year 3 ____________

Discounted minimum cost of years 4, 5, & 6 ____________

Total ____________

b. What is the minimum present value of the total construction cost to meet the requirements? __________

c. What is the optimal schedule for adding plants?

| *Year t* | *# plants to add* | *Cumulative # plants added* | *# plants required* |
|---|---|---|---|
| 1 | | | 1 |
| 2 | | | 2 |
| 3 | | | 3 |
| 4 | | | 5 |
| 5 | | | 7 |
| 6 | | | 8 |

d. Suppose that (for unspecified reasons) the number of plants added during the first year is one (not optimal!). What is the best schedule for adding capacity during the remaining five years?

| *Year t* | *# plants to add* | *Cumulative # plants added* | *# plants required* |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | | | 2 |
| 3 | | | 3 |
| 4 | | | 5 |
| 5 | | | 7 |
| 6 | | | 8 |

| 56:171 Operations Research |
|---|
| Homework #11 – due 5 December 2001 |

1. ***Redistricting Problem*** A state is to be allocated twenty representatives (Reps) to be sent to the national legislature. There are nine districts within the state, whose boundaries are fixed. Every district should be assigned at least one representative. The allocation should be done according to the population (Pop) of the districts:

| *District* | *1* | *2* | *3* | *4* | *5* | *6* | *7* | *8* | *9* |
|---|---|---|---|---|---|---|---|---|---|
| **Population** | 50 | 60 | 70 | 50 | 70 | 100 | 20 | 70 | 40 |

The "target allocation" of district i is Reps times Pop[i] divided by the population of the state, but this target is generally non-integer. The objective is the assign the representatives to the districts in such a way that the maximum absolute deviation from the targets is as small as possible.
In the DP model, there is a stage for each district, and the state of the system is the number of representatives not yet assigned when we decide upon the allocation to district i. The optimal value function is defined by a forward recursion:

$$\begin{cases} f_n(s) = \underset{x\in\{1,2,3,4\}}{\text{minimum}} \ \max\left\{\left|\alpha_n - x\right|, f_{n+1}(s-x)\right\} \\ f_0(0) = 0 \ \& \ f_0(s) = +\infty \text{ for } s > 0 \end{cases}$$

That is, the optimal value function $f_n(s)$ at stage $n$ with state $s$ is the smallest possible value of the maximum absolute deviations from the targets $\alpha$ of the allocation to districts $n$, $n$+1, …. 9 if the total number of representatives available to those districts is given by the state $s$.

a. What is the "target" allocation $\alpha_i$ of each district?

| District | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **Population** | 50 | 60 | 70 | 50 | 70 | 100 | 20 | 70 | 40 |
| **Target** | | | | | | | | | |

b. Compute the missing values in the table below for stage 3.

c. There are three optimal solutions to this problem. For each solution, what are the optimal allocations of representatives to districts? (Enter in tables below.)

***Solution #1:***

| District | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **Allocation** | | | | | | | | | |
| **Deviation** | | | | | | | | | |

***Solution #2:***

| District | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **Allocation** | | | | | | | | | |
| **Deviation** | | | | | | | | | |

***Solution #3:***

| District | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **Allocation** | | | | | | | | | |
| **Deviation** | | | | | | | | | |

d. Does one of the three solutions seem "better" than the others with respect to some other considerations?

Select one of the three solutions (#____ ), and answer the following:
e. Which district has the largest positive deviation from its target allocation? _______

f. Which district has the largest negative deviation from its target allocation? _______

---Stage 9---

| s \ x: | 1 | 2 | 3 | 4 | Min |
|---|---|---|---|---|---|
| 1 | 0.51 | 999.99 | 999.99 | 999.99 | 0.51 |
| 2 | 999.99 | 0.49 | 999.99 | 999.99 | 0.49 |
| 3 | 999.99 | 999.99 | 1.49 | 999.99 | 1.49 |
| 4 | 999.99 | 999.99 | 999.99 | 2.49 | 2.49 |

---Stage 8---

| s \ x: | 1 | 2 | 3 | 4 | Min |
|---|---|---|---|---|---|
| 2 | 1.64 | 999.99 | 999.99 | 999.99 | 1.64 |
| 3 | 1.64 | 0.64 | 999.99 | 999.99 | 0.64 |
| 4 | 1.64 | 0.64 | 0.51 | 999.99 | 0.51 |
| 5 | 2.49 | 1.49 | 0.49 | 1.36 | 0.49 |
| 6 | 999.99 | 2.49 | 1.49 | 1.36 | 1.36 |
| 7 | 999.99 | 999.99 | 2.49 | 1.49 | 1.49 |
| 8 | 999.99 | 999.99 | 999.99 | 2.49 | 2.49 |

---Stage 7---

| s \ x: | 1 | 2 | 3 | 4 | Min |
|---|---|---|---|---|---|
| 3 | 1.64 | 999.99 | 999.99 | 999.99 | 1.64 |
| 4 | 0.64 | 1.64 | 999.99 | 999.99 | 0.64 |
| 5 | 0.51 | 1.25 | 2.25 | 999.99 | 0.51 |
| 6 | 0.49 | 1.25 | 2.25 | 3.25 | 0.49 |
| 7 | 1.36 | 1.25 | 2.25 | 3.25 | 1.25 |
| 8 | 1.49 | 1.36 | 2.25 | 3.25 | 1.36 |
| 9 | 2.49 | 1.49 | 2.25 | 3.25 | 1.49 |
| 10 | 999.99 | 2.49 | 2.25 | 3.25 | 2.25 |
| 11 | 999.99 | 999.99 | 2.49 | 3.25 | 2.49 |
| 12 | 999.99 | 999.99 | 999.99 | 3.25 | 3.25 |

---Stage 6---

| s \ x: | 1 | 2 | 3 | 4 | Min |
|---|---|---|---|---|---|
| 4 | 2.77 | 999.99 | 999.99 | 999.99 | 2.77 |
| 5 | 2.77 | 1.77 | 999.99 | 999.99 | 1.77 |
| 6 | 2.77 | 1.77 | 1.64 | 999.99 | 1.64 |
| 7 | 2.77 | 1.77 | 0.77 | 1.64 | 0.77 |
| 8 | 2.77 | 1.77 | 0.77 | 0.64 | 0.64 |
| 9 | 2.77 | 1.77 | 0.77 | 0.51 | 0.51 |
| 10 | 2.77 | 1.77 | 1.25 | 0.49 | 0.49 |
| 11 | 2.77 | 1.77 | 1.36 | 1.25 | 1.25 |
| 12 | 2.77 | 2.25 | 1.49 | 1.36 | 1.36 |
| 13 | 3.25 | 2.49 | 2.25 | 1.49 | 1.49 |
| 14 | 999.99 | 3.25 | 2.49 | 2.25 | 2.25 |
| 15 | 999.99 | 999.99 | 3.25 | 2.49 | 2.49 |

---Stage 5---

| s \ x: | 1 | 2 | 3 | 4 | Min |
|---|---|---|---|---|---|
| 5 | 2.77 | 999.99 | 999.99 | 999.99 | 2.77 |
| 6 | 1.77 | 2.77 | 999.99 | 999.99 | 1.77 |
| 7 | 1.64 | 1.77 | 2.77 | 999.99 | 1.64 |
| 8 | 1.64 | 1.64 | 1.77 | 2.77 | 1.64 |
| 9 | 1.64 | 0.77 | 1.64 | 1.77 | 0.77 |
| 10 | 1.64 | 0.64 | 0.77 | 1.64 | 0.64 |
| 11 | 1.64 | 0.64 | 0.64 | 1.36 | 0.64 |
| 12 | 1.64 | 0.64 | 0.51 | 1.36 | 0.51 |
| 13 | 1.64 | 1.25 | 0.49 | 1.36 | 0.49 |
| 14 | 1.64 | 1.36 | 1.25 | 1.36 | 1.25 |
| 15 | 2.25 | 1.49 | 1.36 | 1.36 | 1.36 |
| 16 | 2.49 | 2.25 | 1.49 | 1.36 | 1.36 |

---Stage 4---

| s \ x: | 1 | 2 | 3 | 4 | Min |
|---|---|---|---|---|---|
| 8 | 1.64 | 1.77 | 2.77 | 999.99 | 1.64 |
| 9 | 1.64 | 1.64 | 1.77 | 2.77 | 1.64 |
| 10 | 0.89 | 1.64 | 1.64 | 2.11 | 0.89 |
| 11 | 0.89 | 0.77 | 1.64 | 2.11 | 0.77 |
| 12 | 0.89 | 0.64 | 1.11 | 2.11 | 0.64 |
| 13 | 0.89 | 0.64 | 1.11 | 2.11 | 0.64 |
| 14 | 0.89 | 0.51 | 1.11 | 2.11 | 0.51 |
| 15 | 1.25 | 0.49 | 1.11 | 2.11 | 0.49 |
| 16 | 1.36 | 1.25 | 1.11 | 2.11 | 1.11 |
| 17 | 1.36 | 1.36 | 1.25 | 2.11 | 1.25 |

---Stage 3---

| s \ x: | 1 | 2 | 3 | 4 | Min |
|---|---|---|---|---|---|
| 12 | | | | | 0.89 |
| 13 | | | | | 0.77 |
| 14 | 1.64 | 0.64 | 0.77 | 1.36 | 0.64 |
| 15 | 1.64 | 0.64 | 0.64 | 1.36 | 0.64 |
| 16 | 1.64 | 0.64 | 0.64 | 1.36 | 0.64 |
| 17 | 1.64 | 0.64 | 0.51 | 1.36 | 0.51 |
| 18 | 1.64 | 1.11 | 0.49 | 1.36 | 0.49 |

---Stage 2---

| s \ x: | 1 | 2 | 3 | 4 | Min |
|---|---|---|---|---|---|
| 16 | 1.26 | 0.64 | 0.77 | 1.74 | 0.64 |
| 17 | 1.26 | 0.64 | 0.74 | 1.74 | 0.64 |
| 18 | 1.26 | 0.64 | 0.74 | 1.74 | 0.64 |
| 19 | 1.26 | 0.51 | 0.74 | 1.74 | 0.51 |

---Stage 1---

| s \ x: | 1 | 2 | 3 | 4 | Min |
|---|---|---|---|---|---|
| 20 | 0.89 | 0.64 | 1.11 | 2.11 | 0.64 |

2. ***(Deterministic) Equipment Replacement.*** The optimal policy for replacement of a machine over the next ten years is required. The cost of a new machine is $10,000. The table below indicates the annual operating cost of the machine, and the trade-in value, according to its age. The policy is to keep a machine for no more than six years. A new machine has just been purchased (whose cost should not be considered), and at the end of the ten-year planning period, a new machine is required.

| Age of machine (yrs) | Operating cost/year ($) | Trade-in value ($) |
|---|---|---|
| 1 | 1400 | 7500 |
| 2 | 1800 | 6000 |
| 3 | 2400 | 5000 |
| 4 | 3000 | 4200 |
| 5 | 3500 | 3500 |
| 6 | 4000 | 2500 |

*The operating cost is for the year ending. For example, if we keep the original machine 3 years, the operating cost during that three-year interval would be $1400+1800+2400 = $5600, and the replacement cost would be $10000 - $5000 = $5000, a total of $10600, and we would have a new machine with 7 years remaining in the planning period.*

a. What is the total cost of the policy which replaces the machine at the end of year 5 and year 10?

Define the functions

g(n) = minimum total operating & replacement cost (including trade-in value) if, with n years remaining in the planning period, you have a new machine

y(n) = optimal age at which the machine is to be replaced if, with n years remaining in the planning period, you have a new machine.

| *Yrs to go (n)* | *g(n)* | *y(n)* |
|---|---|---|
| 10 | | |
| 9 | | |
| 8 | 18400 | 2 or 3 |
| 7 | 15000 | 2, 3, or 4 |
| 6 | 11200 | 3 |
| 5 | 7800 | 2 or 3 |
| 4 | 4400 | 2 or 4 |
| 3 | 600 | 3 |
| 2 | −2800 | 2 |
| 1 | −6100 | 1 |
| 0 | 0 | |

For example, if n=6 years remain with a new machine, which is replaced at the end of 3 years, i.e., y(6)=3, the total cost is:

| | |
|---|---|
| operating cost: | 1400+1800+2400 = 5600 |
| cost of new machine: | 10000 |
| trade-in value: | −5000 |
| cost of final 3 years: g(3): | 600 |
| Total: | 11200 |

b. Complete the computations in the table above.

c. What is the minimum total cost of owning and operating the machine during the ten-year planning period?

d. If "now" is January 1, 2002, what are the optimal dates at which the machine should be replaced?
Note: there will be several optimal solutions-- find at least two of them!

3. ***Stochastic Production Planning.*** The state of the system is the "inventory position", which if positive is the stock on hand, and if negative is the number of backorders which have accumulated, which is observed at the end of the business day. The decision is the number of units to be produced, which is no more than 3. There is a setup cost of $10 if any units are produced, plus $4 per unit. We assume that production is completed in time to meet any demand that occurs the next day.
The demand is a discrete random variable with stationary distribution

| D | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| P{D} | 0.1 | 0.2 | 0.3 | 0.2 | 0.2 |

In addition, there is a storage cost of $1 per unit, based upon the end-of-day inventory, and a shortage cost of $15 per unit, based upon any backorders. Finally, at the end of the planning period (5 days), a salvage value of $2 per unit is received for any remaining inventory. If any shortage exists at the end of the planning period, enough must be produced to satisfy that shortage.

A backward recursion is used, where $F_N(S)$ is the minimum expected cost of the final N days of the planning period if the initial inventory position is S.
Initially, with 5 days remaining (i.e., stage 5) there are 2 units in inventory, so we wish to compute $F_5(2)$.

Below are the tables used to compute the optimal production policy.

a. What is the missing value in the table for stage 1? ___________

b. What is the missing value in the table for stage 5? ___________

c. What is the optimal production decision at the initial stage (stage 5)? _______

d. What is the minimum expected cost (total of production, storage, and shortage costs) for the five-day period?
_________

e. Suppose that the demand in the first day (i.e., stage 5) is 1. What is the optimal production decision for day 2 (i.e., stage 4)? ______

---Stage 1---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| ¯3 | 9999.99 | 9999.99 | 9999.99 | 114.00 | 114.00 |
| ¯2 | 9999.99 | 9999.99 | 95.00 | 83.50 | 83.50 |
| ¯1 | 9999.99 | 76.00 | 64.50 | 51.60 | 51.60 |
| 0 | 47.00 | 45.50 | 32.60 | 25.80 | 25.80 |
| 1 | 32.50 | 29.60 |  | 19.40 | 19.40 |
| 2 | 16.60 | 19.80 | 16.40 | 18.40 | 16.40 |
| 3 | 6.80 | 13.40 | 15.40 | 17.40 | 6.80 |
| 4 | 0.40 | 12.40 | 14.40 | 16.60 | 0.40 |
| 5 | ¯0.60 | 11.40 | 13.60 | 16.20 | ¯0.60 |
| 6 | ¯1.60 | 10.60 | 13.20 | 16.40 | ¯1.60 |

---Stage 2---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| ¯3 | 9999.99 | 9999.99 | 9999.99 | 150.55 | 150.55 |
| ¯2 | 9999.99 | 9999.99 | 131.55 | 114.08 | 114.08 |
| ¯1 | 9999.99 | 112.55 | 95.08 | 77.28 | 77.28 |
| 0 | 83.55 | 76.08 | 58.28 | 47.26 | 47.26 |
| 1 | 63.08 | 55.28 | 44.26 | 38.36 | 38.36 |
| 2 | 42.28 | 41.26 | 35.36 | 33.22 | 33.22 |
| 3 | 28.26 | 32.36 | 30.22 | 29.48 | 28.26 |
| 4 | 19.36 | 27.22 | 26.48 | 26.78 | 19.36 |
| 5 | 14.22 | 23.48 | 23.78 | 26.00 | 14.22 |
| 6 | 10.48 | 20.78 | 23.00 | 26.60 | 10.48 |

---Stage 3---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| ¯3 | 9999.99 | 9999.99 | 9999.99 | 181.63 | 181.63 |
| ¯2 | 9999.99 | 9999.99 | 162.63 | 141.40 | 141.40 |
| ¯1 | 9999.99 | 143.63 | 122.40 | 100.44 | 100.44 |
| 0 | 114.63 | 103.40 | 81.44 | 67.89 | 67.89 |
| 1 | 90.40 | 78.44 | 64.89 | 57.68 | 57.68 |
| 2 | 65.44 | 61.89 | 54.68 | 52.09 | 52.09 |
| 3 | 48.89 | 51.68 | 49.09 | 47.00 | 47.00 |
| 4 | 38.68 | 46.09 | 44.00 | 42.93 | 38.68 |
| 5 | 33.09 | 41.00 | 39.93 | 40.00 | 33.09 |
| 6 | 28.00 | 36.93 | 37.00 | 39.23 | 28.00 |

---Stage 4---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| ¯3 | 9999.99 | 9999.99 | 9999.99 | 208.95 | 208.95 |
| ¯2 | 9999.99 | 9999.99 | 189.95 | 166.08 | 166.08 |
| ¯1 | 9999.99 | 170.95 | 147.08 | 122.48 | 122.48 |
| 0 | 141.95 | 128.08 | 103.48 | 88.09 | 88.09 |
| 1 | 115.08 | 100.48 | 85.09 | 77.01 | 77.01 |
| 2 | 87.48 | 82.09 | 74.01 | 71.10 | 71.10 |
| 3 | 69.09 | 71.01 | 68.10 | 65.84 | 65.84 |
| 4 | 58.01 | 65.10 | 62.84 | 61.46 | 58.01 |
| 5 | 52.10 | 59.84 | 58.46 | 58.15 | 52.10 |
| 6 | 46.84 | 55.46 | 55.15 | 57.01 | 46.84 |

---Stage 5---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| 2 | 108.65 | 102.02 |  | 90.18 | 90.18 |

56:171 Operations Research

# Homework #12 – due 12 December 2001

1. **Quiz Show** A person has been invited to be a contestant on a TV quiz show, in which there are six stages (1,...6).

At any stage i, the contestant may choose to quit and receive her accumulated winnings.
If the person chooses to continue, she is presented with a question which, if correctly answered, allows her to advance to the next stage (i+1), but if not correctly answered, forces her to quit with no payoff, i.e., she loses everything.
The questions become progressively more difficult at each stage, of course, and she estimates that the probability that she can answer the question at stage i to be P[i] where P[i+1] < P[i].
If she correctly answers the sixth question, she receives her total winnings ($1+2+4+8+…+64= $127).

| *Question* | *1* | *2* | *3* | *4* | *5* | *6* | *7* |
|---|---|---|---|---|---|---|---|
| Prize | 1 | 2 | 4 | 8 | 16 | 32 | 64 |
| P{correct} | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 |

Having recently taken an O.R. course, she does an analysis using dynamic programming to determine her optimal strategy prior to appearing on the quiz show. She defines the optimal value function $f_n(s)$ = maximum expected future payoff if, at stage (question) n, she is in state s (s=1 for "active", 0 for "inactive"):

$$f_n(s) = \begin{cases} \sum_{i=1}^{n-1} R_i & \text{if s=1 "active" \& x=0 "quit"} \\ p_n f_{n+1}(1) + (1-p_n) f_{n+1}(0) & \text{if x=1 "continue"} \end{cases}$$

$$f_8(s) = 0$$

```
Stage 7
    s \ x: 1       0   | Maximum
   1   |  18.90  63.00|  63.00
   0   |   0.00   0.00|   0.00

Stage 6
    s \ x: 1       0   | Maximum
   1   |  25.20  31.00|  31.00
   0   |   0.00   0.00|   0.00

Stage 5
    s \ x: 1       0   | Maximum
   1   |  15.50  15.00|  15.50
   0   |   0.00   0.00|   0.00

Stage 4
    s \ x: 1       0   | Maximum
   1   |   9.30   7.00|   9.30
   0   |   0.00   0.00|   0.00

Stage 3
    s \ x: 1       0   | Maximum
   1   |   6.51   3.00|   6.51
   0   |   0.00   0.00|   0.00

Stage 2
    s \ x: 1       0   | Maximum
   1   |   5.21   1.00|   5.21
   0   |   0.00   0.00|   0.00

Stage 1
    s \ x: 1       0   | Maximum
   1   |   4.69   0.00|   4.69
   0   |   0.00   0.00|   0.00
```

a. Explain the contestant's optimal strategy: at what stage should she quit and keep her earnings?

b. Assume that she is motivated by economic values alone. A bus ticket to the studio of the TV station will cost her $5. Should she accept the invitation? (Explain.)

2. Vladimir Ulanowsky is playing Keith Smithson in a two-game chess match. Winning a game scores 1 match point, and drawing a game scores $^1/_2$ match point. After the two games are played, the player with more match points is declared the champion. If the two players are tied after two games, there is a “sudden death” playoff, i.e., they continue playing until someone wins a game (the winner of that game will then be the champion).

During each game, Ulanowsky can play one of two ways: **boldly** or **conservatively**. If he plays **boldly**, he has a 45% chance of winning the game and a 55% chance of losing the game. If he plays **conservatively**, he has a 90% chance of drawing the game and a 10% chance of losing the game. *Note that if the match enters a “sudden death” playoff, his obvious strategy is to play boldly at that time, since he has no chance to win otherwise.*

Ulanowsky’s goal is to maximize his probability of winning the match. Use dynamic programming to help him accomplish this goal. (If this problem is solved correctly, even though Ulanowsky is the inferior player, his chance of winning the match is over $^1/_2$ .)

3. **Casino Problem** Consider the “Casino Problem” presented in the lectures, but with six plays of the game, and the goal being to accumulate at least six chips, beginning with 3 chips, where the probability of winning at each play of the game is 55%.
In the DP model with results presented below, the recursion is “forward”, i.e., the stages range from n=1 (first play of the game) to n=6 (final play of the game). The state is the number of chips accumulated, and the decision is the number of chips to bet at the current play of the game.
a. Compute the missing number in the table for stage 1. ___________
b. What is the probability that six chips can be accumulated at the end of six plays of the game? _______%
c. How many chips should be bet at the first play of the game? ________ (If more than one value is optimal, choose an answer arbitrarily.)
d. If one bets the amount you selected in (c) and the first play of the game is won, what should be the bet at the second play of the game? _____ If the first play of the game is lost, what should be the bet at the second play of the game? _____

---Stage 6---

| s \ | x: | 1 | 2 | 3 | 4 | 5 | 6 | Max |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.00 | XXXXXXXXXXXXXXXXXXXXXXXXXXXXX | | | | | | 0.00 |
| 1 | 0.00 | 0.00 | XXXXXXXXXXXXXXXXXXXXXXXX | | | | | 0.00 |
| 2 | 0.00 | 0.00 | 0.00 | XXXXXXXXXXXXXXXXXXX | | | | 0.00 |
| 3 | 0.00 | 0.00 | 0.00 | 0.55 | XXXXXXXXXXXXXX | | | 0.55 |
| 4 | 0.00 | 0.00 | 0.55 | 0.55 | 0.55 | XXXXXXXXX | | 0.55 |
| 5 | 0.00 | 0.55 | 0.55 | 0.55 | 0.55 | 0.55 | XXXX | 0.55 |
| 6 | 1.00 | 0.55 | 0.55 | 0.55 | 0.55 | 0.55 | 0.55 | 1.00 |

---Stage 5---

| s \ | x: | 1 | 2 | 3 | 4 | 5 | 6 | Max |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.00 | XXXXXXXXXXXXXXXXXXXXXXXXXXXXX | | | | | | 0.00 |
| 1 | 0.00 | 0.00 | XXXXXXXXXXXXXXXXXXXXXXXX | | | | | 0.00 |
| 2 | 0.00 | 0.30 | 0.30 | XXXXXXXXXXXXXXXXXXX | | | | 0.30 |
| 3 | 0.55 | 0.30 | 0.30 | 0.55 | XXXXXXXXXXXXXX | | | 0.55 |
| 4 | 0.55 | 0.55 | 0.55 | 0.55 | 0.55 | XXXXXXXXX | | 0.55 |
| 5 | 0.55 | 0.80 | 0.80 | 0.55 | 0.55 | 0.55 | XXXX | 0.80 |
| 6 | 1.00 | 0.80 | 0.80 | 0.80 | 0.55 | 0.55 | 0.55 | 1.00 |

---Stage 4---

| s \ | x: | 1 | 2 | 3 | 4 | 5 | 6 | Max |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.00 | XXXXXXXXXXXXXXXXXXXXXXXXXXXXX | | | | | | 0.00 |
| 1 | 0.00 | 0.17 | XXXXXXXXXXXXXXXXXXXXXXXX | | | | | 0.17 |
| 2 | 0.30 | 0.30 | 0.30 | XXXXXXXXXXXXXXXXXXX | | | | 0.30 |
| 3 | 0.55 | 0.44 | 0.44 | 0.55 | XXXXXXXXXXXXXX | | | 0.55 |
| 4 | 0.55 | 0.69 | 0.69 | 0.55 | 0.55 | XXXXXXXXX | | 0.69 |
| 5 | 0.80 | 0.80 | 0.80 | 0.69 | 0.55 | 0.55 | XXXX | 0.80 |
| 6 | 1.00 | 0.91 | 0.80 | 0.80 | 0.69 | 0.55 | 0.55 | 1.00 |

---Stage 3---

| s \ x: | | 1 | 2 | 3 | 4 | 5 | 6 | Max |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.00 | XXXXXXXXXXXXXXXXXXXXXXXXXXXXXX | | | | | | 0.00 |
| 1 | 0.17 | 0.17 | XXXXXXXXXXXXXXXXXXXXXXXXX | | | | | 0.17 |
| 2 | 0.30 | 0.38 | 0.38 | XXXXXXXXXXXXXXXXXXXX | | | | 0.38 |
| 3 | 0.55 | 0.51 | 0.51 | 0.55 | XXXXXXXXXXXXXXX | | | 0.55 |
| 4 | 0.69 | 0.69 | 0.69 | 0.62 | 0.55 | XXXXXXXXXX | | 0.69 |
| 5 | 0.80 | 0.86 | 0.80 | 0.69 | 0.62 | 0.55 | XXXX | 0.86 |
| 6 | 1.00 | 0.91 | 0.86 | 0.80 | 0.69 | 0.62 | 0.55 | 1.00 |

---Stage 2---

| s \ x: | 0 | 1 | 2 | 3 | 4 | 5 | 6 | Max |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.00 | XXXXXXXXXXXXXXXXXXXXXXXXXXXXXX | | | | | | 0.00 |
| 1 | 0.17 | 0.21 | XXXXXXXXXXXXXXXXXXXXXXXXX | | | | | 0.21 |
| 2 | 0.38 | 0.38 | 0.38 | XXXXXXXXXXXXXXXXXXXX | | | | 0.38 |
| 3 | 0.55 | 0.55 | 0.55 | 0.55 | XXXXXXXXXXXXXXX | | | 0.55 |
| 4 | 0.69 | 0.72 | 0.72 | 0.62 | 0.55 | XXXXXXXXXX | | 0.72 |
| 5 | 0.86 | 0.86 | 0.80 | 0.72 | 0.62 | 0.55 | XXXX | 0.86 |
| 6 | 1.00 | 0.94 | 0.86 | 0.80 | 0.72 | 0.62 | 0.55 | 1.00 |

---Stage 1---

| s \ x: | 0 | 1 | 2 | 3 | Max |
|---|---|---|---|---|---|
| 3 | 0.55 | | 0.57 | 0.55 | 0.57 |

56:171 Operations Research
HW#13 – Due December 19, 2001

1. The ***Green Valley Christmas Tree Farm*** owns a plot of land with 6000 evergreen trees. Each year they allow individuals to select and cut Christmas trees. However, they protect small trees (usually less than 4 feet tall) so that they will grow and be available for sale in future years. Currently 2000 trees are classified as protected trees, while the remaining 4000 are available for cutting. However, even though a tree is available for cutting in a given year, it possibly might not be selected for cutting until future years. While most trees not cut in a given year live until the next year (protected or unprotected), approximately 15% are lost to disease. Each year, approximately 60% of the unprotected trees are cut, and 30% of the protected trees surviving from the previous year have matured sufficiently to be made available for cutting.

   (a.) Define a Markov chain model of the system consisting of a **single** tree. Sketch the transition diagram and write down the probability matrix.

   (b.) What are the absorbing states of this model?

   (c.) What is the probability that a tree which is protected is eventually sold? that it eventually dies of disease?

   (d.) How many of the farm's 6000 trees are expected to be sold eventually, and how many will be lost to disease?

   (e.) If a tree is initially protected, what is the expected number of years until it is either sold or dies?

**2. Reservoir control**

A water reservoir with random inflows is to be controled by selecting, for each of NN=8 periods (stages) the amount of water to be released, which we will assume to be restricted to values 0, 10, 20, 30, 40, and 50. (Units might be in acre-feet, or millions of cubic feet, etc.)

There are "target" values for both the amount of water in the reservoir and the amount of water released at each stage. These targets vary by stage:

| *Stage* | *1* | *2* | *3* | *4* | *5* | *6* | *7* | *8* | *Final* |
|---|---|---|---|---|---|---|---|---|---|
| Storage target | 25 | 30 | 35 | 30 | 25 | 30 | 35 | 30 | 25 |
| Release target | 25 | 25 | 20 | 20 | 25 | 25 | 20 | 20 | --- |

(The release targets depend upon downstream demand for irrigation water and navigation, while the storage targets depend upon the desirability of recreational use of the reservoir.)

The objective is to *minimize the sum of the squares of the deviations from the targets*.

The state of the system at stage N is the quantity of water in the reservoir in that time period (discrete: 0, 10, 20, ...50) and the decision is the amount of water to be released (0, 10, ...50).

The recursion is forward, with optimal value function at stage N which is a vector-valued function equal to the minimum cost for stages N, N+1, ...N given the current state (amount of water in the reservoir).

Assume that the probability distribution of inflows into the reservoir are the same in every stage:

| *Inflow* | *10* | *20* | *30* | *40* |
|---|---|---|---|---|
| Probability | 0.2 | 0.4 | 0.3 | 0.1 |

A stochastic DP model was built, with the optimal value function $f_n(s)$ equal to the minimum sum of the squares of deviations from the target storage volumes and the target releases from stage n through stage 8, plus the square of the deviation from the target storage volume at stage 9, if the reservoir contains s units of water at the beginning of stage n. At the beginning of the first stage, the storage volume is 40.

The results of the DP model are shown below

---Stage 8---

| s \ x: | 0 | 10 | 20 | 30 | 40 | 50 | Minimum |
|---|---|---|---|---|---|---|---|
| 0 | 1385.0 | 1001224.0 | 1201218.8 | 1601208.4 | 1901360.6 | 2001798.0 | 1385.0 |
| 10 | 945.0 | 585.0 | 1000624.0 | 1200818.8 | 1601008.4 | 1901360.6 | 585.0 |
| 20 | 100782.4 | $\boxed{\phantom{0000000}}$ | 185.0 | 1000424.0 | 1200818.8 | 1601208.4 | 185.0 |
| 30 | 400694.6 | 100382.4 | 145.0 | 185.0 | 1000624.0 | 1201218.8 | 145.0 |
| 40 | 800624.2 | 400494.6 | 100382.4 | 345.0 | 585.0 | 1001224.0 | 345.0 |
| 50 | 1000799.0 | 800624.2 | 400694.6 | 100782.4 | 945.0 | 1385.0 | 945.0 |

---Stage 7---

| s \ x: | 0 | 10 | 20 | 30 | 40 | 50 | Minimum |
|---|---|---|---|---|---|---|---|
| 0 | 1894.0 | 1001905.0 | 1201971.8 | 1601797.4 | 1901761.6 | 2002123.0 | 1894.0 |
| 10 | 1318.0 | 994.0 | 1001205.0 | 1201471.8 | 1601497.4 | 1901661.6 | 994.0 |
| 20 | 101075.4 | 618.0 | 494.0 | 1000905.0 | 1201371.8 | 1601597.4 | 494.0 |
| 30 | 400871.6 | 100575.4 | 318.0 | 394.0 | 1001005.0 | 1201671.8 | 318.0 |
| 40 | 800613.2 | 400571.6 | 100475.4 | 418.0 | 694.0 | 1001505.0 | 418.0 |
| 50 | 1000624.0 | 800513.2 | 400671.6 | 100775.4 | 918.0 | 1394.0 | 918.0 |

---Stage 6---

| s \ x: | 0 | 10 | 20 | 30 | 40 | 50 | Minimum |
|---|---|---|---|---|---|---|---|
| 0 | 2058.6 | 1002080.4 | 1202029.0 | 1601591.0 | 1901312.5 | 2001523.0 | 2058.6 |
| 10 | 1468.2 | 1158.6 | 1001380.4 | 1201529.0 | 1601291.0 | 1901212.5 | 1158.6 |
| 20 | 101231.1 | 768.2 | 658.6 | 1001080.4 | 1201429.0 | 1601391.0 | 658.6 |
| 30 | 401075.4 | 100731.1 | 468.2 | 558.6 | 1001180.4 | 1201729.0 | 468.2 |
| 40 | 800907.8 | 400775.4 | 100631.1 | 568.2 | 858.6 | 1001680.4 | 568.2 |
| 50 | 1001024.0 | 800807.8 | 400875.4 | 100931.1 | 1068.2 | 1558.6 | 1068.2 |

---Stage 5---

| s \ x: | 0 | 10 | 20 | 30 | 40 | 50 | Minimum |
|---|---|---|---|---|---|---|---|
| 0 | 1942.4 | 1001968.6 | 1201885.7 | 1601381.8 | 1901054.0 | 2001248.0 | 1942.4 |
| 10 | 1446.3 | 1142.4 | 1001368.6 | 1201485.7 | 1601181.8 | 1901054.0 | 1142.4 |
| 20 | 101291.3 | 846.3 | 742.4 | 1001168.6 | 1201485.7 | 1601381.8 | 742.4 |
| 30 | 401190.5 | 100891.3 | 646.3 | 742.4 | 1001368.6 | 1201885.7 | 646.3 |
| 40 | 801062.8 | 400990.5 | 100891.3 | 846.3 | 1142.4 | 1001968.6 | 846.3 |
| 50 | 1001249.0 | 801062.8 | 401190.5 | 101291.3 | 1446.3 | 1942.4 | 1446.3 |

---Stage 4---

| s \ x: | 0 | 10 | 20 | 30 | 40 | 50 | Minimum |
|---|---|---|---|---|---|---|---|
| 0 | 2104.0 | 1002131.8 | 1202092.8 | 1601695.4 | 1901492.3 | 2001798.0 | 2104.0 |
| 10 | 1605.5 | 1304.0 | 1001531.8 | 1201692.8 | 1601495.4 | 1901492.3 | 1304.0 |
| 20 | 101401.6 | 1005.5 | 904.0 | 1001331.8 | 1201692.8 | 1601695.4 | 904.0 |
| 30 | 401147.4 | 101001.6 | 805.5 | 904.0 | 1001531.8 | 1202092.8 | 805.5 |
| 40 | 800788.5 | 400947.4 | 101001.6 | 1005.5 | 1304.0 | 1002131.8 | 1005.5 |
| 50 | 1000799.0 | 800788.5 | 401147.4 | 101401.6 | 1605.5 | 2104.0 | 1605.5 |

---Stage 3---

| s \ x: | 0 | 10 | 20 | 30 | 40 | 50 | Minimum |
|---|---|---|---|---|---|---|---|
| 0 | 2589.6 | 1002618.1 | 1202547.0 | 1602085.0 | 1901833.5 | 2002123.0 | 2589.6 |
| 10 | 1990.2 | 1689.6 | 1001918.1 | 1202047.0 | 1601785.0 | 1901733.5 | 1689.6 |
| 20 | 101669.9 | 1290.2 | 1189.6 | 1001618.1 | 1201947.0 | 1601885.0 | 1189.6 |
| 30 | 401267.9 | 101169.9 | 990.2 | 1089.6 | 1001718.1 | 1202247.0 | 990.2 |
| 40 | 800745.3 | 400967.9 | 101069.9 | 1090.2 | 1389.6 | 1002218.1 | 1090.2 |
| 50 | 1000624.0 | 800645.3 | 401067.9 | 101369.9 | 1590.2 | 2089.6 | 1590.2 |

---Stage 2---

| s \ x: | 0 | 10 | 20 | 30 | 40 | 50 | Minimum |
|---|---|---|---|---|---|---|---|
| 0 | 2744.8 | 1002773.7 | 1202585.5 | 1601869.2 | 1901382.1 | 2001523.0 | 2744.8 |
| 10 | 2145.1 | 1844.8 | 1002073.7 | 1202085.5 | 1601569.2 | 1901282.1 | 1844.8 |
| 20 | 101836.1 | 1445.1 | 1344.8 | 1001773.7 | 1201985.5 | 1601669.2 | 1344.8 |
| 30 | 401478.7 | 101336.1 | 1145.1 | 1244.8 | 1001873.7 | 1202285.5 | 1145.1 |
| 40 | 801042.2 | 401178.7 | 101236.1 | 1245.1 | 1544.8 | 1002373.7 | 1245.1 |
| 50 | 1001024.0 | 800942.2 | 401278.7 | 101536.1 | 1745.1 | 2244.8 | 1745.1 |

---Stage 1---

| s \ x: | 0 | 10 | 20 | 30 | 40 | 50 | Minimum |
|---|---|---|---|---|---|---|---|
| 40 | 801198.2 | 401396.6 | 101500.5 | 1525.0 | $\boxed{\phantom{0000}}$ | 1002653.9 | 1525.0 |

a. Two values are missing in the table in the tables above--- compute them.
b. What is the optimal amount to be released in stage 1? _________
c. Suppose that the inflow in stage 1 happens to be the most likely value (20). What should then be the release in stage 2? ______