# 56:171

# Fall 2002

# Operations Research

# Homework Assignments

Instructor: D. L. Bricker

University of Iowa

Dept. of Mechanical & Industrial Engineering

| 56:171 Operations Research<br>Homework #1 - Due Wednesday, September 4, 2002 |
|---|

1. The Keyesport Quarry has two different pits from which it obtains rock. The rock is run through a crusher to produce two products: concrete grade stone and road surface chat. Each ton of rock from the South pit converts into 0.75 tons of stone and 0.25 tons of chat when crushed. Rock from the North pit is of different quality. When it is crushed it produces a "50-50" split of sone and chat. The Quarry has contracts for 60 tons of stone and 40 tons of chat this planning period. The cost per ton of extracting and crushing rock from the South pit is 1.6 times as costly as from the North pit.
   a. What are the decision variables in the problem? *Be sure to give their definitions, not just their names!*
   b. There are two constraints for this problem.
      - State them in words.
      - State them in equation or inequality form.
   c. State the objective function.
   d. Graph the feasible region (in 2 dimensions) for this problem.
   e. Draw an appropriate objective function line on the graph and indicate graphically and numerically the optimal solution.
   f. Use LINDO (or other appropriate LP solver) to compute the optimal solution.

2. a. Draw the feasible region of the following LP:

$$\begin{array}{ll} \text{Maximize} & 5X_1 + 2X_2 \\ \text{subject to} & 4X_1 + 3X_2 \leq 24 \\ & X_1 + X_2 \leq 8 \\ & 3X_1 + X_2 \leq 9 \\ & X_1 \geq 0, X_2 \geq 0 \end{array}$$

   b. Indicate on the graph the optimal solution.

3. a. Compute the inverse of the matrix (showing your computational steps):

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ -2 & -1 & 1 \end{bmatrix}$$

   b. Find a solution (if one exists) of the equations:

$$\begin{cases} X_1 + 2X_2 - X_3 = 4 \\ 2X_1 - X_2 + 2X_3 = 15 \\ 3X_2 - 2X_3 = -5 \end{cases}$$

56:171 Operations Research
Homework #2 - Due Friday, September 12, 2002

1. (Exercise 3.4-18, page 98, of Hillier&Lieberman text, 7th edition)
"Oxbridge University maintains a powerful mainframe computer for research use by its faculty, Ph.D. students, and research associates. During all working hours, an operator must be able to operate and maintain the computer, as well as to perform some programming services. Beryl Ingram, the director of the computer facility, oversees the operation.
It is now the beginning of the fall semester, and Beryl is confronted with the problem of assigning different working hours to her operators. Because all the operators are currently enrolled in the university, they are available to work only a limited number of hours each day, as shown in the following table.

| | | Maximum # hours available | | | | |
|---|---|---|---|---|---|---|
| Name | Wage $/hr | Mon | Tues | Wed | Thur | Fri |
| K.C. | 10.00 | 6 | 0 | 6 | 0 | 6 |
| D.H. | 10.10 | 0 | 6 | 0 | 6 | 0 |
| H.B. | 9.90 | 4 | 8 | 4 | 0 | 4 |
| S.C. | 9.80 | 5 | 5 | 5 | 0 | 5 |
| K.S. | 10.80 | 3 | 0 | 3 | 8 | 0 |
| N.K. | 11.30 | 0 | 0 | 0 | 6 | 2 |

There are six operators (four undergraduate students and two graduate students). They all have different wage rates because of differences in their experience with computers and in their programming ability. The above table shows their wage rates, along with the maximum number of hours that each can work each day.
Each operator is guaranteed a certain minimum number of hours per week that will maintain an adequate knowledge of the operation. This level is set arbitrarily at 8 hours per week for the undergraduate students (K.C,, D.H, H.B, and S.C.) and 7 hours per week for the graduate students (K.S. and N.K.).
The computer facility is to be open for operation from 8 a.m. to 10 p.m. Monday through Friday with exactly one operator on duty during these hours. On Saturdays and Sundays, the computer is to be operated by other staff.
Because of a tight budget, Beryl has to minimize cost. She wishes to determine the number of hours she should assign to each operator on each day."

a. Formulate a linear programming model for this problem. Be sure to define your variables!
b. Use an LP solver (e.g. LINDO or LINGO) to find the optimal solution.

2. (Exercise 4.4-9, page 176, of Hillier&Lieberman text, 7th edition)
Work through the simplex method step by step (in tabular form) to solve the following problem:

Maximize $Z=2X_1 - X_2 + X_3$
subject to
$3X_1 + X_2 + X_3 \leq 6$
$X_1 - X_2 + 2X_3 \leq 1$
$X_1 + X_2 - X_3 \leq 2$
and
$X_1 \geq 0, X_2 \geq 0, X_3 \geq 0$

3. (Exercise 3.5-3, p. 99, of Hillier&Lieberman text, 7th edition)
Read the 1986 article footnoted in Sec. 2.1 that describes the third case study presented in Section 3.5: "Planning Supply, Distribution, and Marketing at Citgo Petroleum Corporation," by D. Klingman, N. Phillips, D. Steiger, R. Wirth, & W. Young, *Interfaces*, Vol. 16 no. 3 (May-June 1986), pp. 1-19.

a. What happened during the years preceding this OR study that made it vastly more important to control the amount of capital tied up in inventory?
b. What geographical area is spanned by Citgo's distribution network of pipelines, tankers, and barges? Where do they market their products?
c. What time periods are included in the model?
d. Which computer did Citgo use to solve the model? What were typical run times?
e. Who are the four types of model users?
f. List the major types of reports generated by the SDM system.

56:171 Operations Research
Homework #3 - Due Friday, September 20, 2002

1. **Revised Simplex Method** Consider the LP problem

$$\text{Maximize } z = 3x_1 - x_2 + 2x_3$$

subject to

$$\begin{cases} x_1 + x_2 + x_3 \le 15 \\ 2x_1 - x_2 + x_3 \le 2 \\ -x_1 + x_2 + x_3 \le 4 \end{cases}$$

$$\& \ x_j \ge 0, j = 1,2,3$$

a. Let $x_4$, $x_5$, & $x_6$ denote the slack variables for the three constraints, and write the LP with equality constraints.

After several iterations of the revised simplex method, the basis B={4,3,2} and the basis inverse matrix is

$$\left(A^B\right)^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

b. Proceed with one iteration of the revised simplex method, by
   i. computing the simplex multiplier vector $\pi$
   ii. "pricing", i.e., computing the "relative profits", of the nonbasic columns
   iii. selecting the column to enter the basis
   iv. computing the substitution rates of the entering column
   v. selecting the variable to leave the basis
   vi. updating the basis inverse matrix.

c. Write the dual of the above LP (i.e. with equality constraints & slack variables) in (a).

d. Substitute the vector $\pi$ which you computed above in step (i) above to test whether it is feasible in the dual LP. Which constraint(s) if any are violated? How does this relate to the results in step (ii) above?

2. **LP formulation:** ***Staffing a Call Center*** (Case 3.3, pages 106-108, *Intro. to O.R.* by Hillier & Lieberman) Answer parts (a), (b), & (c) on page 108, using LINGO with sets to enter the model.

"California Children's Hospital has been receiving numerous customer complaints because of its confusing, decentralized appointment and registration process. When customers want to make appointments or register child patients, they must contact the clinic or department they plan to visit. Several problems exist with this current strategy. Parents do not always know the most appropriate clinic or department they must visit to address their children's ailments. They therefore spend a significant amount of time on the phone being transferred from clinic to clinic until they reach the most appropriate clinic for their needs. The hospital also does not publish the phone numbers of all clinics and departments, and parents must therefore invest a large amount of time in detective work to track down the correct phone number. Finally, the various clinics and departments do not communicate with each other. For example, when a doctor schedules a referral with a colleague located in another department or clinic that department or clinic almost never receives word of the referral. The parent must contact the correct department or clinic and provide the needed referral information.

In efforts to reengineer and improve its appointment and registration process, the children's hospital has decided to centralize the process by establishing one call center devoted exclusively to appointments and registration. The hospital is currently in the middle of the planning stages for the call center. Lenny Davis, the hospital manager, plans to operate the call center from 7 a.m. to 9 p.m. during the weekdays.

Several months ago, the hospital hired an ambitious management consulting firm, Creative Chaos Consultants, to forecast the number of calls the call center would receive each hour of the day. Since all appointment and registration- related calls would be received by the call center, the consultants decided that they could forecast the calls at the call center by totaling the number of appointment and registration-related calls received by all clinics and departments. The team members visited all the clinics and departments, where they diligently recorded every call relating to appointments and registration. they then totaled these calls and altered the totals to account for calls missed during data collection. They also altered totals to account for repeat calls that occurred when the same parent called the hospital many times because of the confusion surrounding the decentralized process. Creative Chaos Consultants determined the average number of calls the call center should expect during each hour of a weekday. The following table provides the forecasts:

| *Work Shift* | *Average Number of Calls per hour* |
|---|---|
| 7 a.m. — 9 a.m. | 40 |
| 9 a.m. —11 a.m. | 85 |
| 11 a.m. —1 p.m. | 70 |
| 1 p.m.—3 p.m. | 95 |
| 3 p.m. —5 p.m. | 80 |
| 5 p.m.—7 p.m. | 35 |
| 7 p.m.—9 p.m. | 10 |

After the consultants submitted these forecasts, Lenny became interested in the percentage of calls from Spanish speakers since the hospital services many Spanish-speaking patients. Lenny knows that he has to hire some operators who speak Spanish to handle these calls. The consultants performed further data collection and determined that on average, 20 percent of the calls were from Spanish speakers.

Given these call forecasts, Lenny must now decide how to staff the call center during each 2 hour shift of a weekday. During the forecasting project, Creative Chaos Consultants closely observed the operators working at the individual clinics and departments and determined the number of calls that operators could process per hour. The consultants informed Lenny that an operator is able to process an average of six calls per hour. Lenny also knows that he has both full-time and part-time workers available to staff the call center. A full-time employee works 8 hours per day, but because of paperwork that must be completed, the employee spends only 4 hours per day on the phone. To balance the schedule, the employee alternates the 2-hour shifts between answering phones and completing paperwork. Full-time employees can start their day either by answering phones or by completing paperwork on the first shift. The full-time employees speak either Spanish or English, but none of them are bilingual. Both Spanish-speaking and English-speaking employees are paid \$10 per hour for work before 5 p.m. and \$12 per hour for work after 5 p.m. The full-time employees can begin work at the beginning of the 7 a.m. to 9 a.m. shift, 9 a.m. to 11 a.m. shift, 11 a.m. to 1 p.m. shift, or 1 p.m. to 3 p.m. shift. The part-time employees work for 4 hours, only answer calls, and only speak English. They can start work at the beginning of the 3 p.m.—5 p.m. shift or the 5 p.m.—7 p.m. shift, and like the full-time employees, they are paid \$10 per hour for work before 5 p.m. and \$12 per hour for work after 5 p.m.

For the following analysis, consider only the labor cost for the time employees spend answering phones. The cost for paperwork time is charged to other cost centers.
a. How many Spanish-speaking operators and how many English-speaking operators does the hospital need to staff the call center during each 21-hour shift of the day in order to answer all calls? Pleas provide an integer number since half a human operator makes no sense

b. Lenny needs to determine how many full-time employees who speak Spanish, full-time employees who speak English, and part-time employees he should hire to begin on each shift. Creative Chaos Consultants advise him that linear programming can be used to do this in such a way as to minimize operating costs while answering all calls. Formulate a linear programming model of this problem.

c. Obtain an optimal solution for the LP model formulated in part (b) to guide Lenny's decision.

**3. Sensitivity Analysis** (exercise 6.7-18, pages 296-297, *Intro. to O.R.* by Hillier & Lieberman) Answer parts (a) through (e), using the information shown on page 296.
Ken and Larry, Inc., supplies its ice cream parlors with three flavors of ice cream: chocolate, vanilla, and banana. Because of extremely hot weather and a high demand for its products, the company has run short of its supply of ingredients: milk, sugar, & cream. Hence, they will not be able to fill all the orders received from their retail outlets, the ice cream parlors. Owing to these circumstances, the company has decided to choose the amount of each product to produce that will maximize total profit, given the constraints on supply of the basic ingredients. The chocolate, vanilla, and banana flavors generate, respectively, $1.00, $0.90, and $0.95 per profit per gallon sold. The company has only 200 gallons of milk, 150 pounds of sugar, and 60 gallons of cream left in its inventory. The LP formulation for this problem has variables C, V, and B representing gallons of chocolate, vanilla, and banana ice cream produced, respectively.

```
!    Ken & Larry Ice Cream – from Intro to O.R.
!    Hillier &(7th ed) p. 296

MAXIMIZE  C+0.9V+0.95B
ST

0.45C + 0.50V + 0.40B <= 200  ! milk resource
0.50C + 0.40V + 0.40B <= 150  ! sugar resource
0.10C + 0.15V + 0.20B <= 60   ! cream resource

END
```

LINDO model

```
       OBJECTIVE FUNCTION VALUE
       1)      341.2500

  VARIABLE        VALUE          REDUCED COST
         C         0.000000          0.037500
         V       300.000000          0.000000
         B        75.000000          0.000000

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)        20.000000          0.000000
        3)         0.000000          1.875000
        4)         0.000000          1.000000
```

```
RANGES IN WHICH THE BASIS IS UNCHANGED:

                           OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT         ALLOWABLE        ALLOWABLE
                    COEF          INCREASE         DECREASE
        C        1.000000          0.037500         INFINITY
        V        0.900000          0.050000         0.012500
        B        0.950000          0.021429         0.050000

                           RIGHTHAND SIDE RANGES
      ROW         CURRENT         ALLOWABLE        ALLOWABLE
                      RHS         INCREASE         DECREASE
        2      200.000000          INFINITY        20.000000
        3      150.000000         10.000000        30.000000
        4       60.000000         15.000000         3.750000
```

a. What is the optimal profit and the optimal solution?

b. Suppose the profit per gallon of banana changes to $1.00. Will the optimal solution change, and what can be said about the effect on total profit?

c. Suppose the profit per gallon of banana changes to 92 cents. Will the optimal solution change, and what can be said about the effect on total profit?

d. Suppose the company discovers that 3 gallons of cream have gone sour and so must be thrown out. Will the optimal solution change, and what can be said about the effect on total profit?

e. Suppose that the company has the opportuntiy to buy an additional 15 pounds of sugar at a total cost of $15. Should they buy it? Explain!

56:171 Operations Research
Homework #4 - Due Friday, September 27, 2002

**1. Ken & Larry's Ice Cream**, *continued.* Refer to the problem description in last week's homework (HW#3). The optimal LP tableau provided by LINDO is as shown below.

```
THE TABLEAU

     ROW  (BASIS)        C          V          B      SLK 2      SLK 3
       1   ART        0.038      0.000      0.000      0.000      1.875
       2   SLK  2    -0.350      0.000      0.000      1.000     -2.000
       3        V     3.000      1.000      0.000      0.000     10.000
       4        B    -1.750      0.000      1.000      0.000     -7.500

     ROW     SLK 4
       1     1.000   341.250
       2     2.000    20.000
       3   -20.000   300.000
       4    20.000    75.000
```

a. Chocolate ice cream is not included in the optimal production plan. If one gallon of chocolate ice cream were to be produced, how would it change the quantity
   …of vanilla ice cream produced?
   …of banana ice cream produced?
   …of milk used?
   …of sugar used?
   …of cream used?

b. In last week's homework, you were asked about the effect on profit of a reduction in the quantity of available cream due to spoilage. That is, the effect of an increase in the unused cream (slack in the available cream constraint). According to the substitution rates in the tableau, what would be the effect of this spoilage on the quantity
   …of vanilla ice cream produced?
   …of banana ice cream produced?
   …of milk used?
   …of sugar used?

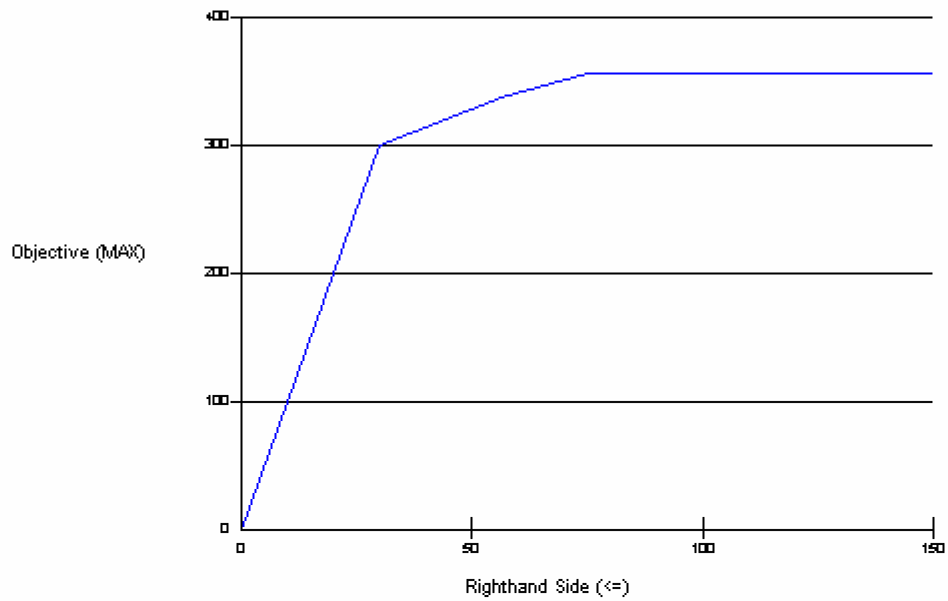
The right-hand-side of row #4 (available cream) was changed to zero, and then parametric analaysis performed with the right-hand-side increasing to 150 gallons, with the results below.

```
RIGHTHANDSIDE PARAMETRICS REPORT FOR ROW: 4

   VAR    VAR  PIVOT   RHS   DUAL PRICE   OBJ
   OUT    IN    ROW    VAL  BEFORE PIVOT  VAL

                     0.0000   10.0000    0.000
 SLK 3      V   4   30.0000   10.0000  300.000
     C      B   3   56.2500    1.42857 337.500
     V  SLK 4   4   75.0000    1.00000 356.250
                   150.000     0.0000  356.250
```

The plot of optimal value vs gallons of cream available was also prepared by LINDO:




c. Using LINDO's report, indicate on the graph above the <u>slope</u> of each linear segment and the <u>coordinates</u> of each break-point (profit & gallons of cream).

**2. LP model formulation.** Buster Sod's younger brother, Marky Dee, operates three ranches in Texas. the acreage and irrigation water available for the three farms are shown below:

| FARM | ACREAGE | WATER AVAILABLE (ACRE-FT) |
|---|---|---|
| 1 | 400 | 1500 |
| 2 | 600 | 2000 |
| 3 | 300 | 900 |

Three crops can be grown. However, the maximum acreage that can be grown of each crop is limited by the amount of appropriate harvesting equipment available. The three crops are described below. Any combination of crops may be grown on a farm.

| CROP | TOTAL HARVESTING CAPACITY (IN ACRES) | WATER REQMTS (ACRE-FT PER ACRE) | EXPECTED PROFIT ($/ACRE) |
|---|---|---|---|
| Milo | 700 | 6 | 400 |
| Cotton | 800 | 4 | 300 |
| Wheat | 300 | 2 | 100 |

Using **LINGO**, the following sets were defined, with decision variables:

$X_{ij}$ = # acreas of crop j planted on farm i.

```
MODEL:   ! MARKY DEE SOD'S RANCHES;

SETS:
      FARM/1..3/:ACREAGE, H20_AVAIL;
      CROP/MILO, COTTON, WHEAT/:CAPACITY, H20_RQMT, PROFIT;
      COMBO(FARM,CROP):X;
ENDSETS

DATA:
      ACREAGE   =  400  600 300;
      H20_AVAIL = 1500 2000 900;
      CAPACITY  =   700  800 300;
      H20_RQMT  =     6    4   2;
      PROFIT    =   400  300 100;
ENDDATA

!   INSERT OBJECTIVE & CONSTRAINTS HERE ;

END
```

a. Using LINGO, formulate the LP model to maximize the total expected profit of the three ranches.

b. Add the statements to the accompanying file (HW4_2.lg4) , and solve.

**56:171 Operations Research**
**Homework #5 - Due Friday, October 4, 2002**

1. Consider the transportation tableau:

| dstn→ ↓source | 1 | 2 | 3 | 4 | 5 | Supply |
|---|---|---|---|---|---|---|
| **A** | 12 | 8 | 9 | 15 | 11 | 9 |
| **B** | 10 | 11 | 12 | 11 | 14 | 7 |
| **C** | 9 | 7 | 11 | 14 | 8 | 4 |
| **D** | 13 | 12 | 13 | 12 | 12 | 7 |
| **E** | 8 | 9 | 10 | 9 | 10 | 3 |
| **Demand=** | 4 | 7 | 5 | 5 | 9 | |

a. Use the initial basic solution: $X_{A3}=5$, $X_{A5}=4$, $X_{B1}=4$, $X_{B4}=3$, $X_{C4}=X_{C5}=2$, $X_{D2}=7$, $X_{E5}=3$ & _____ = 0. *(Choose one more variable to complete the basis. Any choice is valid except one that would create a "cycle" of basic cells in the tableau!)*

b. Compute two different sets of values for the dual variables U & V (*simplex multipliers*) for this basis.

c. Using each set of simplex multipliers, price all of the nonbasic cells. How do the reduced costs depend upon the choice of dual variables? Select the variable having the "most negative" reduced cost to enter the basis.

d. What variable will enter the basis as the new variable enters the basis?

e. Complete the computation of the optimal solution, using the transportation simplex method.

2. **Production scheduling** (adapted from O.R. text by Hillier & Lieberman, 7th edition, page 394) The MLK Manufacturing Company must produce two products in sufficient quantity to meet contracted sales in each of the next three months. The two products share the same production facilities, and each unit of both products requires the same amount of production capacity. The available production and storage facilities are changing month by month, so the production capacities, unit production costs, and unit storage costs vary by month. Therefore, it may be worthwhile to overproduce one or both products in some months and store them until needed.

For each of the three months, the second column of the following table gives the maximum number of units of the two products combined that can be produced in Regular Time (RT) and in Overtime (OT). For each of the two products, the subsequent columns give (1) the number of units needed for the contracted sales, (2) the cost (in thousands of dollars) per unit produced in regular time, (3) the cost (in thousands of dollars) per unit produced in overtime, and (4) the cost (in thousands of dollars) of storing each extra unit that is held over into the next month.

In each case, the numbers for the two products are separated by a slash /, with the number for product 1 on the left and the number for product 2 on the right.

| | Max combined production | | | Unit cost of production ($K) | | |
|---|---|---|---|---|---|---|
| Month | RT | OT | Sales | RT | OT | Storage cost ($K) |
| 1 | 10 | 3 | 5/3 | 15/16 | 18/20 | 1/2 |
| 2 | 8 | 2 | 3/5 | 17/15 | 20/18 | 2/1 |
| 3 | 10 | 3 | 4/4 | 19/17 | 22/22 | |

The production manager wants a schedule developed for the number of units of each of the two products to be produced in regular time and (if regular time production capacity is used up) in overtime in each of the three months. The objective is to minimize the total of the production and storage costs while meeting the contracted sales for each month. There is no initial inventory, and no final inventory is desired after the three months.

a. Formulate this problem as a balanced transportation problem by constructing the appropriate transportation tableau.

b. Use the Northwest Corner Method to find an initial basic feasible solution. Is it degenerate?

c. Use the transportation simplex algorithm to find the optimal solution. Is it degenerate? Are there multiple optima?

**3. Assignment Problem.** (adapted from O.R. text by Hillier & Lieberman, 7th edition, page 399.) Four cargo ships will be used for shipping goods from one port to four other ports (labeled 1, 2, 3, 4). Any ship can be used for making any one of these four trips. However, because of differences in the ships and cargoes, the total cost of loading, transporting, and unloading the goods for the different ship-port combinations varies considerably, as shown in the following table:

| PORT→<br>↓SHIP | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | $500 | $400 | $600 | $700 |
| 2 | $600 | $600 | $700 | $500 |
| 3 | $700 | $500 | $700 | $600 |
| 4 | $500 | $400 | $600 | $600 |

The objective is to assign the four ships to four different ports in such a way as to minimize the total cost for all four shipments.

a. Use the Hungarian method to find an optimal solution.

b. Reformulate this as an equivalent transportation problem.

| dstn→ ↓source | 1 | 2 | 3 | 4 | Supply= |
|---|---|---|---|---|---|
| 1 | | | | | |
| 2 | | | | | |
| 3 | | | | | |
| 4 | | | | | |
| Demand= | | | | | |

c. Use the Northwest Corner Method to obtain an initial basic feasible solution. (This will be a *degenerate* solution. Be sure to specify which variables are basic!)

d. Use the transportation simplex method to find the optimal solution.

e. In how many iterations was the solution degenerate?

f. How many iterations produce a change in the *values* of the variables?

g. How many iterations leave the variables *unchanged in value* (although the basis changes)?

**4. Return of Marky D. Sod** Recall the LP model for this problem in HW#4:
Buster Sod's younger brother, Marky Dee, operates three ranches in Texas. the acreage and irrigation water available for the three farms are shown below:

| FARM | ACREAGE | WATER AVAILABLE (ACRE-FT) |
|---|---|---|
| 1 | 400 | 1500 |
| 2 | 600 | 2000 |
| 3 | 300 | 900 |

Three crops can be grown. However, the maximum acreage that can be grown of each crop is limited by the amount of appropriate harvesting equipment available. The three crops are described below. Any combination of crops may be grown on a farm.

| CROP | TOTAL HARVESTING CAPACITY (IN ACRES) | WATER REQMTS (ACRE-FT PER ACRE) | EXPECTED PROFIT ($/ACRE) |
|---|---|---|---|
| Milo | 700 | 6 | 400 |
| Cotton | 800 | 4 | 300 |
| Wheat | 300 | 2 | 100 |

Decisionvariables: $X_{ij}$ = # acreas of crop j planted on farm i.
The LINDO model (generated by LINGO) is:

```
MAX     400 X1MILO + 300 X1COTTON + 100 X1WHEAT + 400 X2MILO
      + 300 X2COTTON + 100 X2WHEAT + 400 X3MILO + 300 X3COTTON + 100 X3WHEAT
SUBJECT TO
```

```
        2)   X1MILO + X1COTTON + X1WHEAT <=   400
        3)   6 X1MILO + 4 X1COTTON + 2 X1WHEAT <=   1500
        4)   X2MILO + X2COTTON + X2WHEAT <=   600
        5)   6 X2MILO + 4 X2COTTON + 2 X2WHEAT <=   2000
        6)   X3MILO + X3COTTON + X3WHEAT <=   300
        7)   6 X3MILO + 4 X3COTTON + 2 X3WHEAT <=   900
        8)   X1MILO + X2MILO + X3MILO <=   700
        9)   X1COTTON + X2COTTON + X3COTTON <=   800
       10)   X1WHEAT + X2WHEAT + X3WHEAT <=   300
 END

       1)      320000.0

 VARIABLE        VALUE          REDUCED COST
   X1MILO         0.000000         0.000000
 X1COTTON       375.000000         0.000000
  X1WHEAT         0.000000        33.333332
   X2MILO        50.000000         0.000000
 X2COTTON       425.000000         0.000000
  X2WHEAT         0.000000        33.333332
   X3MILO       150.000000         0.000000
 X3COTTON         0.000000         0.000000
  X3WHEAT         0.000000        33.333332

      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)        25.000000         0.000000
       3)         0.000000        66.666664
       4)       125.000000         0.000000
       5)         0.000000        66.666664
       6)       150.000000         0.000000
       7)         0.000000        66.666664
       8)       500.000000         0.000000
       9)         0.000000        33.333332
      10)       300.000000         0.000000

 RANGES IN WHICH THE BASIS IS UNCHANGED:

                           OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                    COEF          INCREASE         DECREASE
   X1MILO      400.000000         0.000000         INFINITY
 X1COTTON      300.000000         INFINITY         0.000000
  X1WHEAT      100.000000        33.333328         INFINITY
   X2MILO      400.000000         0.000000         0.000000
 X2COTTON      300.000000         0.000000         0.000000
  X2WHEAT      100.000000        33.333328         INFINITY
   X3MILO      400.000000         INFINITY         0.000000
 X3COTTON      300.000000         0.000000         INFINITY
  X3WHEAT      100.000000        33.333328         INFINITY

                           RIGHTHAND SIDE RANGES
      ROW         CURRENT        ALLOWABLE        ALLOWABLE
                    RHS           INCREASE         DECREASE
        2      400.000000         INFINITY        25.000000
        3     1500.000000       100.000000       300.000000
        4      600.000000         INFINITY       125.000000
        5     2000.000000       750.000000       300.000000
        6      300.000000         INFINITY       150.000000
        7      900.000000       900.000000       900.000000
        8      700.000000         INFINITY       500.000000
        9      800.000000        75.000000       425.000000
       10      300.000000         INFINITY       300.000000
```

THE TABLEAU:

| ROW | (BASIS) | X1MILO | X1COTTON | X1WHEAT | X2MILO | X2COTTON | X2WHEAT |
|---|---|---|---|---|---|---|---|
| 1 | ART | 0.000 | 0.000 | 33.333 | 0.000 | 0.000 | 33.333 |
| 2 | SLK 2 | -0.500 | 0.000 | 0.500 | 0.000 | 0.000 | 0.000 |
| 3 | X1COTTON | 1.500 | 1.000 | 0.500 | 0.000 | 0.000 | 0.000 |
| 4 | SLK 4 | 0.500 | 0.000 | 0.167 | 0.000 | 0.000 | 0.667 |
| 5 | X2MILO | 1.000 | 0.000 | 0.333 | 1.000 | 0.000 | 0.333 |
| 6 | SLK 6 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 7 | X3MILO | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 8 | SLK 8 | 0.000 | 0.000 | -0.333 | 0.000 | 0.000 | -0.333 |
| 9 | X2COTTON | -1.500 | 0.000 | -0.500 | 0.000 | 1.000 | 0.000 |
| 10 | SLK 10 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 1.000 |

| ROW | X3MILO | X3COTTON | X3WHEAT | SLK 2 | SLK 3 | SLK 4 | SLK 5 |
|---|---|---|---|---|---|---|---|
| 1 | 0.000 | 0.000 | 33.333 | 0.000 | 66.667 | 0.000 | 66.667 |
| 2 | 0.000 | 0.000 | 0.000 | 1.000 | -0.250 | 0.000 | 0.000 |
| 3 | 0.000 | 0.000 | 0.000 | 0.000 | 0.250 | 0.000 | 0.000 |
| 4 | 0.000 | -0.333 | 0.000 | 0.000 | 0.083 | 1.000 | -0.167 |
| 5 | 0.000 | -0.667 | 0.000 | 0.000 | 0.167 | 0.000 | 0.167 |
| 6 | 0.000 | 0.333 | 0.667 | 0.000 | 0.000 | 0.000 | 0.000 |
| 7 | 1.000 | 0.667 | 0.333 | 0.000 | 0.000 | 0.000 | 0.000 |
| 8 | 0.000 | 0.000 | -0.333 | 0.000 | -0.167 | 0.000 | -0.167 |
| 9 | 0.000 | 1.000 | 0.000 | 0.000 | -0.250 | 0.000 | 0.000 |
| 10 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| ROW | SLK 6 | SLK 7 | SLK 8 | SLK 9 | SLK 10 | |
|---|---|---|---|---|---|---|
| 1 | 0.00E+00 | 67. | 0.00E+00 | 33. | 0.00E+00 | 0.32E+06 |
| 2 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 25.000 |
| 3 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 375.000 |
| 4 | 0.000 | 0.000 | 0.000 | -0.333 | 0.000 | 125.000 |
| 5 | 0.000 | 0.000 | 0.000 | -0.667 | 0.000 | 50.000 |
| 6 | 1.000 | -0.167 | 0.000 | 0.000 | 0.000 | 150.000 |
| 7 | 0.000 | 0.167 | 0.000 | 0.000 | 0.000 | 150.000 |
| 8 | 0.000 | -0.167 | 1.000 | 0.667 | 0.000 | 500.000 |
| 9 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 425.000 |
| 10 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 300.000 |

a. Another farmer whose farm adjoins Sod Farm #3 might be willing to sell Marky a portion of his water rights. How much should Marky offer, and for how many acre-feet?

b. What increase in the profit per acre for wheat is required in order for it to be profitable for Marky to plant any?

c. If Marky were to plant 100 acres of wheat on Farm #1, how should he best adjust the optimal plan above?

d. Is there another optimal basic solution, besides the one given above? If so, how does it differ from that given above?

# 56:171 Operations Research
## Homework #6 - Due Friday, October 11, 2002

**1.** (From *O.R. Applications & Algorithms*, by Wayne Winston)
During the month of July, Pittsburgh resident B. Fly must make four round-trip flights between Pittsburgh and Chicago. The dates of the trips are as shown below.

| Leave Pittsburgh | Leave Chicago |
|---|---|
| Monday, July 1 | Friday, July 5 |
| Tuesday, July 9 | Thursday, July 11 |
| Monday, July 15 | Friday, July 19 |
| Wednesday, July 24 | Thursday, July 25 |

B. Fly must purchase four round-trip tickets. Without a discounted fare, a round-trip ticket between Pittsburgh and Chicago costs $500. If Fly's stay in a city includes a weekend, he gets a 20% discount on the round-trip fare. If his stay in a city is at least 21 days, he receives a 35% discount, and if his stay is more than 10 days, he receives a 30% discount. Of course, only one discount can be applied toward the purchase of any ticket.
Formulate and solve an assignment problem that minimizes the total cost of purchasing the four round-trip tickets. (Hint: Let $X_{ij} = 1$ if a round-trip ticket is purchased for use on the ith flight out of Pittsburgh and the jth flight out of Chicago. Also think about where Fly should buy a ticket if, for example, $X_{21} = 1$.)

2. (From *Practical Management Science*, by W. Winston & S. C. Albright)
The city of Spring View is taking bids from six bus companies on the eight routes that must be driven in the surrounding school district. Each company enters a bid on how much it will charge to drive selected routes, although not all companies bid on all routes. (Blank cells in the table indicates routes on chich a company does not bid.)

| Company | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| A | | 8200 | 7800 | 5400 | | 3900 | | |
| B | 7800 | 8200 | | 6300 | | 3300 | 4900 | |
| C | | 4800 | | | | 4400 | 5600 | 3600 |
| D | | | 8000 | 5000 | 6800 | | 6700 | 4200 |
| E | 7200 | 6400 | | 3900 | 6400 | 2800 | | 3000 |
| F | 7000 | 5800 | 7500 | 4500 | 5600 | | 6000 | 4200 |

The city needs to select which companies to assign to which routes with the specifications that
(1) if a company does not bid on a route, it cannot be assigned to that route,
(2) exactly one company must be assigned to each route, and
(3) a company can be assigned to at most two routes.
The objective is to minimize the total cost of covering all routes. Formulate this as an assignment problem, and use the Hungarian method to solve it.

**3.** (Based on exercises 10.2-2, & 10.3-3, page 515f, *Introduction to O.R.,* Hillier & Lieberman (7th ed.))
Christine Phillips is in charge of planning and coordinating next spring's sales management training program for her company. Christine has listed the following activity information for this project. (Durations are given in weeks.)

| Activity | Description | Immediate predecessors | Expected Duration | Estimated variance |
|---|---|---|---|---|
| A | Select location | -- | 2 | 1 |
| B | Obtain keynote speaker | -- | 1 | 0.2 |
| C | Obtain other speakers | B | 2 | 1 |
| D | Make travel plans for keynote speaker | A,B | 2 | 1 |
| E | Make travel plans for other speakers | A, C | 3 | 1 |
| F | Make food arrangements | A | 2 | 1 |
| G | Negotiate hotel rates | A | 1 | 0.2 |
| H | Prepare brochure | C, G | 1 | 0.2 |
| I | Mail brochure | H | 1 | 0.2 |
| J | Take reservations | I | 3 | 1 |
| K | Prepare handouts | C, F | 4 | 4 |

a. Draw the AON (activity-on-node) project network.
b. Draw the AOA (activity-on-arrow) project network.
c. Label the nodes (events) of the AOA network so that if there is an arrow i→j then i<j.

In parts (d)-(h), assume that the expected durations are the actual durations.

d. Do a forward pass to compute the ET (earliest time) for each node.
e. What is the earliest completion time for this project?
f. Do a backward pass to compute the LT (latest time) for each node.
g. Complete the table below of ES (earliest start), EF (earliest finish), LS (latest start), LF (latest finish), and slack for each activity:

| Activity | Description | ES | EF | LS | LF | Slack |
|---|---|---|---|---|---|---|
| A | Select location | | | | | |
| B | Obtain keynote speaker | | | | | |
| C | Obtain other speakers | | | | | |
| D | Make travel plans for keynote speaker | | | | | |
| E | Make travel plans for other speakers | | | | | |
| F | Make food arrangements | | | | | |
| G | Negotiate hotel rates | | | | | |
| H | Prepare brochure | | | | | |
| I | Mail brochure | | | | | |
| J | Take reservations | | | | | |
| K | Prepare handouts | | | | | |

h. Which activities are critical?

Now assume that the durations are random with the variance shown above.

i. What is the expected completion time of the project?
j. What is the standard deviation of the completion time of the project?
k. Assuming that the project completion time has normal distribution, what is the probability that the project will be completed not later than two weeks after the expected completion time?

56:171 Operations Research
Homework #7 -- Due October 25, 2002

1. **Decision Analysis** (adapted from Exercise 15.2-7, page 784, *Operations Research*, 7th edition, by Hillier & Lieberman.)
Dwight Moody is the manager of a large farm with 1,000 acres of arable land. For greater efficiency, Dwight always devotes the farm to growing one crop at a time. He now needs to make a decision on which one of four crops to grow during the upcoming growing season. For each of these crops, Dwight has obtained the following estimates of crop yields and net incomes per bushel under various weather conditions.

| Weather | Crop 1 | Crop 2 | Crop 3 | Crop 4 |
|---|---|---|---|---|
| Dry | 20 | 15 | 30 | 40 |
| Moderate | 35 | 20 | 25 | 40 |
| Damp | 40 | 30 | 25 | 40 |
| Net income/bushel | $1.00 | $1.50 | $1.00 | $0.50 |

After referring to historical meteorological records, Dwight also estimated the following probabilities for the weather during the growing season:

| | |
|---|---|
| Dry | 0.3 |
| Moderate | 0.5 |
| Damp | 0.2 |

Using the criterion of "Maximize expected payoff", determine which crop to grow.

2. **Bayes' Rule** (Exercise 15.3-15, pp. 788-789, *Operations Research*, 7th edition, by Hillier & Lieberman)
There are two biased coins, coin A with probability of landing heads equal to 0.8 and the coin B with probability of heads equal to 0.4. One coin is chosen at random (each with probability 50%) to be tossed twice. You are to receive $100 if you correctly predict how many heads will occur in two tosses of this coin.
   a. Using the "Maximum Expected Payoff" criterion, what is the optimal prediction, and what is the corresponding expected payoff?

   Suppose now that you may observe a preliminary toss of the chosen coin before predicting.
   b. Determine your optimal prediction after observing a head in the preliminary toss.
   …after observing a tail in the preliminary toss.
   c. What is the expected value of the preliminary toss?

3. **Integer Programming Model** (based upon Case 12.3, page 649-653 of Operations Research, 7th edition, by Hillier & Lieberman. See the text for the complete case description. What follows is a condensed version.)
Brenda Sims, the saleswoman on the floor at Furniture City, understood that Furniture City required a new inventory policy. Not only was the megastore losing money by making customers unhappy with delivery delays, but it was also losing money by wasting warehouse space. By changing the inventory policy to stock only popular items and replenish them immediately when they are sold, Furniture City would ensure that the majority of customers receive their furniture immediately and that the valuable warehouse space was utilized effectively.

   She decided… to use her kitchen department as a model for the new inventory policy. She would identify all kitchen sets comprising 85% of customers orders. Given the fixed amount of warehouse space allocated to the kitchen department, she would identify the items Furniture City should stock in order to satisfy the greatest number of customer orders.

Brenda analyzed her records over the past three years and determined that 20 kitchen sets were responsible for 85% of customer orders. These 20 kitchen sets were composed of up to eight features, usually with four styles of each feature (except for the dishwashers, with two styles.)

- Floor tile: styles T1, T2, T3, T4
- Wallpaper: styles W1, W2, W3, W4
- Light fixtures: styles L1, L2, L3, L4
- Cabinets: styles C1, C2, C3, C4
- Countertops: styles O1, O2, O3, O4
- Dishwashers: styles D1, D2
- Sinks: styles S1, S2, S3, S4
- Ranges: styles R1, R2, R3, R4

(Sets, 14 through 20, however, do not include dishwashers.)

The warehouse could hold 50 $ft^2$ of tile and 12 rolls of wallpaper in the inventory bins. the inventory shelves could hold two light fixtures, two cabinets, three countertops, and two sinks. Dishwashers and ranges are similar in size, so Furniture City stored them in similar locations. The warehouse floor could hold a total of four dishwashers and ranges.

Every kitchen set includes exactly 20 $ft^2$ of tile and exactly 5 rolls of wallpaper. Therefore, 20 $ft^2$ of a particular style of tile and five rolls of a particular style of wallpaper are required for the styles to be in stock.

a. Formulate and use LINGO to solve a binary integer programming model which will maximize the total number of kitchen sets (and thus the number of customer orders) Furniture City stocks in the local warehouse. Assume that when a customer orders a kitchen set, all the particular items composing that kitchen set are replenished at the local warehouse immediately. (The sets and data section of a LINGO model may be downloaded with this homework assignment.)

b. How many of each feature and style should Furniture City stock in the local warehouse? How many different kitchen sets are in stock?

Furniture City decides to discontinue carrying nursery sets, and the warehouse space previously allocated to the nursery department is divided between the existing departments at Furniture City. The kitchen department receives enough additional space to allow it to stock both styles of dishwashers and three of the four styles of ranges.

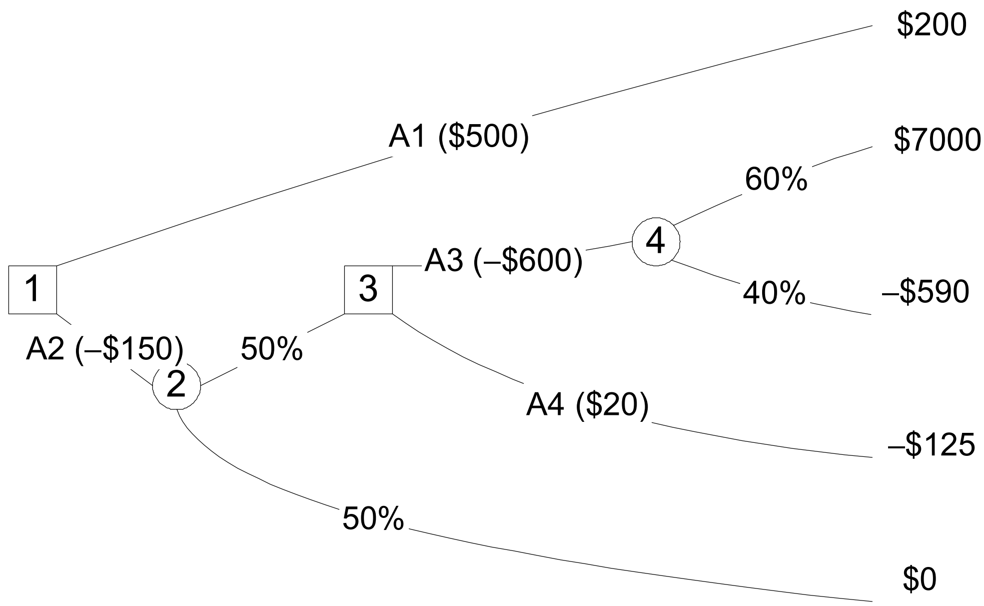
c. How does the optimal inventory policy for the kitchen department change with this additional warehouse space?

| | T1 | T2 | T3 | T4 | W1 | W2 | W3 | W4 | L1 | L2 | L3 | L4 | C1 | C2 | C3 | C4 | O1 | O2 | O3 | O4 | D1 | D2 | S1 | S2 | S3 | S4 | R1 | R2 | R3 | R4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Set 1 | | X | | | | X | | | | | | X | | X | | | | | | X | | X | | X | | | | X | | |
| Set 2 | | X | | | X | | | | X | | | | | | | X | | | | X | | X | | | | X | | X | | |
| Set 3 | X | | | | | | X | | | X | | | X | | | | X | | | | X | | | | X | | | | X | |
| Set 4 | | | X | | | | X | | | | X | | | | X | | | | X | | X | | X | | | | X | | | |
| Set 5 | | | | X | | | | X | X | | | | X | | | | | X | | | X | | | X | | | X | | | |
| Set 6 | | X | | | | X | | | | X | | | | | | X | | | | X | | X | | | X | | | | | X |
| Set 7 | X | | | | | | X | | | | | X | | | X | | | X | | | X | | X | | | | X | | | |
| Set 8 | | X | | | X | | | | | | X | | X | | | | X | | | | | X | | | X | | | | | X |
| Set 9 | | X | | | X | | | | | X | | | | | X | | | X | | | | X | | X | | | | X | | |
| Set10 | X | | | | X | | | | X | | | | X | | | | | | X | | X | | | | | X | | | X | |
| Set11 | | | X | | X | | | | | | X | | | | X | | X | | | | X | | X | | | | | | X | |
| Set12 | | X | | | | X | | | X | | | | | X | | | | X | | | | X | | | | X | | X | | |
| Set13 | | | | X | | | | X | | | X | | | | X | | X | | | | X | | | X | | | | | X | |
| Set14 | | | | X | | | | X | | | | X | X | | | | | | X | | | | X | | | | X | | | |
| Set15 | | | X | | | | X | | X | | | | X | | | | X | | | | | | | | X | | | | X | |
| Set16 | | | X | | | | X | | | | | X | X | | | | | | X | | | | | X | | | X | | | |
| Set17 | X | | | | | | | X | | X | | | | | X | | | | X | | | | | | | X | | | X | |
| Set18 | | X | | | | | X | | | | X | | | X | | | | | | X | | | X | | | | | X | | |
| Set19 | | X | | | | | | X | | | | X | | | | X | | | | X | | | | X | | | | | | X |
| Set20 | | X | | | | | X | | X | | | | X | | | | | X | | | | | | | X | | | | | X |

Table: Features composing each of twenty kitchen sets.

**4. Decision Trees.** Consider the decision tree below: On each decision branch, the immediate payoff (if +) or cost (if −) is shown. The probability is shown on each random branch. On the far right is the final payoff or cost.

a. "Fold back" the decision tree.
b. What is the *expected payoff* at node 1?
c. What is the *optimal decision* at node 1?



56:171 Operations Research
Homework #8 -- Due November 1, 2002

1. **Decision Analysis** (an exercise from *Operations Research: a Practical Introduction*, by M. Carter & C. Price)
Suppose that you are in the position of having to buy a used car, and you have narrowed down your choices to two possible models: one car is a private sale and the other is from a dealer. You must now choose between them. The cars are similar, and the only criterion is to minimize expected cost. The dealer car is more expensive, but it comes with a one-year warranty which would cover all costs of repairs. You decide that, if the car will last for 1 year, you can sell it again and recover a large part of your investment. If it falls apart, it will not be worth fixing. After test driving both cars and checking for obvious flaws, you make the following evaluation of probably resale value:

| Car | Purchase price | Probability of lasting one year | Estimated resale price |
|---|---|---|---|
| A: Private | $800 | 0.3 | $600 |
| B: Dealer | $1500 | 0.9 | $1000 |

a. Which car would you buy?

b. What is the expected value of perfect information (EVPI)?

Suppose you have the opportunity to take car A to an independent mechanic, who will charge you $50 to do a complete inspection and offer you an opinion as to whether the car will last 1 year. For various subjective reasons, you assign the following probabiliities to the accuracy of the mechanic's opinion:

| Given: | Mechanic says Yes | Mechanic says No |
|---|---|---|
| A car that will last 1 year | 70% | 30% |
| A car that will not last 1 year | 10% | 90% |

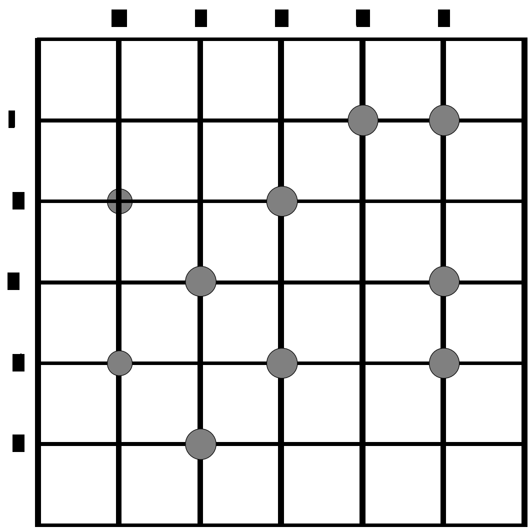
*(For example, if a car that will last 1 year is taken to the mechanic, there is 70% probability that he will give you the opinion that it will last a year.)*

c. Assuming that you must buy one of these two cars, formulate this problem as a decision tree problem.

d. What is the expected value of the mechanic's advice? Is it worth asking for the mechanic's opinion? What is you optimal decision strategy?

*Note: it is not necessary to ask for advice on car B because its problems could be repaired under the warranty!*

2. **Integer Programming** A convenience store chain is planning to enter a growing market and must determine where to open several new stores. The map shows the major streets in the area being considered. (Adjacent streets are 1 mile apart. A Avenue, B Avenue, etc. are N-S streets (with A Ave. being the westernmost) while 1st Street, 2nd Street, etc. are E-W streets (with 1st Street being the furthest north.) . The symbol ● indicates possible store locations. All travel must follow the street network, so distance is determined with a rectilinear metric. For instance, the distance between corners A1 and C2 is 3 miles.



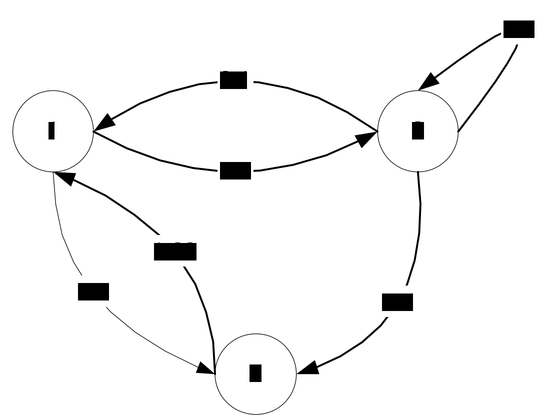
- The costs of purchasing property & constructing stores at the various locations are as follows:

| Location | A2 | A4 | B3 | B5 | C2 | C4 | D1 | E1 | E3 | E4 |
|---|---|---|---|---|---|---|---|---|---|---|
| Cost | 100 | 80 | 90 | 50 | 80 | 90 | 100 | 70 | 90 | 80 |

- No two stores can be on the same street (either north-south or east-west).
- Sttores must be <u>at least</u> 3 miles apart.
- Every grid point (A1, B2, etc.) must be <u>no more than</u> 3 miles from a store.

a. Set up an integer programming model that can be used to find the optimal store locations.
b. Find the optimal locations and the minimum cost..

3. **Discrete-time Markov Chains** A stochastic process with three states has the transition probabilities shown below:



a. Write the transition probability matrix P.

Suppose that the system begins in state 1, and is in state 3 after two steps.

b. What are the possible sequences of two transitions that might have occurred?

c. What are the probabilities of each of these sequences?

d. What is the probability $p_{13}^{(2)}$ ?

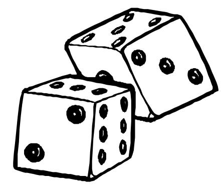
c. Write the equations which determine $\pi$, the steadystate probability distribution.

d. Compute the steadystate probability distribution $\pi$.

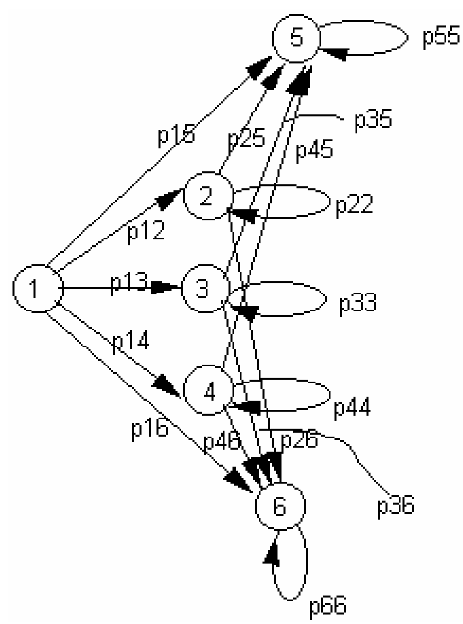
56:171 Operations Research
Homework #9 -- Due November 8, 2002

*Computation may be done with the MARKOV workspace of APL functions, or any other software that you may wish to use.*

1. ***Markov Chains***. In the game of "craps", one player rolls a pair of six-sided dice one or more times, until he or she either wins or loses. Suppose that we are the "roller":



- On the first throw, if we roll a 7 or an 11, we *win* right away.
- If we roll a 2, 3, or 12, we *lose* right away.
- If we first roll a 4, 5, 6, 8, 9, or 10, we keep rolling the dice until we get either a 7 or the total rolled on the first throw:
    - If we get a 7, we *lose*.
    - If we roll the same total as the first throw, we *win*.




Define the states of a Markov chain model as :
state 1 : start the game,
state 2 : roll a 4 or 10,
state 3 : roll a 5 or 9,
state 4 : roll a 6 or 8,
state 5 : win,
state 6 : loss.
(We have simplified the model by combining states: for example, whether a 4 or 10 is rolled, the probability of "making one's point" is the same.)

There are 6×6=36 different outcomes (equally likely) of throwing the dice:

```
+|1 2 3  4  5  6
1|2 3 4  5  6  7
2|3 4 5  6  7  8
3|4 5 6  7  8  9
4|5 6 7  8  9 10
5|6 7 8  9 10 11
6|7 8 9 10 11 12
```

a. Complete the table with the probability distribution of the outcome of a toss:

| Sum: | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Probability | | | | | | | | | | | |

The transition probability matrix is :

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| **1** | 0 | 6/36 | 8/36 | 10/36 | 8/36 | 4/36 |
| **2** | 0 | 27/36 | 0 | 0 | 3/36 | 6/36 |
| **3** | 0 | 0 | 26/36 | 0 | 4/36 | 6/36 |
| **4** | 0 | 0 | 0 | 25/36 | 5/36 | 6/36 |
| **5** | 0 | 0 | 0 | 0 | 1 | 0 |
| **6** | 0 | 0 | 0 | 0 | 0 | 1 |

b. Identify the following matrices:

**Q =**

| From\to | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |

**R=**

| From\to | 5 | 6 |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |

**E = (I–Q)$^{-1}$ = *Expected number of visits***

```
   \  1      2          3           4
1  |  1   0.6666667  0.8    0.90909091
2  |  0   4          0      0
3  |  0   0          3.6    0
4  |  0   0          0      3.2727273
```

**A = ER = *Absorption probabilities***

```
      \        5            6
  1   |  0.49292929  0.50707071
  2   |  0.33333333  0.66666667
  3   |  0.4         0.6
  4   |  0.45454545  0.54545455
```

c. What is the probability of winning for the roller (the person rolling the dice)?

d. What is the expected number of rolls of the dice in a crap game?

e. If the first roll is a "4", what is the roller's probability of winning?

f. If the first roll of the dice is a "4", how many additional rolls are expected before the game ends?

2. **Light Bulb Replacement** A bowling center has just purchased a new outdoor sign containing 1000 light bulbs. Based upon historical data, the manager has the following probability distribution of a bulb's failure. Failed bulbs are replaced monthly. The age of a bulb is its age at the beginning of a month, so when t=0, the bulb has just been placed in service and according to the table, has a 50% probability of failing during its first month of operation. (There is a high "infant mortality" rate.)

| Age t (months) | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Failure probability p(t) | 0.5 | 0.1 | 0.1 | 0.1 | 0.2 |
| Cumulative probability F(t) | 0.5 | 0.6 | 0.7 | 0.8 | 1.0 |
| Conditional probability of failure, $f(t\|n\geq t)$ | 0.5 | 0.2 | 0.25 | 0.333 | 1.0 |

For example, the probability that a bulb fails during the next month, if it has reached the age 2 months, is $f(2|n\geq 2) = 0.1/(0.1+0.1+0.2) = 0.25$.
We assume that a 4-month-old bulb is certain to fail during its next month of operation.

Define a discrete-time Markov chain model for one of positions in the sign,where the state t=0, 1, …4 is the age of the bulb in that position at the end of the month *after* all failed bulbs have been replaced.

a. Draw a transition diagram and write the transition probability matrix.

| From\to | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | | | | | |
| 1 | | | | | |
| 2 | | | | | |
| 3 | | | | | |
| 4 | | | | | |

b. What is the steadystate probability distribution?

| t | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\pi_t$ | | | | | |

c. According to this model, what is the expected number of replacements when the sign is examined at the end of each month?

d. If each bulb costs $2 to replace, what is the monthly replacement cost?

Because of the high "infant mortality" rate, the manager is considering "burn-in"—for $2.40 the manufacturer will burn them for one month at the plant. The more expensive bulb will have one less month of life, but the first failure-prone month is eliminated. She wants to know if the extra $0.40 is worth it.

e. Define a new Markov chain model with states t=1, 2, 3, 4, and write the transition probability matrix:

| From\to | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |

f. What is the steadystate distribution of the new Markov chain?

| t | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\pi_t$ | | | | |

g. What is the expected number of failed bulbs each month?

h. What is the expected monthly replacement cost for the sign?

**3.** Consider a **two-stage production flow line**. At each stage, an operation is performed on the part being processed. Parts are introduced into the system at stage 1. Processing is serial, and each stage can hold only one part at a time.

For purposes of analysis, we discretize time into 1-minute intervals. At the beginning of an interval, stage 1 is empty, working, or blocked whereas stage 2 is either working or empty. The system operates under the following rules:

- If stage 1 is empty at the beginning of an intgerval, a new part is introduced into stage 1 with probability 0.9. Work begins on the part; however, it cannot be completed during the minute it is introduced. If stage 1 is working or blocked, no part enters.

- If stage 2 is working at the beginning of an interval, the part will be completed and will leave the system with probability 0.8. Alternatively, it will remain in stage 2 with probability 0.2.
- If stage 1 is working at the beginning of an interval, the part will be completed with probability 0.6. A completed part will move to stage 2 if stage 2 is empty. Otherwise, it is blocked and remains in stage 1 until stage 2 becomes empty.

Develop a discrete-time Markov chain model of this situation, defining the states

1. (empty, idle)
2. (working, empty)
3. (working, working)
4. (empty, working)
5. (blocked, working)

a. Draw the transition diagram, and write the transition probability matrix.

| From\to | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | | | | | |
| 2 | | | | | |
| 3 | | | | | |
| 4 | | | | | |
| 5 | | | | | |

b. Compute the steadystate probability distribution of this process.

| i | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\pi_i$ | | | | | |

c. What is the expected idle time (empty or blocked) for each stage?

d. What is the expected throughput, i.e., # parts completed per hour?

---

Problems 2 & 3 are based upon exercises in the textbook *Operations Research Models & Methods*, by Paul Jensen & Jonathan Bard.

**56:171 Operations Research**
**Homework #10 –due November 15, 2002**

1. (#18.4-3, page 927, of *Intro to O.R.* by Hillier & Lieberman) The Garrett-Tompkins Company provides three copy machines in its copying room for the use of its employees. However, due to recent complaints about considerable time being wasted waiting for a copier to become free, management is considering adding one or more additional copy machines.

During the 2,000 working hours per year, employees arrive at the copying room according to a Poisson process at a mean rate of 30 per hour. The time each employee needs with a copy machine is believed to have an exponential distribution with a mean of 5 minutes. The lost productivity due to an employee spending time in the copying room is estimated to cost the company an average of $25 per hour. Each copy machine is leased for $3,000 per year. (Assume that the number of employees is so large that it could be assumed to be infinite.)

a. According to this model, what is currently the average number of employees waiting to use a copier?

b. What is currently the number of person-hours per year that is lost because of the waiting line?

c. Determine how many copy machines the company should have in order to minimize its expected total cost per hour.




2. (#17.6-14, page 898, of *Intro to O.R.* by Hillier & Lieberman) Airplanes arrive for takeoff at the runway of an airport according to a Poisson process at a mean rate of 20/hour (i.e., time between arrivals has exponential distribution with mean 1/20 hour = 3 minutes). The time required for an airplane to take off has an exponential distribution with a mean of 2 minutes, and this process must be completed before the next airplane can begin to take off.

Because a brief thunderstorm has just begun, all airplanes which have not commenced takeoff have just been grounded temporarily. However, airplanes continue to arrive at the runway during the thunderstorm to await its end.

Assuming steady-state operation before the thunderstorm, determine the expected number of airplanes that will be waiting to take off at the end of the thunderstorm if it lasts 30 minutes.

3. A production system consists of two machines which may operate simultaneously but independently. The machines fail randomly, with mean time to failure having exponential distribution with mean 15 hours. There is a single repairman whose reponsibility is to return the machines to operating condition. Most of the failures require only an average of 1 hour to be made operational again, but 20% of the failures are severe and require an average of 8 hours. Assume the repair times have exponential distributions.

a. List the states of a continuous-time Markov chain model of this system.

b. Draw the transition diagram, with transition rates.

c. Write down $\Lambda$, the transition rate matrix.

d. Write down one of the balance equations which describe $\pi$, the steady state distribution.

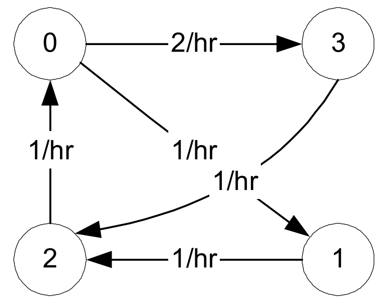
e. Compute the steady state distribution. (You may use the APL workspace CTMC, or solve the system of linear equations using other software.)

f. What is the utilization of the machines, i.e., the fraction of the time that a machine is operable?

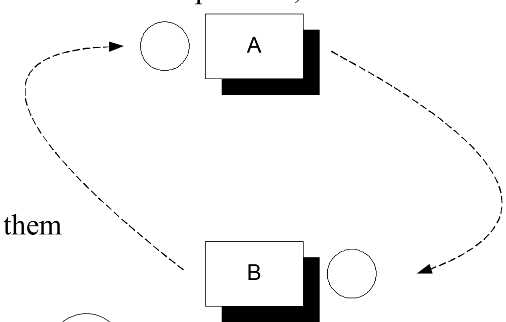
g. What is the utilization of the repairman, i.e., the fraction of the time that he is busy?

## 56:171 Operations Research
## Homework #11 –due November 22, 2002

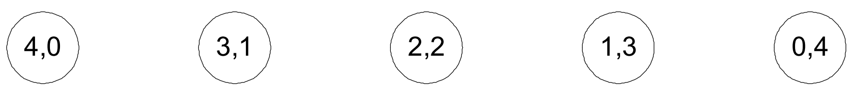
1. Consider the continuous-time Markov chain at the right.
   a. Write the matrix $\Lambda$ of transition rates.
   b. Write the system of linear equations which determine the steady-state probabilities.
   c. Solve the equations to obtain the steady-state distribution.




2. Four customers circulate between two single-server systems, i.e., all customers leaving server A enter the queue of server B, and vice versa. Server A works can serve 2 customers per hour, while server B works at half the rate of server A.
   Define the states of a continuous-time Markov chain model of this system to be
   (i,j) where i=# customers in queue or being served at A, and j=# customers in queue or being served at B.




   a. Indicate the possible transitions from each state, and label them with the transition rates:




   b. Is this a birth-death process?

   c. Compute the steady-state probability distribution to find the expected number of customers in subsystem A and subsystem B.

3. At a taxi stand, taxis looking for customers and customers looking for taxis arrive according to Poisson processes with rates $\lambda_t$ = 1/minute and $\lambda_c$ = 2/minute, respectively.
A taxi will always wait if no customers are at the stand. However, an arriving customer waits only if there are 3 or fewer customers already waiting.
Define a continuous-time Markov chain model with states:
   (−3): three customers waiting
   (−2): two customers waiting
   (−1): one customer waiting
   ( 0 ): neither customer nor taxi waiting
   (+1): one taxi waiting
   (+2): two taxis waiting
   (+3): three taxis waiting
   *etc.*
Draw the transition diagram.

b. Compute the steady-state probability distribution.

c. What fraction of the time will customers be waiting for taxis?

**4. Deterministic Dynamic Programming** Production is to be scheduled for the next 8 weeks. You are given the quantity required during each of those weeks:

| Week# | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Demand | 2 | 3 | 1 | 2 | 3 | 1 | 1 | 3 |

The production capacity limits the number produced during each week to no more than 4. There is a \$10 setup cost which is incurred during each week that production is scheduled, and a marginal cost of \$2 per unit produced for the first three units produced per week, and \$3 for the fourth unit.

The state of the system is the inventory level at the end of the week, after any production has occurred and demand has been satisfied. The maximum quantity which may be stored is 6, and a holding cost is incurred of \$1 per unit per week in storage for the first 4 units and \$2 each for the fifth and sixth unit.. Shortages are not allowed. At the end of the 8 week planning period, the item is obsolete and has a salvage value of \$2 per unit for 3 or fewer, and only \$1 per unit for any units in excess of 3.

In the tables below, the stages are defined in the natural way, i.e., stage 1 is the first week and stage 8 is the final week. The value "9999.99" in the table indicates an infeasible combination of state & decision variables.

a. Two values are missing in the tables (stages 1 & 2). Compute their values and insert them into the tables.

Suppose that initially you have 2 units in inventory.

b. Determine the number of units to be produced during each of the 8 weeks *(where week #1 is the first week, i.e., stage 1, etc.)*

| Week# | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Beginning Inventory | **2** | | | | | | | |
| Production Quantity | | | | | | | | |

c. What is the minimum total cost (production cost + storage cost – salvage value) for the 8-week period?

---Stage 8---

| s \ x: | 0 | 1 | 2 | 3 | 4 | Minimum |
|---|---|---|---|---|---|---|
| 0 | 9999.99 | 9999.99 | 9999.99 | 16.00 | 17.00 | 16.00 |
| 1 | 9999.99 | 9999.99 | 15.00 | 15.00 | 16.00 | 15.00 |
| 2 | 9999.99 | 14.00 | 14.00 | 14.00 | 15.00 | 14.00 |
| 3 | 3.00 | 13.00 | 13.00 | 13.00 | 15.00 | 3.00 |
| 4 | 2.00 | 12.00 | 12.00 | 13.00 | 15.00 | 2.00 |
| 5 | 2.00 | 12.00 | 13.00 | 14.00 | 16.00 | 2.00 |
| 6 | 2.00 | 13.00 | 14.00 | 15.00 | 9999.99 | 2.00 |

---Stage 7---

| s \ x: | 0 | 1 | 2 | 3 | 4 | Minimum |
|---|---|---|---|---|---|---|
| 0 | 9999.99 | 28.00 | 29.00 | 30.00 | 22.00 | 22.00 |
| 1 | 17.00 | 28.00 | 29.00 | 20.00 | 22.00 | 17.00 |
| 2 | 17.00 | 28.00 | 19.00 | 20.00 | 23.00 | 17.00 |
| 3 | 17.00 | 18.00 | 19.00 | 21.00 | 24.00 | 17.00 |
| 4 | 7.00 | 18.00 | 20.00 | 22.00 | 9999.99 | 7.00 |
| 5 | 8.00 | 20.00 | 22.00 | 9999.99 | 9999.99 | 8.00 |
| 6 | 10.00 | 22.00 | 9999.99 | 9999.99 | 9999.99 | 10.00 |

---Stage 6---

| s \ x: | 0 | 1 | 2 | 3 | 4 | Minimum |
|---|---|---|---|---|---|---|
| 0 | 9999.99 | 34.00 | 31.00 | 33.00 | 36.00 | 31.00 |
| 1 | 23.00 | 30.00 | 32.00 | 34.00 | 27.00 | 23.00 |
| 2 | 19.00 | 31.00 | 33.00 | 25.00 | 29.00 | 19.00 |
| 3 | 20.00 | 32.00 | 24.00 | 27.00 | 32.00 | 20.00 |
| 4 | 21.00 | 23.00 | 26.00 | 30.00 | 9999.99 | 21.00 |
| 5 | 13.00 | 26.00 | 30.00 | 9999.99 | 9999.99 | 13.00 |
| 6 | 16.00 | 30.00 | 9999.99 | 9999.99 | 9999.99 | 16.00 |

---Stage 5---

| s \ x: | 0 | 1 | 2 | 3 | 4 | Minimum |
|---|---|---|---|---|---|---|
| 0 | 9999.99 | 9999.99 | 9999.99 | 47.00 | 42.00 | 42.00 |
| 1 | 9999.99 | 9999.99 | 46.00 | 40.00 | 39.00 | 39.00 |
| 2 | 9999.99 | 45.00 | 39.00 | 37.00 | 41.00 | 37.00 |
| 3 | 34.00 | 38.00 | 36.00 | 39.00 | 43.00 | 34.00 |
| 4 | 27.00 | 35.00 | 38.00 | 41.00 | 36.00 | 27.00 |
| 5 | 25.00 | 38.00 | 41.00 | 35.00 | 41.00 | 25.00 |
| 6 | 28.00 | 41.00 | 35.00 | 40.00 | 9999.99 | 28.00 |

---Stage 4---

| s \ x: | 0 | 1 | 2 | 3 | 4 | Minimum |
|---|---|---|---|---|---|---|
| 0 | 9999.99 | 9999.99 | 56.00 | 55.00 | 56.00 | 55.00 |
| 1 | 9999.99 | 55.00 | 54.00 | 54.00 | 54.00 | 54.00 |
| 2 | 44.00 | 53.00 | 53.00 | 52.00 | 48.00 | 44.00 |
| 3 | 42.00 | 52.00 | 51.00 | 46.00 | 47.00 | 42.00 |
| 4 | 41.00 | 50.00 | 45.00 | 45.00 | 51.00 | 41.00 |
| 5 | 40.00 | 45.00 | 45.00 | 50.00 | 9999.99 | 40.00 |
| 6 | 35.00 | 45.00 | 50.00 | 9999.99 | 9999.99 | 35.00 |

---Stage 3---

| s \ x: | 0 | 1 | 2 | 3 | 4 | Minimum |
|---|---|---|---|---|---|---|
| 0 | 9999.99 | 67.00 | 68.00 | 60.00 | 61.00 | 60.00 |
| 1 | 56.00 | 67.00 | 59.00 | 59.00 | 61.00 | 56.00 |
| 2 | 56.00 | 58.00 | 58.00 | 59.00 | 61.00 | 56.00 |
| 3 | 47.00 | 57.00 | 58.00 | 59.00 | 57.00 | 47.00 |
| 4 | 46.00 | 57.00 | 58.00 | 55.00 | 9999.99 | 46.00 |
| 5 | 47.00 | 58.00 | 55.00 | 9999.99 | 9999.99 | 47.00 |
| 6 | 48.00 | 55.00 | 9999.99 | 9999.99 | 9999.99 | 48.00 |

---Stage 2---

| s \ x: | 0 | 1 | 2 | 3 | 4 | Minimum |
|---|---|---|---|---|---|---|
| 0 | 9999.99 | 9999.99 | 9999.99 | 76.00 | 75.00 | 75.00 |
| 1 | 9999.99 | 9999.99 | 75.00 | 73.00 | 76.00 | 73.00 |
| 2 | 9999.99 | 74.00 | 72.00 | 74.00 | 68.00 | 68.00 |
| 3 | 63.00 | 71.00 | 73.00 | 66.00 | 68.00 | 63.00 |
| 4 | 60.00 | 72.00 | 65.00 | 66.00 | 70.00 | 60.00 |
| 5 | 62.00 | 65.00 | 66.00 | 69.00 | 73.00 | |
| 6 | 55.00 | 66.00 | 69.00 | 72.00 | 9999.99 | 55.00 |

---Stage 1---

| s \ x: | 0 | 1 | 2 | 3 | 4 | Minimum |
|---|---|---|---|---|---|---|
| 0 | 9999.99 | 9999.99 | 89.00 | 89.00 | 87.00 | 87.00 |
| 1 | 9999.99 | 88.00 | 88.00 | 85.00 | 83.00 | 83.00 |
| 2 | 77.00 | 87.00 | | 81.00 | 81.00 | 77.00 |
| 3 | 76.00 | 83.00 | 80.00 | 79.00 | 84.00 | 76.00 |
| 4 | 72.00 | 79.00 | 78.00 | 82.00 | 78.00 | 72.00 |
| 5 | 69.00 | 78.00 | 82.00 | 77.00 | 9999.99 | 69.00 |
| 6 | 68.00 | 82.00 | 77.00 | 9999.99 | 9999.99 | 68.00 |

**56:171 Operations Research**
**Homework #12 --Due Friday, 6 December 2002**

**1.** ***Power Plant Construction.*** The construction of eight power plants is to be scheduled over an eight-year period, with a restriction that **no more than three** may be built during each one-year period. The cost per plant is 125M$. In addition, a cost of 100M$ is incurred in each year that construction is scheduled. (Assume for sake of simplicity that completion of the plant occurs

| Year | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| # plants required at end of year | 2 | 3 | 3 | 4 | 5 | 5 | 7 | 8 |

The table above gives the *cumulative* # of plants required by the end of each year, i.e., a total of 8 additional plants are required by the end of year #8.

The utility company wants to minimize the present value of the construction costs, using an internal rate-of-return of 20%.

A deterministic dynamic programming model was developed, with stages = years (stage 1 = year 1, etc.), state = cumulative # of plants built, and decision = # plants to be added in each stage. The tables below show the tables of computations of the optimal schedule:

---Stage 8---

| s \ x: | 0 | 1 | Minimum |
|---|---|---|---|
| 7 | 99999.999 | 225.000 | 225.000 |
| 8 | 0.000 | 99999.999 | 0.000 |

---Stage 7---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| 5 | 99999.999 | 99999.999 | 537.500 | 475.000 | 475.000 |
| 6 | 99999.999 | 412.500 | 350.000 | 99999.999 | 350.000 |
| 7 | 187.500 | 225.000 | 99999.999 | 99999.999 | 187.500 |
| 8 | 0.000 | 99999.999 | 99999.999 | 99999.999 | 0.000 |

---Stage 6---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| 5 | 395.833 | 516.667 | 506.250 | 475.000 | 395.833 |
| 6 | 291.667 | _______ | 350.000 | 99999.999 | 291.667 |
| 7 | 156.250 | 225.000 | 99999.999 | 99999.999 | 156.250 |
| 8 | 0.000 | 99999.999 | 99999.999 | 99999.999 | 0.000 |

---Stage 5---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| 4 | 99999.999 | 554.861 | 593.056 | 605.208 | 554.861 |
| 5 | 329.861 | 468.056 | 480.208 | 475.000 | 329.861 |
| 6 | 243.056 | 355.208 | 350.000 | 99999.999 | 243.056 |
| 7 | 130.208 | 225.000 | 99999.999 | 99999.999 | 130.208 |
| 8 | 0.000 | 99999.999 | 99999.999 | 99999.999 | 0.000 |

---Stage 4---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| 3 | 99999.999 | 687.384 | 624.884 | 677.546 | 624.884 |
| 4 | 462.384 | 499.884 | 552.546 | 583.507 | 462.384 |
| 5 | 274.884 | 427.546 | 458.507 | 475.000 | 274.884 |
| 6 | 202.546 | 333.507 | 350.000 | 99999.999 | 202.546 |
| 7 | 108.507 | 225.000 | 99999.999 | 99999.999 | 108.507 |
| 8 | 0.000 | 99999.999 | 99999.999 | 99999.999 | 0.000 |

---Stage 3---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| 3 | 520.737 | 610.320 | 579.070 | 643.789 | 520.737 |
| 4 | 385.320 | 454.070 | 518.789 | 565.422 | 385.320 |
| 5 | 229.070 | 393.789 | 440.422 | 475.000 | 229.070 |
| 6 | 168.789 | 315.422 | 350.000 | 99999.999 | 168.789 |
| 7 | 90.422 | 225.000 | 99999.999 | 99999.999 | 90.422 |
| 8 | 0.000 | 99999.999 | 99999.999 | 99999.999 | 0.000 |

---Stage 2---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| 2 | 99999.999 | 658.947 | 671.100 | _______ | 658.947 |
| 3 | 433.947 | 546.100 | 540.892 | 615.657 | 433.947 |
| 4 | 321.100 | 415.892 | 490.657 | 550.352 | 321.100 |
| 5 | 190.892 | 365.657 | 425.352 | 475.000 | 190.892 |
| 6 | 140.657 | 300.352 | 350.000 | 99999.999 | 140.657 |
| 7 | 75.352 | 225.000 | 99999.999 | 99999.999 | 75.352 |
| 8 | 0.000 | 99999.999 | 99999.999 | 99999.999 | 0.000 |

---Stage 1---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| 0 | 99999.999 | 99999.999 | 899.123 | 836.623 | 836.623 |
| 1 | 99999.999 | 774.123 | 711.623 | 742.583 | 711.623 |
| 2 | 549.123 | 586.623 | 617.583 | 634.077 | 549.123 |
| 3 | 361.623 | 492.583 | 509.077 | 592.214 | 361.623 |
| 4 | 267.583 | 384.077 | 467.214 | 537.793 | 267.583 |
| 5 | 159.077 | 342.214 | 412.793 | 475.000 | 159.077 |
| 6 | 117.214 | 287.793 | 350.000 | 99999.999 | 117.214 |
| 7 | 62.793 | 225.000 | 99999.999 | 99999.999 | 62.793 |
| 8 | 0.000 | 99999.999 | 99999.999 | 99999.999 | 0.000 |

a. What are the two values which are missing in the tables above, for stages 7 and 2?

b. What is the present value of the construction costs of the optimal schedule?

c. What is the optimal construction schedule?

| **Year** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** |
|---|---|---|---|---|---|---|---|---|
| **# plants required at end of year** | **2** | **3** | **3** | **4** | **5** | **5** | **7** | **8** |
| **# plants to be constructed** | __ | __ | __ | __ | __ | __ | __ | __ |
| **cumulative # of plants added** | | | | | | | | |

❑ ❑ ❑ ❑ ❑ ❑ ❑ ❑ ❑ ❑

**2. *System Reliability*** A system is to be composed of five unreliable components with unit weights and reliabilities as shown in the table below.
The designer wishes to select the level of redundancy to maximize the system reliability, subject to a restriction that the total weight of the system cannot be more than 15 kg. (Note that the system requires *at least one* unit of each component for successful operation.)

| Component | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Weight (kg) | 1 | 2 | 3 | 1 | 2 |
| Reliability (%) | 80 | 85 | 90 | 75 | 70 |

A deterministic dynamic programming model was developed, in which the components are assumed to be considered one at a time, beginning with component 5. The optimal value $f_n(s_n)$ is defined to be the maximum reliability which can be achieved for the subsystem of components n, n-1, …1 if $s_n$ kilograms can be allocated to these n components. The tables below show the computation of the optimal design. (The value –99.9999 in the table indicates an infeasible combination of state and decision.)

---Stage 1---

| s \ x: | 1 | 2 | 3 | Maximum |
|---|---|---|---|---|
| 1 | 0.8000 | -99.9999 | -99.9999 | 0.8000 |
| 2 | 0.8000 | 0.9600 | -99.9999 | 0.9600 |
| 3 | 0.8000 | 0.9600 | 0.9920 | 0.9920 |

---Stage 2---

| s \ x: | 1 | 2 | 3 | Maximum |
|---|---|---|---|---|
| 3 | 0.6800 | -99.9999 | -99.9999 | 0.6800 |
| 4 | 0.8160 | -99.9999 | -99.9999 | 0.8160 |
| 5 | 0.8432 | 0.7820 | -99.9999 | 0.8432 |
| 6 | 0.8432 | 0.9384 | -99.9999 | 0.9384 |
| 7 | 0.8432 | 0.9697 | 0.7973 | 0.9697 |
| 8 | 0.8432 | 0.9697 | ______ | 0.9697 |
| 9 | 0.8432 | 0.9697 | 0.9887 | 0.9887 |

---Stage 3---

| s \ x: | 1 | 2 | 3 | Maximum |
|---|---|---|---|---|
| 6 | 0.6120 | -99.9999 | -99.9999 | 0.6120 |
| 7 | 0.7344 | -99.9999 | -99.9999 | 0.7344 |
| 8 | 0.7589 | -99.9999 | -99.9999 | 0.7589 |
| 9 | 0.8446 | 0.6732 | -99.9999 | 0.8446 |
| 10 | 0.8727 | 0.8078 | -99.9999 | 0.8727 |
| 11 | 0.8727 | 0.8348 | -99.9999 | 0.8727 |
| 12 | 0.8898 | 0.9290 | 0.6793 | 0.9290 |

---Stage 4---

| s \ x: | 1 | 2 | 3 | Maximum |
|---|---|---|---|---|
| 7 | 0.4590 | -99.9999 | -99.9999 | 0.4590 |
| 8 | 0.5508 | 0.5738 | -99.9999 | 0.5738 |
| 9 | 0.5692 | 0.6885 | 0.6024 | 0.6885 |
| 10 | 0.6334 | 0.7115 | 0.7229 | 0.7229 |
| 11 | 0.6545 | 0.7918 | 0.7470 | 0.7918 |
| 12 | 0.6545 | 0.8182 | 0.8314 | 0.8314 |
| 13 | 0.6968 | 0.8182 | 0.8591 | 0.8591 |

---Stage 5---

| s \ x: | 1 | 2 | 3 | Maximum |
|---|---|---|---|---|
| 9 | 0.3213 | -99.9999 | -99.9999 | 0.3213 |
| 10 | 0.4016 | -99.9999 | -99.9999 | 0.4016 |
| 11 | 0.4820 | 0.4177 | -99.9999 | 0.4820 |
| 12 | 0.5060 | 0.5221 | -99.9999 | 0.5221 |
| 13 | ______ | 0.6265 | 0.4466 | 0.6265 |
| 14 | 0.5820 | 0.6579 | 0.5583 | 0.6579 |
| 15 | 0.6014 | 0.7205 | 0.6699 | 0.7205 |

a. What is the reliability of the system if only a single unit of each component were included?

b. Compute the two missing values in the tables above (in stages 2 & 5).

c. What is the maximum reliability of a system weighing 15 kg?

d. What is the optimal number of units of each component?

| Component | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| # units | | | | | |

❑ ❑ ❑ ❑ ❑ ❑ ❑ ❑ ❑ ❑

**3. *Stochastic Production Planning*** An optimal production policy is required for a product with random daily demand:

| Demand d | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Probability{d} | 20% | 30% | 30% | 20% |

The state of the system is the "inventory position" at the end of the day, which if positive is the stock on hand, and if negative is the number of backorders which have accumulated. A maximum of 1 backorder is allowed, and a maximum of 4 units may be held in storage at the end of each day. (For simplicity, assume that any inventory in excess of 4 units is discarded.) There is a storage cost of $1 per unit, based upon the end-of-day inventory, and a shortage cost of $15 per unit, based upon any backorders at the end of the day.

The next day's production must be scheduled after the inventory position is determined. Assume that production on the next day is completed in time to meet any demand that occurs that same day. The maximum number of units which can be produced each day is 3. If production is scheduled, there is a setup cost of $10, plus a $3 marginal cost per unit.

Finally, at the end of the planning period (6 days), a salvage value of $2 per unit is received for any remaining inventory.

A backward recursion is used, where the stage **n** is the number of days remaining in the planning period, the state $s_n$ is the inventory position (–1 = "1 backorder", +3 = "three units in stock", etc.), and optimal value function $f_n(s_n)$ = minimum expected cost for stages n, n-1, ...2, 1 if the inventory position at stage n is $s_n$.

---Stage 1---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| ¯1 | 99999.999 | 50.400 | 44.600 | 38.200 | 38.200 |
| 0 | 22.400 | 26.600 | 20.200 | 16.000 | 16.000 |
| 1 | 14.600 | 18.200 | 14.000 | 15.000 | 14.000 |
| 2 | 6.200 | 12.000 | 13.000 | 14.400 | 6.200 |
| 3 | 0.000 | 11.000 | 12.400 | 14.400 | 0.000 |
| 4 | ¯1.000 | 10.400 | 12.400 | 15.000 | ¯1.000 |

---Stage 2---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| ¯1 | 99999.999 | 61.760 | 57.700 | 51.880 | 51.880 |
| 0 | 33.760 | 39.700 | 33.880 | 28.260 | 28.260 |
| 1 | 27.700 | 31.880 | 26.260 | 24.460 | 24.460 |
| 2 | 19.880 | 24.260 | 22.460 | 21.740 | 19.880 |
| 3 | 12.260 | 20.460 | 19.740 | 21.200 | 12.260 |
| 4 | 8.460 | 17.740 | 19.200 | 22.000 | 8.460 |

---Stage 3---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| ¯1 | 99999.999 | 75.156 | 70.310 | 64.168 | 64.168 |
| 0 | 47.156 | 52.310 | 46.168 | 40.406 | 40.406 |
| 1 | 40.310 | 44.168 | 38.406 | 36.226 | 36.226 |
| 2 | 32.168 | 36.406 | 34.226 | 32.884 | 32.168 |
| 3 | 24.406 | 32.226 | 30.884 | 31.220 | 24.406 |
| 4 | 20.226 | 28.884 | 29.220 | 31.460 | 20.226 |

---Stage 4---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| ¯1 | 99999.999 | 87.416 | 82.451 | 76.257 | 76.257 |
| 0 | 59.416 | 64.451 | 58.257 | 52.481 | 52.481 |
| 1 | 52.451 | 56.257 | 50.481 | 48.263 | 48.263 |
| 2 | 44.257 | 48.481 | 46.263 | 44.868 | 44.257 |
| 3 | 36.481 | 44.263 | 42.868 | 43.062 | 36.481 |
| 4 | 32.263 | 40.868 | 41.062 | 43.226 | 32.263 |

---Stage 5---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| ¯1 | 99999.999 | 99.502 | 94.525 | 88.326 | 88.326 |

| | | | | | |
|---|---|---|---|---|---|
| 0 | 71.502 | 76.525 | 70.326 | 64.548 | 64.548 |
| 1 | 64.525 | 68.326 | 62.548 | 60.326 | 60.326 |
| 2 | 56.326 | 60.548 | 58.326 | 56.927 | 56.326 |
| 3 | 48.548 | 56.326 | 54.927 | 55.106 | 48.548 |
| 4 | 44.326 | 52.927 | 53.106 | 55.263 | 44.326 |

---Stage 6---

| s \ x: | 0 | 1 | 2 | 3 | Minimum |
|---|---|---|---|---|---|
| -1 | 99999.999 | 111.570 | 106.593 | 100.393 | 100.393 |
| 0 | 83.570 | 88.593 | 82.393 | 76.615 | 76.615 |
| 1 | 76.593 | 80.393 | ______ | 72.393 | 72.393 |
| 2 | 68.393 | 72.615 | 70.393 | 68.993 | 68.393 |
| 3 | 60.615 | 68.393 | 66.993 | 67.171 | 60.615 |
| 4 | 56.393 | 64.993 | 65.171 | 67.326 | 56.393 |

a. Compute the missing value in the table for stage 6 (the first day of the planning period).

Suppose that initially there is one unit in stock.

b. What is the expected total cost of production, storage, & shortages during the six days, minus salvage value received for any final inventory?

c. What is the optimal production quantity for the first day (stage 6)?

d. What is the optimal production policy for each of the first five days?

| If inventory position is… | Produce… |
|---|---|
| -1 | |
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |

e. What is the optimal production policy for the final day (stage 1)?

| If inventory position is… | Produce… |
|---|---|
| -1 | |
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |

❑ ❑ ❑ ❑ ❑ ❑ ❑ ❑ ❑ ❑

**4.** ***Equipment Replacement*** A machine initially costs $5000. The annual operating cost increases with the age of the machine, while the trade-in value decreases, according to the following table (assume that management policy is to keep no machine more than four years) :

| **Year of operation** | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| **Operating cost($)** | 1000 | 1200 | 1500 | 1800 |
| **Trade-in value at end of year ($)** | 3500 | 3000 | 2200 | 1500 |

Assume an interest rate of 12%, that initially you have a new machine, and that the current machine at the end of eight years will be traded in, but not replaced.

Determine the replacement strategy for the next 8 years which will minimize the present value of replacement & operating costs, minus trade-in value.

What is the present value of the costs for the eight-year period