56:171
Operations Research
Fall 1998

Quiz Solutions

Dept of Mechanical & Industrial Engineering
University of Iowa

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<><><><><><><><><><> Quiz #1 Solutions - September 9, 1998 <><><><><><><><><><>

Part I: For each statement, indicate "+"=**true** or "o"=**false**.

Morning section:

__o_ a. After the LP model has been entered, the **SOLVE** command finds the optimal solution.

__+_ b. In the simplex method (as described in the lectures, not the textbook), the quantity $-Z$ serves as a basic variable, where Z is the value of the objective function.

__o_ c. To begin your entry of the LP model into LINDO, you should use the command "**ENTER**".

__+_ d. In a basic LP solution, the nonbasic variables equal zero.

__o_ e. If a slack variable S_i for row i is basic in the optimal solution, then variable X_i cannot be basic.

__o_ f. The "minimum ratio test" is used to select the pivot column in the simplex method for linear programming.

__o_ g. LINDO does not accept "strict inequalities" such as $X + 2Y < 10$, so that you must enter $X + 2Y <= 10$ instead.

__+_ h. When entering your LP model, the last constraint which you enter should be followed by "**END**".

__o_ i. The **PRINT** command may be used to print any LINDO output which has previously appeared on the monitor.

__+_ j. LINDO assumes that all variables are restricted to be nonnegative, so that you need not explicitly enter these constraints.

__o_ k. After you have entered the objective function, you must enter "**SUBJECT TO**" before entering the first constraint.

__o_ l. When you enter an LP formulation into LINDO, you must first convert all inequalities to equations.

__+_ m. When you enter an LP formulation into LINDO, you must manipulate your constraints so that all variables appear on the left, and all constants on the right.

Afternoon section:

__+_ a. . A "pivot" in a nonbasic column of a tableau will make it a basic column.

__o_ b. If a zero appears on the right-hand-side of row i of an LP tableau, then at the next iteration you *cannot* pivot in row i.

__+_ c. In the simplex method, every variable of the LP is either basic or nonbasic.

__+_ d. The number of *rows* in the LP model is always greater than the number of constraints which you have entered.

__o_ e. The restriction that X1 be nonnegative should be entered into LINDO as the constraint X1 >= 0.

__o_ f. If a zero appears on the right-hand-side of row i of an LP tableau, then at the next iteration you *must* pivot in row i.

__+_ g. A "pivot" in the simplex method corresponds to a move from one corner point of the feasible region to another.

__o_ h. The feasible region is the set of all points that satisfy *at least* one constraint.

__o_ i. Adding constraints to an LP may improve the optimal objective function value.

__+_ j. The number of basic variables in an LP is equal to the number of rows, *including* the objective function row.

__o_ k. A "pivot" in row i of the column for variable X_j will increase the number of basic variables.

__+_ l. Basic solutions of an LP with constraints $Ax \leq b$, $x \geq 0$ correspond to "corner" points of the feasible region.

__+_ m. In the simplex tableau, the objective row is written in the form of an equation.

Multiple-Choice:

__c_n. If you make a mistake in choosing the pivot column in the simplex method, the solution in the next tableau

(a) will be nonbasic (c) will have a worse objective value
(b) will be infeasible (d) *None of the above*

__b_o. If you make a mistake in choosing the pivot row in the simplex method, the solution in the next tableau

(a) will be nonbasic (c) will have a worse objective value
(b) will be infeasible (d) *None of the above*

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Part II: Investment problem

Consider the following investment problem: You now have $100 available for investment (beginning of year #1). Your objective is to maximize the value of this initial investment after four years, i.e., the end of year #4 or equivalently, the beginning of year #5. The available investments are:

- Investment **A** is available <u>only at the beginning of years 1 and 2</u>; each $1 invested in A will be returned in two equal payments of $0.70 at the beginning of each of the following 2 years. (For example, if you invest $1 now, at the beginning of year 1, then you receive $0.70 at the beginning of year 2 <u>and</u> another $0.70 at the beginning of year 3.)
- Investment **B** is available only once, <u>at the beginning of year 2</u>; each $1 invested in B at the beginning of year 2 returns $2 after 3 years, i.e., the beginning of year 5.
- A Money Market fund (**R**) is <u>available every year</u>; each $1 invested in this way will return $1.10 after 1 year.

The following table displays these cash flows. For example, -1 indicates $1 put into the investment, and +0.70 indicates $0.70 received from the investment.

begin year #	A1	A2	B	R1	R2	R3	R4
1	-1			-1			
2	+0.7	-1	-1	+1.1	-1		
3	+0.7	+0.7			+1.1	-1	
4		+0.7				+1.1	-1
5			+2				+1.1

p. Complete the equation: $0.7A2 + 1.1R3 - R4 =$ ____0______

q. The objective should be to maximize *(select one):*

- ___ 1.4A2 + 2B + 1.1R4
- _X_ 2B + 1.1R4
- ___ 1.4A1 + 1.4A2 + 2B + 1.1R1 + 1.1R2 + 1.1R3 + 1.1R4
- ___ 0.4A2 + 1B + 0.1R4
- ___ none of the above

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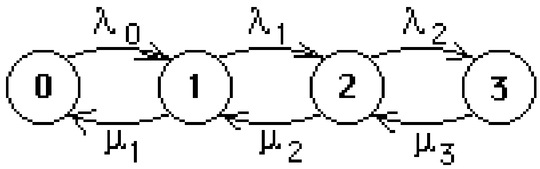
A machine operator has sole responsibility for keeping three semi-automatic machines busy. The time required to ready a machine (unloading & reloading) has exponential distribution with mean 15 minutes. The machine will then run unattended for an average of 1 hour (but with actual time having exponential distribution) before it requires the operator's attention again.



Note: The state of the system (population of the birth/death process) is the number of machines which are either being serviced or waiting to be serviced.)

_b,c,g_ 1. The Markov chain model diagrammed above is *(select one or more)*:

- a. a discrete-time Markov chain
- b. a continuous-time Markov chain
- c. a Birth-Death process
- d. an M/M/1 queue
- e. an M/M/3 queue
- f. an M/M/1/3 queue
- g. an M/M/1/3/3 queue
- h. a Poisson process

Note: This is a queue with the M/M/1/3/3 classification, i.e., 1 server, maximum of 3 in the "system", and 3 in the "source population".

_a_ 2. The value of λ_2 is

- a. 1/hr.
- b. 2/hr.
- c. 3/hr.
- d. 4/hr.
- e. 0.5/hr.
- f. *none of the above*

Note: In state 2, 2 machines are shut down and one is processing a job. Therefore, the "arrival" rate, i.e., the rate at which machines are requiring service, is 1 machine per hour.

_f_ 3. The value of μ_2 is

- a. 1/hr.
- b. 2/hr.
- c. 3/hr.
- d. 0.5/hr.
- e. 0.5/hr.
- f. *none of the above*

Note: the service rate, independently of the number of "customers in the system", is 1 "customer"/15 minutes = 4 / hr.

_c_ 4. The value of λ_0 is

- a. 1/hr.
- b. 2/hr.
- c. 3/hr.
- d. 0.5/hr.
- e. 0.5/hr.
- f. *none of the above*

Note: each machine "arrives" at the rate of once per hour, so when three machines are running, they jointly complete their jobs at the rate of 3/hr.

_b_ 5. The steady-state probability π_0 is computed by solving

- a. $\frac{1}{\pi_0} = 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 \approx \frac{1}{0.366}$
- b. $\boxed{\frac{1}{\pi_0} = 1 + \frac{3}{4} + \frac{1}{2}\times\frac{3}{4} + \frac{1}{4}\times\frac{1}{2}\times\frac{3}{4} \approx \frac{1}{0.451}}$
- c. $\frac{1}{\pi_0} = 1 + \frac{3}{4} + \frac{1}{2} + \frac{1}{4} \approx \frac{1}{0.4}$
- d. $\frac{1}{\pi_0} = 1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 \approx \frac{1}{0.753}$
- e. $\frac{1}{\pi_0} = 1 + \frac{3}{4} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^3 \approx \frac{1}{0.496}$
- f. *none of the above*

c (or d) 6. The operator will be busy what fraction of the time? *(choose nearest value)*

- a. 40%
- b. 45%
- c. 50%
- d. 60%
- e. 65%
- f. 75%

Note: $\pi_0 = 0.451$ *is the probability that there are no machines shut down, i.e., the fraction of the time that the operator is idle. The fraction of the time that he is busy is therefore 1-0.451 = 0.549.*

_c_ 7. What fraction of the time will the operator be busy but with no machine waiting to be serviced? *(choose nearest value)*

- a. 10%
- b. 20%

$\boxed{\text{c. 30\%}}$ d. 40%
e. 50% f. 60%

Note: $\pi_1 = 0.75\pi_0 = 0.338$. (Recall that π_1 is found by multiplying π_0 by the second term in the sum in (5) above, π_2 is found by multiplying π_0 by the third term, etc.)

_a_ 8. Approximately 2.2 machines per hour require the operator's attention. What is the average length of time that a machine waits before the operator begins to ready the machine for the next job? *(select nearest value)*

$\boxed{\text{a. 0.1 hr. (i.e.,6 min.)}}$ b. 0.15 hr. (i.e., 9 min.)
c. 0.2 hr. (i.e., 12 min.) d. 0.25 hr. (i.e., 15 min.)
e. 0.3 hr. (i.e., 18 min.) f. greater than 0.33 hr. (i.e., >20 min.)

Note: The average "arrival" rate, λ, is stated to be 2.2/hour, and $L = \sum_{n=0}^{3} n\,\pi_n = \pi_1 + 2\pi_2 + 3\pi_3 \approx 0.34 + 2 \times 0.17 + 3 \times 0.04 = 0.8$

Little's Law then says $L = \lambda W \Rightarrow W = L/\lambda \approx 0.8/2.2\,/hr = 0.364$, *the average time spent both waiting and being serviced. Since an average of 15 minutes* = 0.25 *hr. is spent being serviced, the average time waiting is* 0.364 - 0.25 = 0.114 hr.

An alternate computation would use $L_q = \lambda W_q \Rightarrow W_q = L_q/\lambda \approx 0.25/2.2\,/hr = 0.114$,

where $L_q = 0\pi_0 + 0\pi_1 + 1\pi_2 + 2\pi_3 \approx 0.17 + 2(0.04) = 0.25$ *is the average number of machines waiting to be serviced (not including any being serviced) and* W_q *is the average time waiting (excluding service time).*

_f_ 9. What will be the utilization of this group of 3 machines? *(choose nearest value)*

a. 50% b. 55%
c. 60% d. 65%
e. 70% $\boxed{\text{f. greater than 75\%}}$

Note: The average number of machines running in steady state is $3\pi_0 + 2\pi_1 + 1\pi_2 + 0\pi_3 \approx 2.2$ *and so the average fraction of the time that any single machine is running is* 2.2/3 = 73.3%.

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Consider the gasoline blending problem in today's homework exercise #1 (as well as in the class notes).

Xij = #barrels/day of raw gas i used in blend j

Yi = #barrels/day of raw gas i sold on market

LINDO output is:

```
 MAX     14.13 X11 + 12 X21 + 8.8 X31 + 6.4 X41 + 11.93 X12 + 9.8 X22
      + 6.6 X32 + 4.2 X42 + 9.97 X13 + 7.84 X23 + 4.64 X33 + 2.24 X43
      + 5.83 Y1 + 3.7 Y2 + 2.6 Y3 + 0.2 Y4
 SUBJECT TO
        2) - 27 X11 - 9 X21 - 4 X31 + 4 X41 >=    0  (octane reqmt blend 1)
        3) - 22 X12 - 4 X22 + X32 + 9 X42 >=    0  (octane reqmt blend 2)
        4) - 17 X13 + X23 + 6 X33 + 14 X43 >=    0  (octane reqmt blend 3)
        5)   X11 + X12 + X13 + Y1 <=   4000   (available raw gas 1)
        6)   X21 + X22 + X23 + Y2 <=   5050  (available raw gas 2)
        7)   X31 + X32 + X33 + Y3 <=   7100  (available raw gas 3)
        8)   X41 + X42 + X43 + Y4 <=   4300  (available raw gas 4)
        9)   X11 + X21 + X31 + X41 <=   10000  (maximum demand of blend 1)
       10)   X13 + X23 + X33 + X43 >=   15000  (requirement of blend 3)
 END
      OBJECTIVE FUNCTION VALUE
      1)      140216.5

 VARIABLE        VALUE          REDUCED COST
      X11         0.000000          0.000000
      X21         0.000000          0.000000
      X31      2453.703613          0.000000
      X41      2453.703613          0.000000
      X12         0.000000          0.000000
      X22         0.000000          0.542424
      X32         0.000000          0.693098
      X42         0.000000          0.934175
      X13      3457.407471          0.000000
      X23      5050.000000          0.000000
      X33      4646.296387          0.000000
      X43      1846.296265          0.000000
       Y1       542.592590          0.000000
       Y2         0.000000          5.533333
       Y3         0.000000          4.970370
       Y4         0.000000          7.429630

      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)         0.000000         -0.307407
       3)         0.000000         -0.277273
       4)         0.000000         -0.307407
       5)         0.000000          5.830000
       6)         0.000000          9.233334
       7)         0.000000          7.570370
       8)         0.000000          7.629630
       9)      5092.592773          0.000000
      10)         0.000000         -1.085926

 RANGES IN WHICH THE BASIS IS UNCHANGED:
                          OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
     X11       14.130000         0.000000         INFINITY
     X21       12.000000         0.000000         INFINITY
     X31        8.800000         INFINITY         0.000000
     X41        6.400000         0.000000         1.627273
     X12       11.930000         2.283539         2.983334
     X22        9.800000         0.542424         INFINITY
     X32        6.600000         0.693098         INFINITY
     X42        4.200000         0.934175         INFINITY
     X13        9.970000         1.627273         0.000000
```

X23	7.840000	INFINITY	0.000000
X33	4.640000	0.000000	1.207331
X43	2.240000	1.627273	0.000000
Y1	5.830000	6.100000	2.932000
Y2	3.700000	5.533334	INFINITY
Y3	2.600000	4.970370	INFINITY
Y4	0.200000	7.429630	INFINITY

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	0.000000	17096.773438	14650.000000
3	0.000000	0.000000	11937.037109
4	0.000000	93350.000000	14650.000000
5	4000.000000	INFINITY	542.592590
6	5050.000000	5538.888672	1627.777710
7	7100.000000	4334.782227	3662.500000
8	4300.000000	3662.500000	4274.193359
9	10000.000000	INFINITY	5092.592773
10	15000.000000	1465.000000	5864.706055

THE TABLEAU

ROW	(BASIS)	X11	X21	X31	X41	X12	X22
1	ART	0.000	0.000	0.000	0.000	0.000	0.542
2	X31	3.875	1.625	1.000	0.000	0.000	0.333
3	X12	0.000	0.000	0.000	0.000	1.000	0.182
4	X13	1.000	0.000	0.000	0.000	0.000	-0.333
5	X33	-3.875	-1.625	0.000	0.000	0.000	-0.333
6	X23	0.000	1.000	0.000	0.000	0.000	1.000
7	X41	-2.875	-0.625	0.000	1.000	0.000	0.333
8	X43	2.875	0.625	0.000	0.000	0.000	-0.333
9	SLK 9	0.000	0.000	0.000	0.000	0.000	-0.667
10	Y1	0.000	0.000	0.000	0.000	0.000	0.152

ROW	X32	X42	X13	X23	X33	X43	Y1
1	0.693	0.934	0.000	0.000	0.000	0.000	0.000
2	0.426	0.574	0.000	0.000	0.000	0.000	0.000
3	-0.045	-0.409	0.000	0.000	0.000	0.000	0.000
4	-0.148	0.148	1.000	0.000	0.000	0.000	0.000
5	0.574	-0.574	0.000	0.000	1.000	0.000	0.000
6	0.000	0.000	0.000	1.000	0.000	0.000	0.000
7	0.426	0.574	0.000	0.000	0.000	0.000	0.000
8	-0.426	0.426	0.000	0.000	0.000	1.000	0.000
9	-0.852	-1.148	0.000	0.000	0.000	0.000	0.000
10	0.194	0.261	0.000	0.000	0.000	0.000	1.000

ROW	Y2	Y3	Y4	SLK 2	SLK 3	SLK 4	SLK 5
1	5.533	4.970	7.430	0.307	0.277	0.307	5.830
2	0.333	0.426	0.574	0.144	0.000	0.019	0.000
3	0.000	0.000	0.000	0.000	0.045	0.000	0.000
4	-0.333	-0.148	0.148	0.037	0.000	0.037	0.000
5	-0.333	0.574	-0.574	-0.144	0.000	-0.019	0.000
6	1.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.333	0.426	0.574	-0.106	0.000	0.019	0.000
8	-0.333	-0.426	0.426	0.106	0.000	-0.019	0.000
9	-0.667	-0.852	-1.148	-0.037	0.000	-0.037	0.000
10	0.333	0.148	-0.148	-0.037	-0.045	-0.037	1.000

ROW	SLK 6	SLK 7	SLK 8	SLK 9	SLK 10	
1	9.2	7.6	7.6	0.00E+00	1.1	0.14E+06
2	0.333	0.426	0.574	0.000	0.315	2453.704
3	0.000	0.000	0.000	0.000	0.000	0.000
4	-0.333	-0.148	0.148	0.000	-0.370	3457.407
5	-0.333	0.574	-0.574	0.000	-0.315	4646.296
6	1.000	0.000	0.000	0.000	0.000	5050.000
7	0.333	0.426	0.574	0.000	0.315	2453.704
8	-0.333	-0.426	0.426	0.000	-0.315	1846.296
9	-0.667	-0.852	-1.148	1.000	-0.630	5092.593
10	0.333	0.148	-0.148	0.000	0.370	542.593

1. Suppose that the requirement for blend #3 were to increase by 100 barrels/day. What would be the effect on the refinery's profit?

Solution: The requirement for blend #3 is the right-hand-side of row #10. The "dual price" of row #10 is (–0.108)defined by LINDO to be the "rate of improvement of the objective relative to increases in the right-hand-side". A negative rate of improvement of a profit function is a decrease, and therefore the answer is (–0.10859)×100 = <u> – $108.59</u>

2. If the requirement for blend #3 were to increase by 100 barrels/day, what would be the change in:

quantity of raw gas #1 sold on market:	<u>- 37 barrels/day</u>
quantity of raw gas #2 added to blend 3:	<u>no change</u>
quantity of raw gas #1 added to blend 3:	<u>+37 barrels/day</u>

Solution: Row 10 is an inequality constraint:

```
X13 + X23 + X33 + X43 >=   15000
```

which LINDO transforms into an equation by subtracting a surplus variable (which LINDO, however, names SLK_10):

```
X13 + X23 + X33 + X43 - SLK_10 = 15000
```

The increase in production (left side of row 10) of 100 means (`X13 + X23 + X33 + X43`) = 15100, and so SLK_10 must *increase* to +100 in order that the equation remain balanced. The substitution rates found in the tableau will indicate the resulting changes in the basic variables:

ROW	BASIS	SLK 10
1	ART	1.1
2	X31	0.315
3	X12	0.000
4	X13	-0.370
5	X33	-0.315
6	X23	0.000
7	X41	0.315
8	X43	-0.315
9	SLK 9	-0.630
10	Y1	0.370

According to these substitution rates, if SLK 10 increases by 100, then X13 (basic in row 4) will *increase* by 100×0.370 = 37 barrels/day, X23 (basic in row 6) will be *unchanged* (substitution rate = 0), and Y1 (basic in row 10) will *decrease* by 100×0.370 = 37 barrels/day. That is, 37 barrels/day of raw gas #1 which previously was sold on the market is now diverted to blend #3, accounting for part of the increase in blend #3 to be produced.

3. The plant manager makes a mistake by adding 10 barrels of raw gas #2 to blend #2. How much profit will the refinery lose as a result of that mistake? <u>$ 5.42424</u>

Solution: The variable X22 is zero & nonbasic in the optimal solution. If it were increased by 10 units, the "reduced cost", which is 0.542424, tells us the rate of "deterioration" in the objective function, according to LINDO's definition. Therefore, the profit will deteriorate, i.e. decrease, by 10×0.542424 = 5.42424 dollars.

4. How should the refinery change the following quantities to compensate for the mistake in the previous question?

quantity of raw gas #1 sold on market:	(+) or (-) _______
quantity of raw gas #2 added to blend 3:	(+) or (-) _______
quantity of raw gas #1 added to blend 3:	(+) or (-) _______

Solution: The substitution rates of the nonbasic variable X22 are found in the tableau:

ROW	(BASIS)	X22
1	ART	0.542
2	X31	0.333
3	X12	0.182
4	X13	-0.333
5	X33	-0.333
6	X23	1.000
7	X41	0.333
8	X43	-0.333
9	SLK 9	-0.667
10	Y1	0.152

The substitution rate of X22 for Y1 is 0.152, i.e. each barrel increase in X22 will "substitute" for 0.152 barrel of Y1, so that Y1 will *decrease* by 1.52 barrels. Likewise, the substitution rate of X22 for X23 is 1, so that X23 will *decrease* by 10 barrels, and the substitution rate of X22 for X13 is -0.333 so that X13 will *increase* by 3.33 barrels. *Note: a positive substitution rate means that the changes in nonbasic & basic variables are opposite in direction, while a negative substitution rate means that the direction of changes will be identical!*

5. Suppose that the selling price of raw gas #1 on the market were to drop by $3/barrel. Would the optimal solution change? <u>YES</u>

Solution: A drop of $3/barrel in selling price would result in a decrease of $3/barrel in the objective function (profit), while the "ALLOWABLE DECREASE" in the objective coefficient is only $2.932. Since the change is outside the allowable range, the basis will change, which would mean a change in the solution.

For each statement, indicate "+"=**true** or "o"=**false**.

__o__ 6. If you increase the right-hand-side of a "≤" constraint in a <u>min</u>imization LP, the optimal objective value will either increase or stay the same. *Since the previous solution remains feasible, one cannot do worse than previously, but perhaps may do better, i.e., lower the cost.*

__+__ 7. If the increase in the cost of a nonbasic variable remains less than the "ALLOWABLE INCREASE" reported by LINDO, then the optimal values of all variables will be unchanged. *The basis will remain unchanged, and since the values of the basic variables depend only on the basis matrix and the right-hand-side, their values will remain unchanged also.*

__+__ 8. The "reduced cost" in an LP solution provides an estimate of the change in the objective value when a nonbasic variable increases.

__o__ 9. In the two-phase simplex method, Phase One computes the optimal dual variables, followed by Phase Two in which the optimal primal variables are computed. *The purpose of Phase One is to find a basic feasible solution to the (primal) problem, by first introducing artificial variables into the basis and then removing them from the basis.*

__+__ 10. At the completion of the revised simplex method applied to an LP, the simplex multipliers give the optimal solution to the dual of the LP.

__o__ 11. If the primal LP has an equality constraint, the corresponding dual variable must be zero. *The dual variable corresponding to an equality constraint is unrestricted in sign, and need not be zero!*

__+__ 12. If a minimization LP problem has a cost which is unbounded below, then its dual problem cannot be feasible. *Since the objective value of any feasible solution of the dual problem provides us with a (lower) bound on the primal objective, there cannot be a feasible solution of the dual problem in this case.*

__o__ 13. If the increase in the right-hand-side of a "tight" constraint remains less than the "ALLOWABLE INCREASE" reported by LINDO, then the optimal values of all variables will be unchanged. *If the increase in the right-hand-side remains within the range between allowable increase & decrease, the basis (and hence the set of basic variables) will be unchanged. However, because the right-hand-side changes, the values of those basic variables (which are computed by multiplying the basis inverse matrix times the right-hand-side) will change.*

__o_ 14. If a minimization LP problem is infeasible, then its dual problem has an objective (to be maximized) which must be unbounded above. *While the converse of this statement is true, the statement itself is not-- it is possible that <u>both</u> primal and dual problems are infeasible.*

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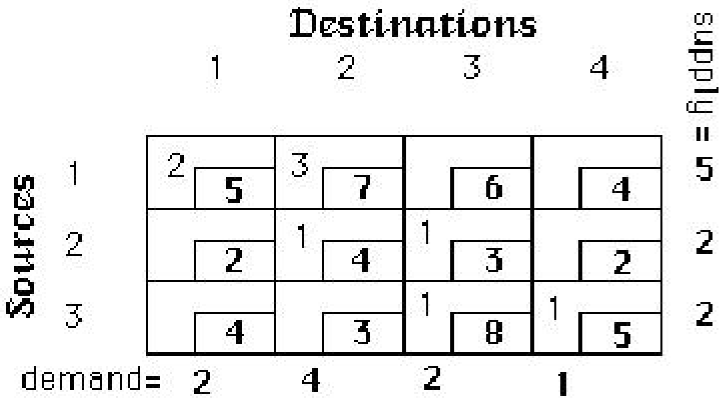

Indicate whether true (+) or false(o):

a. __+_ For every basic solution in the TP tableau above, dual variable V_1 will be larger than U_1.

b. __+_ The transportation problem above is a special case of a linear programming problem.

c. __+_ Variable X_{22} is basic in the set of shipments above.

d. __o_ If one unit were to be shipped from source #2 to destination #4, the result would be a reduction in the total cost . *Note: The reduced cost of X_{24} is +2.*

e. __+_ The number of basic variables for this transportation problem is six

f. __o_ For the solution above, if dual variable U_1 is zero then will U_2 will be strictly positive. *Note: If U_1 is zero, then since $X_{12}>0$, complementary slackness requires that V_2 must be 7 (since C_{12} - $[U_1+V_2]=0$). Then, since $V_2=7$ and $X_{22}>0$, complementary slackness requires that U_2 must therefore be -3 (since C_{22} - $[U_2+V_2]=0$).*

g. __o_ The optimal dual variables for a transportation problem must be nonnegative. *Note: according to LP duality theory, the dual variable corresponding to a primal equality constraint is unrestricted in sign.*

h. __o_ If X_{24} were made a basic variable in the tableau above, then its value would be 2. *Note: If X_{24} is entered into the basis, its value would be 1.*

i. __o_ In the first step of Vogel's method for the above TP tableau, the penalty on column 1 will equal 1. *Note: the penalty is the difference between the two smallest values in that column, namely 4-2=2.*

j. __o_ Vogel's method will always yield an optimal solution, if it is nondegenerate. *Note: Vogel's method yields a feasible solution which is generally "good", but not necessarily optimal.*

k. __o_ According to Complementary Slackness, if X* is optimal in the transportation problem and U* & V* in its dual problem, then if $X_{ij}^* >0$, the slack in the dual constraint $U_i+V_j \leq C_{ij}$ must be positive. *Note: Complementary slackness requires that either $X_{ij}=0$ or the slack in the constraint U_i+V_j£ C_{ij} (or both) must be zero.*

l. __+_ If no degenerate solution is encountered, the transportation simplex method gives an improvement in the objective function at every basis change.

m. __+_ The above transportation problem is "balanced".

n. __+_ The shipments indicated in the above TP tableau constitute a basic solution.

o. __+_ In the first step of Vogel's method for the above TP tableau, the penalty on row 1 will equal 1.

p. __+_ The above transportation problem will have 7 dual variables.

q. __o_ The shipments indicated in the above table are a degenerate solution to this transportation problem. *Note: the number of basic variables must be m+n-1 = 3+4-1=6. Since there are six positive shipments, no basic variable has value 0 and therefore the solution is not degenerate.*

r. __+_ The reduced cost of the variable X_{31} in the tableau above is negative.

r. __o_ If the current solution of a TP is degenerate, the next iteration will not improve the objective function. *Note: while it is possible that the next iteration will not improve the objective function, it is not always the case.*

s. __+_ The shipments indicated in the above table are a feasible solution to this transportation problem.

t. __o_ A "dummy" destination will be required for the transportation problem above. *Note: since the transportation problem is already "balanced", i.e., the sum of the supplies is equal to the sum of the demands, no "dummy" destination is required.*

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Integer Programming Model Formulation. Coach Night is trying to choose the starting linerup for the basketball team. The team consists of seven players who have been rated (on a scale of 1=poor to 3=excellent) according to their ball-handling, shooting, rebounding, and defensive abilities. The positions that each player is allowed to play (G=guard, C=center, F=forward) and the player's abilities are:

Player	Position	Ball-handling	Shooting	Rebounding	Defense
1	G	3	3	1	3
2	C	2	1	3	2
3	G-F	2	3	2	2
4	F-C	1	3	3	1
5	G-F	1	3	1	2
6	F-C	3	1	2	3
7	G-F	3	2	2	1

For each stated constraint, indicate the mathematical expression of that constraint from the list below. *(Note: if more than one answer is correct, only one answer is required.)*

_o_ 1. The starting lineup is to consist of five players.
_t_ 2. At least 2 members must be able to play guard.
_d_ 3. If player 7 starts, then player 3 cannot start.
_s or w 4. If player 2 starts, then player 6 must start.
_p or z 5. Either player 3 or player 5 (or both) must start.
_e_ 6. The average rebounding rating of the starting lineup must be at least 2.
_n_ 7. At least one member must be able to play center.
_h or y 8. At least 2 members must have a rating of 3 in shooting.
_f_ 9. Not more than one of the members may have a rebounding rating of 1.
_a or g_10. At least four members must be able to play at more than one position.

a. $X_1 + X_2 \le 1$

b. $X_1 + X_3 + X_5 + X_7 \le 2$

c. $X_5 \ge X_1$

d. $X_3 + X_7 \le 1$

e. $X_1 + 3X_2 + 2X_3 + 3X_4 + X_5 + 2X_6 + 2X_7 \ge 10$

f. $X_1 + X_5 \le 1$

g. $X_3 + X_4 + X_5 + X_6 + X_7 \ge 4$

h. $X_1 + X_3 + X_4 + X_5 \ge 2$

i. $X_2 \ge X_6$

j. $X_3 + X_7 \ge 1$

k. $X_1 + X_2 \ge 1$

l. $X_1 + 3X_2 + 2X_3 + 3X_4 + X_5 + 2X_6 + 2X_7 \ge 2$

m. $X_1 \ge X_2$

n. $X_2 + X_4 + X_6 \ge 1$

o. $X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 = 5$

p. $X_3 + X_5 \ge 1$

q. $X_3 \ge X_7$

r. $X_3 \le X_7$

s. $X_2 \le X_6$

t. $X_1 + X_3 + X_5 + X_7 \ge 2$

u. $X_2 + X_6 \le 1$

v. $X_1 + X_3 + X_4 + X_5 \le 2$

w. $X_6 \ge X_2$

x. $X_5 \le X_1$

y. $X_1 + X_3 + X_4 + X_5 \ge 2$

z. $X_3 + X_5 \ge 1$

Note that several answers appeared more than once (s&w, p&z, h&y), and that (assuming that the lineup consists of 5 players) a implies g.

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Careful study of a reservoir over the past 20 years has shown that, if the reservoir was full at the beginning of the summer, then the probability it would be full at the beginning of the following summer is **60%**, independent of its status in previous years. On the other hand, if the reservoir was <u>not</u> full at the beginning of one summer, the probability it would be full at the beginning of the following summer is only **20%**. Define a Markov chain model of this reservoir, with the states (1)"full" and (2)"not full". The transition probability matrix and several higher powers, as well as some other computations, are shown below.

____ 1. If the reservoir was full at the beginning of summer 1997, what is the probability that it will be full at the beginning of summer 1999 (rounded to the nearest 10%)?

a. 20%	b. 30%	c. 40%
d. 50%	e. 60%	f. none of the above

Note: $p_{11}^{(2)} = 0.44$

____ 2. Suppose the reservoir was full at the beginning of summer 1997, and consider the first year that follows in which the reservoir is not full at the beginning of the summer. What is the probability that this occurs in 1999? (rounded to the nearest 10%)

a. 10%	b. 20%	c. 30%
d. 40%	e. 50%	f. none of the above

Note: $f_{12}^{(2)} = 0.24$

__B_ 3. If the reservoir was full at the beginning of summer 1997, the expected number of years until it will next be "not full" is: (choose nearest value)

a. 2	b. 2.5	c. 3
d. 3.5	e. 5	f. none of the above

Note: $m_{12} = 2.5$

__F_ 4. If the reservoir was full at the beginning of summer 1997, the probability that 2000 is the first year it is not full is (rounded to the nearest 10%)

a. 20%	b. 30%	c. 40%
d. 50%	e. 60%	f. none of the above

Note: value is $f_{12}^{(3)} = 0.144$ *which rounds to 10%.*

__E_ 5. If the reservoir was full at the beginning of summer 1997, the probability that in 2000 it is not full is (rounded to the nearest 10%)

a. 20%	b. 30%	c. 40%
d. 50%	e. 60%	f. none of the above

Note: $p_{12}^{(3)} = 0.624$

__B_ 6. During the next hundred years, in how many the reservoir be full at the beginning of the summer, as predicted by this model? (choose nearest value)

a. 20	b. 33	c. 40
d. 60	e. 67	f. none of the above

Note: $\pi_1 = 0.333$

____ 7. If the reservoir is full at the beginning of **both** summer 1997 and summer 1998, the probability that it will be full at the beginning of summer 1999 is (rounded to the nearest 10%)

a. 40%	b. 50%	c. 60%
d. 70%	e. 80%	f. none of the above

Note: Because of the memoryless property of the Markov chain, the state in 1997 is not relevant!
$p_{11}^{(2)} = 0.44$

__D_ 8. If the reservoir was full at the beginning of summer 1997, what is the expected number of years until it is expected to be full again? (choose nearest value)

a. 1.5	b. 2	c. 2.5
d. 3	e. 5	f. none of the above

Note: $m_{11} = 3$

__A_ 9. If the reservoir was full at the beginning of summer 1997, the probability that 2000 is the first year it is full again is (rounded to the nearest 10%)

a. 10% b. 20% c. 30%
d. 40% e. 50% f. none of the above

Note: $f_{11}^{(3)} = 0.064$

__F_ 10. Consider a different reservoir: if it was full at the beginning of the summer, then the probability it would be full at the beginning of the following summer is **50%**, while if the reservoir was not full at the beginning of one summer, the probability it would be full at the beginning of the following summer is **25%**. The steadystate probability distribution π for this Markov chain must satisfy the following equation:

a. $\pi_1 + \pi_2 = 0$ b. $\pi_2 = 0.25\pi_1 + 0.75\pi_2$ c. $\pi_2 = 0.75\pi_1 + 0.25\pi_2$
d. $\pi_2 = 0.5\pi_1 + 0.25\pi_2$ e. $0.5\pi_1 + 0.5\pi_2 = 1$ f. none of the above

Note: The transition probability matrix is

$$P = \begin{vmatrix} 0.50 & 0.50 \\ 0.25 & 0.75 \end{vmatrix}$$

The equations p = p*P, together with the restriction that the probabilities must sum to 1, are*

$$\begin{cases} \pi_1 = 0.50\,\pi_1 + 0.25\,\pi_2 \\ \pi_2 = 0.50\,\pi_1 + 0.75\,\pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

You may use the computational results below to answer questions (1) through (9):

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} \quad P^2 = \begin{bmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{bmatrix} \quad P^3 = \begin{bmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{bmatrix} \quad P^4 = \begin{bmatrix} 0.3504 & 0.6496 \\ 0.3248 & 0.6752 \end{bmatrix}$$

First Visit Probabilities

$$F = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} \quad F^{(2)} = \begin{bmatrix} 0.08 & 0.24 \\ 0.16 & 0.08 \end{bmatrix} \quad F^{(3)} = \begin{bmatrix} 0.064 & 0.144 \\ 0.128 & 0.048 \end{bmatrix} \quad F^{(4)} = \begin{bmatrix} 0.0512 & 0.0864 \\ 0.1024 & 0.0288 \end{bmatrix}$$

i	π_i
1	0.33333333
2	0.66666667

$$M = \begin{bmatrix} 3 & 2.5 \\ 5 & 1.5 \end{bmatrix}$$

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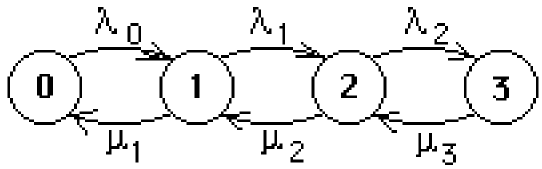
A machine operator has sole responsibility for keeping three semi-automatic machines busy. The time required to ready a machine (unloading & reloading) has exponential distribution with mean 15 minutes. The machine will then run unattended for an average of 1 hour (but with actual time having exponential distribution) before it requires the operator's attention again.



Note: The state of the system (population of the birth/death process) is the number of machines which are either being serviced or waiting to be serviced.)

_b,c,g_ 1. The Markov chain model diagrammed above is *(select one or more):*

- a. a discrete-time Markov chain
- b. a continuous-time Markov chain
- c. a Birth-Death process
- d. an M/M/1 queue
- e. an M/M/3 queue
- f. an M/M/1/3 queue
- g. an M/M/1/3/3 queue
- h. a Poisson process

Note: This is a queue with the M/M/1/3/3 classification, i.e., 1 server, maximum of 3 in the "system", and 3 in the "source population".

_a_ 2. The value of λ_2 is

- a. 1/hr.
- b. 2/hr.
- c. 3/hr.
- d. 4/hr.
- e. 0.5/hr.
- f. *none of the above*

Note: In state 2, 2 machines are shut down and one is processing a job. Therefore, the "arrival" rate, i.e., the rate at which machines are requiring service, is 1 machine per hour.

_f_ 3. The value of μ_2 is

- a. 1/hr.
- b. 2/hr.
- c. 3/hr.
- d. 0.5/hr.
- e. 0.5/hr.
- f. *none of the above*

Note: the service rate, independently of the number of "customers in the system", is 1 "customer"/15 minutes = 4 / hr.

_c_ 4. The value of λ_0 is

- a. 1/hr.
- b. 2/hr.
- c. 3/hr.
- d. 0.5/hr.
- e. 0.5/hr.
- f. *none of the above*

Note: each machine "arrives" at the rate of once per hour, so when three machines are running, they jointly complete their jobs at the rate of 3/hr.

_b_ 5. The steady-state probability π_0 is computed by solving

a. $\frac{1}{\pi_0} = 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 \approx \frac{1}{0.366}$

b. $\boxed{\frac{1}{\pi_0} = 1 + \frac{3}{4} + \frac{1}{2}\times\frac{3}{4} + \frac{1}{4}\times\frac{1}{2}\times\frac{3}{4} \approx \frac{1}{0.451}}$

c. $\frac{1}{\pi_0} = 1 + \frac{3}{4} + \frac{1}{2} + \frac{1}{4} \approx \frac{1}{0.4}$

d. $\frac{1}{\pi_0} = 1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 \approx \frac{1}{0.753}$

e. $\frac{1}{\pi_0} = 1 + \frac{3}{4} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^3 \approx \frac{1}{0.496}$

f. *none of the above*

c (or d) 6. The operator will be busy what fraction of the time? *(choose nearest value)*

- a. 40%
- b. 45%
- c. 50%
- d. 60%
- e. 65%
- f. 75%

Note: $\pi_0 = 0.451$ *is the probability that there are no machines shut down, i.e., the fraction of the time that the operator is idle. The fraction of the time that he is busy is therefore 1-0.451 = 0.549.*

_c_ 7. What fraction of the time will the operator be busy but with no machine waiting to be serviced? *(choose nearest value)*

- a. 10%
- b. 20%

$\boxed{\text{c. 30\%}}$ d. 40%
e. 50% f. 60%

Note: $\pi_1 = 0.75\pi_0 = 0.338$. (Recall that π_1 is found by multiplying π_0 by the second term in the sum in (5) above, π_2 is found by multiplying π_0 by the third term, etc.)

_a_ 8. Approximately 2.2 machines per hour require the operator's attention. What is the average length of time that a machine waits before the operator begins to ready the machine for the next job? *(select nearest value)*

$\boxed{\text{a. 0.1 hr. (i.e.,6 min.)}}$ b. 0.15 hr. (i.e., 9 min.)
c. 0.2 hr. (i.e., 12 min.) d. 0.25 hr. (i.e., 15 min.)
e. 0.3 hr. (i.e., 18 min.) f. greater than 0.33 hr. (i.e., >20 min.)

Note: The average "arrival" rate, $\underline{\lambda}$, is stated to be 2.2/hour, and $L = \sum_{n=0}^{3} n\,\pi_n = \pi_1 + 2\pi_2 + 3\pi_3$
$\approx 0.34 + 2\times 0.17 + 3\times 0.04 = 0.8$

Little's Law then says $L = \underline{\lambda} W \Rightarrow W = L/\underline{\lambda} \approx 0.8/2.2/hr = 0.364$, *the average time spent both waiting and being serviced. Since an average of 15 minutes* = 0.25 *hr. is spent being serviced, the average time waiting is* 0.364 - 0.25 = 0.114 hr.

An alternate computation would use $L_q = \underline{\lambda} W_q \Rightarrow W_q = L_q/\underline{\lambda} \approx 0.25/2.2/hr = 0.114$,
where $L_q = 0\pi_0 + 0\pi_1 + 1\pi_2 + 2\pi_3 \approx 0.17 + 2(0.04) = 0.25$
is the average number of machines waiting to be serviced (not including any being serviced) and W_q *is the average time waiting (excluding service time).*

_f_ 9. What will be the utilization of this group of 3 machines? *(choose nearest value)*

a. 50% b. 55%
c. 60% d. 65%
e. 70% $\boxed{\text{f. greater than 75\%}}$

Note: The average number of machines running in steady state is $3\pi_0 + 2\pi_1 + 1\pi_2 + 0\pi_3 \approx 2.2$ *and so the average fraction of the time that any single machine is running is* 2.2/3 = 73.3%.