56:171
Operations Research
Fall 2001

Quiz Solutions

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56:171 Operations Research Quiz #1 Solution -- Fall 2001

For each statement, indicate "+"=**true** or "o"=**false**.

__o_ 1. When you enter an LP formulation into LINDO, you must first convert all inequalities to equations.
__+_ 2. When you enter an LP formulation into LINDO, you do <u>not</u> need to enter any nonnegativity constraints, for example, X1 >= 0.
__+_ 3. When you enter an LP formulation into LINDO, you must manipulate your constraints so that all variables appear on the left, and all constants on the right.
__+_ 4. LINDO would interpret the constraint "X1 + 2X2 > 10" as "X1 + 2X2 ≥ 10".

<><><><><><><><><><><><>

Cattle feed can be mixed from oats, corn, alfalfa, and peanut hulls. The following table shows the current cost per ton (in dollars) of each of these ingredients, together with the percentage of recommended daily allowances for protein, fat, and fiber that a serving of it fulfills.

	Oats	Corn	Alfalfa	Peanut hulls
% protein	60	80	55	40
% fat	50	70	40	100
% fiber	90	30	60	80
Cost $/ton	200	150	100	75

We want to find a minimum cost way to produce feed that satisfies at least 60% of the daily allowance for protein and fiber while not exceeding 60% of the fat allowance. Define the variables OATS, CORN, ALFALFA, and HULLS to be the quantity (in <u>tons</u>) mixed to obtain a ton of cattle feed. The model & LINDO output are below:

```
MIN     200 OATS + 150 CORN + 100 ALFALFA + 75 HULLS
SUBJECT TO
       2)   0.6 OATS + 0.8 CORN + 0.55 ALFALFA + 0.4 HULLS >=   0.6
       3)   0.9 OATS + 0.3 CORN + 0.6 ALFALFA + 0.8 HULLS >=   0.6
       4)   0.5 OATS + 0.7 CORN + 0.4 ALFALFA + HULLS <=   0.6
       5)   OATS + CORN + ALFALFA + HULLS =    1
END

      OBJECTIVE FUNCTION VALUE
      1)      125.0000

VARIABLE        VALUE          REDUCED COST
    OATS         0.157143          0.000000
    CORN         0.271429          0.000000
 ALFALFA         0.400000          0.000000
   HULLS         0.171429          0.000000

     ROW   SLACK OR SURPLUS     DUAL PRICES
      2)         0.000000       -500.000000
      3)         0.000000       -250.000000
      4)         0.000000          0.000000
      5)         0.000000        325.000000
```

5. The optimal composition of cattle feed is _40_% alfalfa.
6. The minimum cost of a ton of feed is $_125___.
7. There are __4_ basic variables in the optimal solution, in addition to –Z (in the cost equation).

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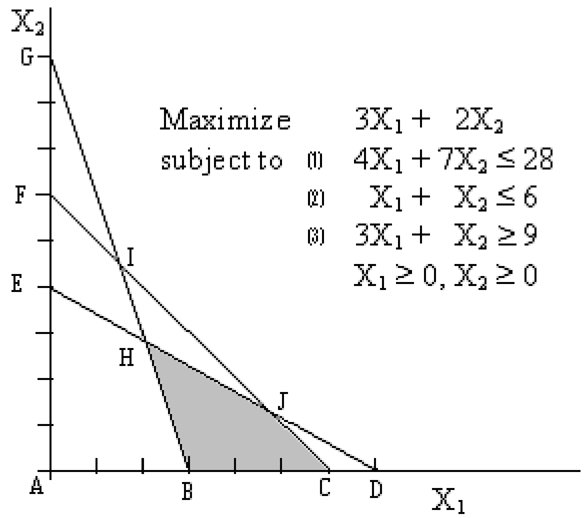
Consider the following LP:



8. The feasible region is bounded by _4_ points, namely _B, C, H, & J____.
9. At point **H**, the slack variable for constraint # _2_ is positive. *($X_1+X_2= 4.88<6$)*

Let X_3, X_4, and X_5 represent the slack (or surplus) in constraints 1, 2, and 3, respectively.

10. The objective coefficients of X_3, X_4, and X_5 in the *initial* simplex tableau are _zero.
11. The optimal solution is at point __C__,
where the basic variables are _X_1, X_3, & X_5__ (plus –Z).

Note: For your convenience, the (X_1, X_2) coordinates of the points labeled above are:

Point	A	B	C	D	E	F	G	H	I	J
X_1	0	3	6	9	0	0	0	2.06	1.5	4.67
X_2	0	0	0	0	4	6	7	2.82	4.5	1.33

The values of the variables X_3, X_4, and X_5, and the cost, are:

Point	A	B	C	D	E	F	G	H	I	J
X1	0.00	3.00	6.00	7.00	0.00	0.00	0.00	2.06	1.50	4.67
X2	0.00	0.00	0.00	0.00	4.00	6.00	9.00	2.82	4.50	1.33
X3	28.00	16.00	4.00	0.00	0.00	-14.00	-35.00	0.00	-9.50	0.00
X4	6.00	3.00	0.00	-1.00	2.00	0.00	-3.00	1.12	3.00	0.00
X5	-9.00	0.00	9.00	12.00	-5.00	-3.00	0.00	0.00	0.00	6.33
Objective	0.00	9.00	18.00	21.00	8.00	12.00	18.00	11.82	13.50	16.67
Feasible?		YES	YES					YES		YES

Below are several simplex tableaus. Assume that the objective in each case is to be **minimized**. Classify each tableau by writing to the right of the tableau a letter **A** through **G**, according to the descriptions below. *Also answer the question accompanying each classification, if any.*

(**A**) Nonoptimal, nondegenerate tableau with bounded solution. *Circle a pivot element which would improve the objective.*

(**B**) Nonoptimal, degenerate tableau with bounded solution. *Circle an appropriate pivot element.*

(**C**) Unique optimal basic solution.

(**D**) Optimal tableau, with alternate optimal basic solution. *Circle a pivot element which would lead to another optimal basic solution.*

(**E**) Objective unbounded (below). *Circle a variable which, when going to infinity, will make the objective arbitrarily low.*

(F) Tableau with infeasible basic solution

Warning: *Some of these classifications might be used for more than one tableau, while others might not be used at all*

(1)

-z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS	
1	5	0	-2	1	0	0	3	2	-17	
0	3	0	4	0	0	1	3	0	3	B
0	-1	1	$\boxed{1}$	-5	0	0	-2	1	0	
0	5	0	0	-2	1	0	-4	3	1	

(2)

-z	x_1	x_2	$\boxed{x_3}$	x_4	x_5	x_6	x_7	x_8	RHS	
1	5	0	-2	1	0	0	3	2	-17	
0	3	0	-4	0	0	1	3	0	3	E
0	-1	1	-1	-5	0	0	-2	1	7	
0	5	0	0	-2	1	0	-4	3	1	

(3)

-z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS	
1	5	0	2	1	0	0	3	2	-17	
0	3	0	-4	0	0	1	3	0	3	C
0	-1	1	-1	-5	0	0	-2	1	7	
0	5	0	0	-2	1	0	-4	3	1	

(4)

-z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS	
1	0	0	2	1	0	0	3	2	-17	
0	3	0	-4	0	0	1	3	0	3	D
0	-1	1	-1	-5	0	0	-2	1	7	
0	$\boxed{5}$	0	0	-2	1	0	-4	3	1	

(5)

-z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS	
1	5	0	-2	1	0	0	3	2	-17	
0	3	0	$\boxed{4}$	0	0	1	3	0	3	A
0	-1	1	1	-5	0	0	-2	1	7	
0	5	0	0	-2	1	0	-4	3	1	

True (+) or False (o)?

__+__ 6. The "minimum ratio test" is used to select the pivot row in the simplex method for linear programming.

__o__ 7. Every feasible solution of an LP is a basic solution.

__+__ 8. If, when solving an LP by the simplex method, you make a mistake in choosing a pivot row, the resulting tableau is infeasible.

__+__ 9. In the simplex method, every variable of the LP is either basic or nonbasic.

__o__ 10. When you enter an LP formulation into LINDO, you must first convert all inequalities to equations.

__o__ 11. The restriction that X1 be nonnegative should be entered into LINDO as the constraint X1 >= 0.

__+__ 12. A "pivot" in a nonbasic column of a tableau will make it a basic column.

Name ______________________________

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Quiz #3 Solutions -- Fall 2001

Consider the following LP problem:

$$\begin{aligned} \text{Min } w = 4Y_1 + 2Y_2 - Y_3 & \\ \text{s.t.} \quad Y_1 + 2Y_2 \qquad & \leq 6 \\ Y_1 - Y_2 + 2Y_3 & = 8 \\ Y_2 \geq 0,\ Y_3 \geq 0 \quad & (Y_1 \text{ unrestricted in sign}) \end{aligned}$$

_a__ 1. The dual objective function is to be
(a) maximized (b) minimized

_b__ 2. The number of dual variables is
(a) one (b) two (c) three (d) four

_c__ 3. The number of dual constraints (excluding sign restrictions such as nonnegativity) is
(a) one (b) two (c) three (d) four

_a__ 4. The first dual constraint is
(a) equation (b) less-than-or-equal (c) greater-than-or-equal

_b__ 5. The right-hand-side of the first constraint is
(a) 2 (b) 4 (c) 6 (d) 8 (e) *other*

_b__ 6. The sign restriction of the first dual variable is
(a) nonnegativity (b) nonpositivity (c) no sign restriction

_c__ 7. The objective coefficient of the first dual variable is
(a) 2 (b) 4 (c) 6 (d) 8 (e) *other*

For each statement, indicate "+"=**true** or "o"=**false**.

_+___ 8. If you increase the right-hand-side of a "≤" constraint in a maximization LP, the optimal objective value will either increase or stay the same.

_o___ 9. The dual variable corresponding to a "≤" constraint in a maximization LP must be nonpositive. *It must be nonnegative!*

_+___ 10. The "reduced cost" in an LP solution provides an estimate of the change (either increase or decrease) in the objective value when a nonbasic variable increases.

_+___ 11. At the completion of the revised simplex method applied to an LP, the simplex multipliers give the optimal solution to the dual of the LP.

_o___ 12. If the increase in the right-hand-side of a "tight" constraint remains less than the "ALLOWABLE INCREASE" reported by LINDO, then the optimal values of all variables will be unchanged. *The basis is unchanged, but the values of the basic variables are given by* $x_B = \left(A^B\right)^{-1} b$ *, so if the right-hand-side b changes, the values* x_B *do also.*

_+___ 13. The "minimum ratio test" is used to select the pivot row in the simplex method for linear programming..

_o___ 14. The "Complementary Slackness" theorem says that if, for example, constraint #1 of the primal problem is "slack", then constraint #1 of the dual problem is "tight". *The theorem says instead that the first dual variable must be zero.*

_+___ 15. If the increase in the cost of a nonbasic variable remains less than the "ALLOWABLE INCREASE" reported by LINDO, then the optimal values of all variables will be unchanged.

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Quiz #4 Solutions -- Fall 2001

Consider again the LP model of PAR, Inc., which manufactures standard and deluxe golf bags:

X1 = number of **STANDARD** golf bags manufactured per quarter
X2 = number of **DELUXE** golf bags manufactured per quarter

Four operations are required, with the time per golf bag as follows:

	STANDARD	DELUXE	Available
Cut-&-Dye	0.7 hr	1 hr	630 hrs.
Sew	0.5 hr	0.8666 hr	600 hrs.
Finish	1 hr	0.6666 hr	708 hrs.
Inspect-&-Pack	0.1 hr	0.25 hr	135 hrs.
Profit ($/bag)	$10	$9	

LINDO provides the following output:

```
MAX     10 X1 + 9 X2
 SUBJECT TO
        2)   0.7 X1 + X2 <=   630
        3)   0.5 X1 + 0.86666 X2 <=   600
        4)   X1 + 0.66666 X2 <=   708
        5)   0.1 X1 + 0.25 X2 <=   135
 END
        OBJECTIVE FUNCTION VALUE
        1)    7668.01200

  VARIABLE        VALUE          REDUCED COST
        X1       540.003110           .000000
        X2       251.997800           .000000

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)          .000000          4.375086
        3)       111.602000           .000000
        4)          .000000          6.937440
        5)        18.000232           .000000

RANGES IN WHICH THE BASIS IS UNCHANGED:
                           OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                    COEF         INCREASE         DECREASE
       X1       10.000000         3.500135         3.700000
       X2        9.000000         5.285715         2.333400

                          RIGHTHAND SIDE RANGES
      ROW         CURRENT        ALLOWABLE        ALLOWABLE
                    RHS          INCREASE         DECREASE
        2      630.000000        52.364582       134.400000
        3      600.000000         INFINITY       111.602000
        4      708.000000       192.000010       128.002800
        5      135.000000         INFINITY        18.000232

THE TABLEAU
ROW (BASIS) X1    X2    SLK 2  SLK 3  SLK 4 SLK 5
  1 ART      .00   .00   4.375   .00   6.937   .00    7668.012
  2 X2       .00 1.00    1.875   .00  -1.312   .00     251.998
  3 SLK 3    .00   .00  -1.000  1.00    .200   .00     111.602
  4 X1     1.00   .00  -1.250   .00   1.875   .00     540.003
  5 SLK 5    .00   .00   -.344   .00    .141  1.00      18.000
```

Enter the correct answer into each blank or check the correct alternative answer, as appropriate. If not sufficient information, write "NSI" in the blank:

1. If the profit on STANDARD bags were to decrease from $10 each to $6 each, the number of STANDARD bags to be produced would
   |__| increase |X| **decrease** |__| remain the same |__| not sufficient info.
2. If the profit on DELUXE bags were to increase from $9 each to $13 each, the number of DELUXE bags to be produced would
   |__| increase |__| decrease |X| **remain the same** |__| not sufficient info.
3. The LP problem above has
   |X| **exactly one optimal solution** |__| exactly two optimal solutions
   |__| an infinite number of optimal solutions
4. If an additional 10 hours were available in the sewing department, PAR would be able to obtain an additional $_0_____ in profits.
5. If an additional 10 hours were available in finishing department, PAR would be able to obtain an additional $_69.37__ in profits.
6. If the variable "SLK 2" were increased, this would be equivalent to
   ___ increasing the hours used in the cut-&-dye department
   _X_ **decreasing the hours used in the cut-&-dye department**
   ___ none of the above
7. If the variable "SLK 2" were increased by 10, X1 would |X| **increase** |__| decrease by _12.5__ STANDARD golf bags/quarter.
8. If the variable "SLK 2" were increased by 10, X2 would |__| increase |X| **decrease** by _18.75____ DELUXE golf bags/quarter.

FYI:

Maximize	Minimize
Type of constraint i: $\leq$ $=$ $\geq$	Sign of variable i: nonnegative unrestricted in sign nonpositive
Sign of variable j: nonnegative unrestricted in sign nonpositive	Type of constraint i: $\geq$ $=$ $\leq$

Data Envelopment Analysis (Note: *DMU* = "decision-making-unit")

_c_ 9. In the *maximization* problem of the primal-dual pair of LP models, the decision variables are:
   a. The amount of each input and output to be used by the DMU
   b. The multipliers for the DMUs which are used to form a linear combination of DMUs in the reference set.
   c. The "prices" assigned to the inputs and outputs.
   c. None of the above

_a_ 10. The "prices" or weights assigned to the input & output variables in the maximization problem must
   a. be nonnegative
   b. sum to 1.0
   c. Both a & b
   d. Neither a nor b.

True (+) or false (o)?

_+_ 11. To perform a complete DEA analysis, an LP must be solved for *every* DMU.
_o_ 12. In the maximization LP form of the problem, there is a constraint for each input and for each output..
_+_ 13. The optimal value of the LP cannot exceed 1.0.
_o_ 14. The number of input and output variables must be equal
_o_ 15. The purpose of the DEA technique is to assist firms in setting market prices for their products.

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Quiz #5 Solutions -- Fall 2001

A company has two plants and three warehouses. The supplies & demands & shipping costs ($/unit) for a particular product is shown in the table:

	Warehouse 1	*Warehouse 2*	*Warehouse 3*	**Supply**
Plant 1	8	9	12	500
Plant 2	9	10	11	200
Demand	150	450	100	

True (+) or false (o)?

_+_ 1. For this problem, the optimal solution found by the simplex method is guaranteed to be integer-valued.
_o_ 2. A dummy plant must be defined so that # sources = # destinations.
_+_ 3. This is a "balanced" transportation problem.
_o_ 4. The "northwest corner method" is a special-purpose algorithm which gives the same result as the simplex algorithm.
_o_ 5. Every basic feasible solution of this problem is degenerate.
_+_ 6. If Plant 2 had 300 units of supply, rather than 200 units, the problem becomes "unbalanced".
_o_ 7. A transportation problem is a special case of an assignment problem.
_o_ 8. The "Hungarian" algorithm can be used to provide an initial basic feasible solution for the transportation problem above.
_+_ 9. Every basic feasible solution of an assignment problem is degenerate.
_o_10. When the transportation simplex algorithm encounters a degenerate solution, the next iteration will not improve the objective function.
_+_11. If 5 machines are to be assigned to 5 jobs, the assignment problem will have 25 variables and 10 linear equations.
_+_12. If no degenerate solution is encountered, the transportation simplex method gives an improvement in the objective function at every basis change.
_o_13. The simplex method applied to the assignment problem might yield non-integer (fractional) solutions.
_o_14. If a zero appears in row 1, column 1 of the cost matrix during row and column reduction in the Hungarian method, then a zero will occupy row 1, column 1 throughout the remaining iterations.
_+_15. If the dual variables of the above transportation problem are (for the sources) U=[0, 1] and (for the destinations) V= [8, 9, 10] , then the reduced costs of all the variables are nonnegative.
_o_16. The above transportation problem has five basic variables. *Note: the number of basic variables will be* $m+n-1 = 2+3-1 = 4$.

The statements below refer to the cost matrix:

Machine \ job	**1**	**2**	**3**	**4**	**5**
1	0	1	0	2	0
2	2	3	1	0	2
3	3	0	4	6	1
4	5	2	1	0	2
5	4	0	5	1	3

_+_17. This cost matrix could possibly result from the row and column reduction steps of the Hungarian method applied to some assignment cost matrix.
_+_18. After the next step of the Hungarian method, all of the elements occupied by zeroes in the above matrix will again be occupied by zeroes. *Note: Three lines are required to cover the zeroes, which intersect only on nonzero costs; therefore, in this situation, only nonzero costs will be increased by the next cost reduction.*
_o_19. After the next step of the Hungarian method, exactly one element which is currently nonzero will be occupied by a zero. *Note: The costs without lines will be reduced by the value 1, and therefore three new zeroes will appear.*
_o_20. The Hungarian method assumes that all costs are integers.

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Quiz #6 Solutions -- Fall 2001

1. ***Integer LP Model*** A court decision has stated that the enrollment of each high school in Metropolis be at least 20% black. The numbers of black and white high school students in each of the city's five school districts, and the distance (in miles) that a student in each district must travel to each high school are:

District	Whites	Blacks
1	80	30
2	70	5
3	90	10
4	50	40
5	60	30

District	HS#1	HS#2
1	1.0	2.0
2	0.5	1.7
3	0.8	0.8
4	1.3	0.4
5	1.5	0.6

School board policy requires that all students in a given district must attend the same school, and that each school must have an enrollment of at least 150 students. We wish to determine how to minimize the total distance that students must travel to high school. Define the binary decision variables

Xij = 1 if students in district *i* are assigned to HS#*j*, 0 otherwise

Put an "X" beside each of the constraints below which would be valid in the integer LP model.

X_ $X_{11}+X_{12}=1$

__ $X_{11}+X_{21}+X_{31}+X_{41}+X_{51}=1$

__ $X_{11}+X_{21}+X_{31}+X_{41}+X_{51}=2$

__ $\sum_{x=1}^{5}\sum_{j=1}^{2}X_{ij}=2$

__ $\sum_{x=1}^{5}\sum_{j=1}^{2}X_{ij}=150$

X_ $110X_{11}+75X_{21}+100X_{31}+90X_{41}+90X_{51}\geq 150$

__ $30X_{11}+5X_{21}+10X_{31}+40X_{41}+30X_{51}\leq 150$

__ $30X_{11}+5X_{21}+10X_{31}+40X_{41}+30X_{51}\geq 30$

__ $X_{11}+X_{21}+X_{31}+X_{41}+X_{51}\geq 150$

__ $30X_{11}+5X_{21}+10X_{31}+40X_{41}+30X_{51}\geq 150$

X_ $30X_{12}+5X_{22}+10X_{32}+40X_{42}+30X_{52}\geq 0.2\left(110X_{12}+75X_{22}+100X_{32}+90X_{42}+90X_{52}\right)$

__ $30X_{12}+5X_{22}+10X_{32}+40X_{42}+30X_{52}\geq 0.2\left(80X_{12}+70X_{22}+90X_{32}+50X_{42}+60X_{52}\right)$

2. Four trucks are available to deliver milk to five grocery stores. Capacities and daily operating costs vary among the trucks. Trucks are to make a single trip per day. Each grocery may be supplied by only one truck, but a truck may deliver to more than one grocery in one trip. No more than $100 may be spent on the trucks.

Truck #	Capacity (gallons)	Daily operating cost ($)
1	400	45
2	500	50
3	600	55
4	900	60

Grocery #	Daily demand (gallons)
1	100
2	200
3	300
4	500
5	800

Define binary variables

Y_i = 1 if truck i is used, 0 otherwise

X_{ij} = 1 if truck i delivers to grocery j, 0 otherwise

Put an "X" beside each of the constraints below which would be valid in the integer LP model.

__ $X_{41}+X_{42}+X_{43}+X_{44}+X_{45}=1$

X_ $X_{13}+X_{23}+X_{33}+X_{43}=1$

__ $X_{41}+X_{42}+X_{43}+X_{44}+X_{45}\leq Y_4$

X_ $X_{43}\leq Y_4$

__ $Y_4\leq X_{43}$

__ $300X_{43}\geq 900Y_4$

*_ $300X_{43}\leq 900Y_4$

X_ $45Y_1+50Y_2+55Y_3+60Y_4\leq 100$

*_ $X_{31}+X_{32}+X_{33}+X_{34}+X_{35}\leq 600Y_3$

__ $400X_{14}+500X_{24}+600X_{34}+900X_{44}\geq 500Y_4$

X_ $X_{41}+X_{42}+X_{43}+X_{44}+X_{45}\leq 5Y_4$

__ $X_{41}+X_{42}+X_{43}+X_{44}+X_{45}\leq 1100$

__ $X_{13}+X_{23}+X_{33}+X_{43}\leq 300Y_3$

__ $X_{13}+X_{23}+X_{33}+X_{43}\leq 4Y_3$

X_ $100X_{41}+200X_{42}+300X_{43}+500X_{44}+800X_{45}\leq 900Y_4$

__ $45Y_1+50Y_2+55Y_3+60Y_4=100$

*: these constraints will be valid, but are redundant, given the other constraints

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Quiz #7 Solution -- Fall 2001

(s,S) Model of Inventory System A periodic inventory replenishment system with reorder point **s**=**2** and order-up to level **S**=**5** is modeled. At the end of each period of demand (day), the inventory is tallied, and if the level is less than or equal to the reorder point (s), enough is ordered (& immediately received) so as to bring the inventory level up to S. The probability distribution is discrete and Poisson, with expected demand 2/day.

The state of the system is the stock-on-hand, i.e., 0 = stockout, 5 = full shelf.

The following output was obtained using the MARKOV workspace (APL code)

c_ 1. Over a long period of time, what is the percent of the days in which you would expect there to be a stockout (zero inventory)? *Choose nearest value!*

a. 5%	c. 15% ***13.36%***	e. 25%	g. 35%
b. 10%	d. 20%	f. 30%	h. >40%

f_ 2. How often (i.e. once every how many days?) will the inventory be full at the end of the day? *Nearest value!*

a. 2 days	c. 6 days	e. 10 days	g. 14 days
b. 4 days	d. 8 days	f. 12 days ***12.74 days***	h. >16 days

b_ 3. How often will the inventory be restocked? *That is, once how many days?*

Note: Probability of restocking is $\boxed{\sum_{j=0}^{2} \pi_j = 0.5015 = \frac{1}{1.994}}$

a. 1 days	c. 3 days	e. 5 days	g. 7 days
b. 2 days ***1.994 days***	d. 4 days	f. 6 days	h. >8 days

c_ 4. If the shelf is full Monday morning, what is the probability that a stockout occurs Friday evening?

a. 5%	c. 15% ***13.35%***	e. 25%	g. 35%
b. 10%	d. 20%	f. 30%	h. >40%

b_ 5. If the shelf is full Monday morning, what is the probability that the *first* stockout occurs Friday evening?

a. 5%	c. 15%	e. 25%	g. 35%
b. 10% ***8.469%***	d. 20%	f. 30%	h. >40%

g_ 6. What is the expected number of days, starting with a full inventory, until a stockout occurs?

a. 1 days	c. 3 days	e. 5 days	g. 7 days ***7.487 days***
b. 2 days	d. 4 days	f. 6 days	h. >8 days

b_ 7. Starting with a full inventory, what is the expected number of stockouts during the first 5 days?

a. 0.25	c. 0.75	e. 1.25	g. 1.75
b. 0.5 ***0.6011***	d. 1	f. 1.5	h. >2

True (+) or False(o)?

+_ 8. In the case of this Markov chain, the rows of the limiting matrix $\lim_{n\to\infty} P^n$ are identical.

+_ 9. The quantity denoted by $f_{ij}^{(n)}$ is a probability

o_10. The inequality $f_{ij}^{(n)} \geq p_{ij}^{(n)}$ is always valid. ***Note:*** $f_{ij}^{(n)} \leq p_{ij}^{(n)}$ ***is always true!***

+_11. The quantity $p_{ij}^{(n)}$ denotes the element in row i & column j of P^n

o_12. The inequality $f_{ij}^{(n)} \geq f_{ij}^{(n+1)}$ is always valid.

o_13. In a Markov chain, the state of the system has the Markov probability distribution.

o_14. For every Markov chain, a steady-state distribution exists.

+_15. The identity matrix is the transition probability matrix of some Markov chain.

o_16. If P is the transition probability matrix of a Markov chain, then the transpose of P is, also. ***Note: only if the column sums are each 1.0!***

o_17. The steadystate probability vector π satisfies $P\pi = 0$ ***Note: π satisfies*** π= π P

o_18. The quantity denoted by m_{ij} is a probability. ***Note: this is the expected value of the random variable*** N_{ij}.

+_19. The sum of each row of a transition probability matrix must always equal 1.0.

o_20. The quantity denoted by N_{ij} is a probability. ***Note: this is a random variable.***

Transition Probability Matrix

	0	1	2	3	4	5
0	0.05265	0.09022	0.1804	0.2707	0.2707	0.1353
1	0.05265	0.09022	0.1804	0.2707	0.2707	0.1353
2	0.05265	0.09022	0.1804	0.2707	0.2707	0.1353
3	0.3233	0.2707	0.2707	0.1353	0	0
4	0.1429	0.1804	0.2707	0.2707	0.1353	0
5	0.05265	0.09022	0.1804	0.2707	0.2707	0.1353

2^{nd} Power

	0	1	2	3	4	5
0	0.1503	0.1635	0.2293	0.234	0.1608	0.06207
1	0.1503	0.1635	0.2293	0.234	0.1608	0.06207
2	0.1503	0.1635	0.2293	0.234	0.1608	0.06207
3	0.08928	0.1146	0.1927	0.2524	0.234	0.117
4	0.1381	0.1513	0.2171	0.234	0.1791	0.08039
5	0.1503	0.1635	0.2293	0.234	0.1608	0.06207

3^{rd} Power

	0	1	2	3	4	5
0	0.1305	0.147	0.2161	0.239	0.1856	0.0819
1	0.1305	0.147	0.2161	0.239	0.1856	0.0819
2	0.1305	0.147	0.2161	0.239	0.1856	0.0819
3	0.1421	0.1569	0.2243	0.2365	0.1707	0.06951
4	0.1322	0.1486	0.2177	0.239	0.1831	0.07942
5	0.1305	0.147	0.2161	0.239	0.1856	0.0819

4^{th} Power

	0	1	2	3	4	5
0	0.1341	0.1501	0.2188	0.2383	0.1809	0.07788
1	0.1341	0.1501	0.2188	0.2383	0.1809	0.07788
2	0.1341	0.1501	0.2188	0.2383	0.1809	0.07788
3	0.1321	0.1483	0.2172	0.2387	0.1836	0.08023
4	0.1339	0.1499	0.2185	0.2383	0.1812	0.07821
5	0.1341	0.1501	0.2188	0.2383	0.1809	0.07788

5^{th} Power

	1	2	3	4	5	6
0	0.1335	0.1495	0.2183	0.2384	0.1817	0.0786
1	0.1335	0.1495	0.2183	0.2384	0.1817	0.0786
2	0.1335	0.1495	0.2183	0.2384	0.1817	0.0786
3	0.1338	0.1498	0.2185	0.2384	0.1812	0.07819
4	0.1335	0.1496	0.2183	0.2384	0.1816	0.07856
5	**0.1335**	0.1495	0.2183	0.2384	0.1817	0.0786

Expected no. of visits during first 5 stages

	0	1	2	3	4	5
0	0.6011	0.7003	1.063	1.22	0.9796	0.4358
1	0.6011	0.7003	1.063	1.22	0.9796	0.4358
2	0.6011	0.7003	1.063	1.22	0.9796	0.4358
3	0.8206	0.8403	1.123	1.101	0.7695	0.3449
4	0.6805	0.7798	1.142	1.22	0.8604	0.3166
5	**0.6011**	0.7003	1.063	1.22	0.9796	0.4358

Steady State Distribution

i	state	PI{i}
0	SOH=zero	**0.1336**
1	SOH=one	0.1496
2	SOH=two	0.2183
3	SOH=three	0.2384
4	SOH=four	0.1816
5	SOH=five	0.0785

n	$f_{5,0}^{(n)}$
1	0.05265
2	0.1476
3	0.1148
4	0.09898
5	**0.08469**
6	0.07244
7	0.06197
8	0.05302
9	0.04536
10	0.0388

Mean First Passage Time Matrix

	0	1	2	3	4	5
0	7.487	6.683	4.58	3.695	4.851	12.74
1	7.487	6.683	4.58	3.695	4.851	12.74
2	7.487	6.683	4.58	3.695	4.851	12.74
3	5.844	5.748	4.303	4.195	6.008	13.9
4	6.892	6.152	4.216	3.695	5.508	14.26
5	**7.487**	6.683	4.58	3.695	4.851	**12.74**

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Quiz #8 Solutions -- Fall 2001

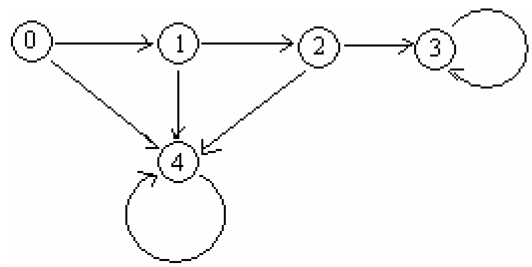
VERSION A: Coldspot manufactures refrigerators. The company has issued a warranty on all refrigerators that requires free replacement of any refrigerator that fails *before it is three years old.* Historical data yields the statistics:

- 4% of all new refrigerators fail during their first year of operation.
- 3% of all 1-year-old refrigerators fail during their second year of operation.
- 6% of all 2-year-old refrigerators fail during their third year of operation.

Under current policy, replacement refrigerators are *not* covered by the warranty!

Define stages to be annual, starting at the day of purchase of the refrigerator, where the state of the system is based upon its age or replacement status:

(0.) new refrigerators
(1.) 1-year-old refrigerators
(2.) 2-year-old refrigerators
(3.) refrigerators that have passed their third anniversary
(4.) replacement refrigerators



Note that, in this model, all ages for *past-warranty refrigerators* are lumped together, as well as all ages for *replacement refrigerators*!

P= transition probability matrix:

	0	1	2	3	4
0	0	0.96	0	0	0.04
1	0	0	0.97	0	0.03
2	0	0	0	0.94	0.06
3	0	0	0	1	0
4	0	0	0	0	1

Match the matrices (**a**, **b**, **c**, & **d**) below with the notation:

_d_ 1. E _a_ 2. Q _b_ 3. A _c_ 4. R

a	0	1	2
0	0	0.96	0
1	0	0	0.97
2	0	0	0

b	3	4
0	0.8753	**0.1247**
1	0.9118	0.0882
2	0.94	0.06

c	3	4
0	0	0.04
1	0	0.03
2	0.93	0.07

d	0	1	2
0	1	0.96	0.9312
1	0	1	0.97
2	0	0	1

_b_ 5. Which states are transient, and which are absorbing?

a. All are transient & none are absorbing b. States {0, 1, 2} are transient & {3, 4} are absorbing
c. All are absorbing & none are transient d. States {0, 1, 2} are absorbing & {3, 4} are transient
e. None of the above

_d_ 6. What fraction of the refrigerators will Coldspot expect to replace? *(Choose nearest value!)*

a. 6% c. 10% e. 14% g. 18%
b. 8% d. 12% $\boxed{a_{04}=12.47\%}$ f. 16% h. 20%

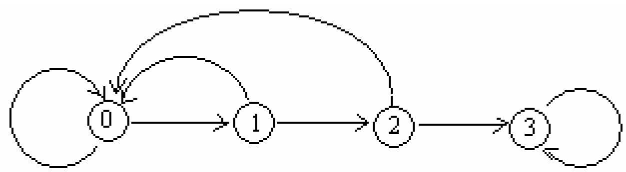
_d_ 7. What fraction of one-year-old refrigerators are expected to survive past the warranty period? *(Choose nearest value!)*

a. 88% c. 90% e. 92% g. 94%
b. 89% d. 91% $\boxed{a_{13}=91.18\%}$ f. 93% h. 95%

_d_ 8. If Coldspot's cost of replacing a refrigerator is $500, what is the expected replacement cost for each refrigerator sold? *Choose nearest value!*

a. $30 c. $50 e. $70 g. $90
b. $40 d. $60 $\boxed{12.47\% \times \$500}$ f. $80 h. $100

Coldspot is considering extending its warranty to cover any replacement refrigerators, and in order to estimate the additional replacement costs, has constructed the following model, where the states are defined as before:



P=transition probabilities

	0	1	2	3
0	0.04	0.96	0	0
1	0.03	0	0.97	0
2	0.06	0	0	0.94
3	0	0	0	1

A =

	3
0	1
1	1
2	1

E=

	0	1	2	row sum
0	1.142	1.097	1.064	3.303
1	0.1008	1.097	1.064	2.261
2	0.06855	0.0658	1.064	1.198

_c_ 9 . Which states are transient, and which are absorbing?

a. All are transient & none are absorbing
b. All are absorbing & none are transient
c. States {0, 1, 2} are transient & {3} is absorbing
d. States {0, 1, 2} are absorbing & {3} is transient
e. None of the above

_e_ 10. In addition to the original refrigerator, what is the expected number of new refrigerators each purchaser will own? (That is, how many times will the system return to state 0?) *(Choose nearest value!)*

a. 0.06
b. 0.08
c. 0.10
d. 0.12
e. $\boxed{0.14\ (e_{00}=1.142)}$
f. 0.16
g. 0.18
h. 0.20

_e_ 11. If Coldspot's cost of replacing a refrigerator is $500, what is the expected replacement cost for each refrigerator sold, under this policy*? (Choose nearest value!)*

a. $30
b. $40
c. $50
d. $60
e. $70 $\boxed{0.142\times\$500}$
f. $80
g. $90
h. $100

_a_ 12. An absorbing state of a Markov chain is one in which the probability of

a. moving out of that state is zero
b. moving out of that state is one.
c. moving into that state is one.
d. moving into that state is zero
e. *NOTA*

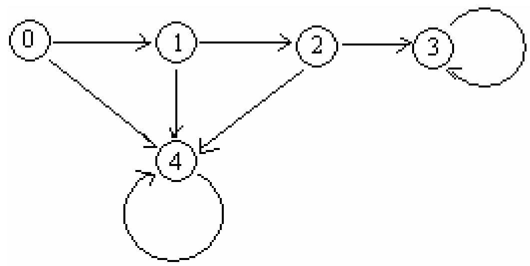
VERSION B: Coldspot manufactures refrigerators. The company has issued a warranty on all refrigerators that requires free replacement of any refrigerator that fails *before it is three years old.* Historical data yields the statistics:

- 2% of all new refrigerators fail during their first year of operation.
- 4% of all 1-year-old refrigerators fail during their second year of operation.
- 7% of all 2-year-old refrigerators fail during their third year of operation.

Under current policy, replacement refrigerators are *not* covered by the warranty!

Define stages to be annual, starting at the day of purchase of the refrigerator, where the state of the system is based upon its age or replacement status:

(0.) new refrigerators
(1.) 1-year-old refrigerators
(2.) 2-year-old refrigerators
(3.) refrigerators that have passed their third anniversary
(4.) replacement refrigerators



Note that, in this model, all ages for *past-warranty refrigerators* are lumped together, as well as all ages for *replacement refrigerators*!

P= transition probability matrix:

	0	1	2	3	4
0	0	0.98	0	0	0.02
1	0	0	0.96	0	0.04
2	0	0	0	0.93	0.07
3	0	0	0	1	0
4	0	0	0	0	1

Match the matrices (**a**, **b**, **c**, & **d**) below with the notation:

_a_ 1. Q _b_ 2. A _c_ 3. R _d_ 4. E

a	0	1	2
0	0	0.98	0
1	0	0	0.96
2	0	0	0

b	3	4
0	0.8749	0.1251
1	0.8928	0.1072
2	0.93	0.07

c	3	4
0	0	0.02
1	0	0.04
2	0.93	0.07

d	0	1	2
0	1	0.98	0.9408
1	0	1	0.96
2	0	0	1

_c_ 5. Which states are transient, and which are absorbing?
a. All are transient & none are absorbing
b. All are absorbing & none are transient
c. States {0, 1, 2} are transient & {3, 4} are absorbing
d. States {0, 1, 2} are absorbing & {3, 4} are transient
e. None of the above

_d_ 6. What fraction of the refrigerators will Coldspot expect to replace? *(Choose nearest value!)*
a. 6% b. 8% c. 10% d. 12% $\boxed{a_{04}=12.51\%}$ e. 14% f. 16% g. 18% h. 20%

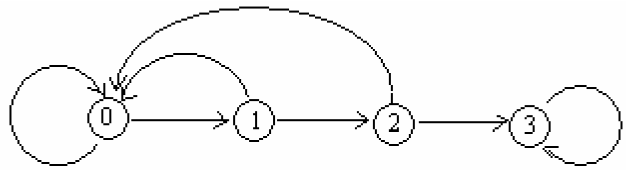
_b_ 7. What fraction of one-year-old refrigerators are expected to survive past the warranty period? *(Choose nearest value!)*
a. 88% b. 89% $\boxed{a_{13}=89.28\%}$ c. 90% d. 91% e. 92% f. 93% g. 94% h. 95%

_d_ 8. If Coldspot's cost of replacing a refrigerator is \$500, what is the expected replacement cost for each refrigerator sold? *Choose nearest value!*
a. \$30 b. \$40 c. \$50 d. \$60 $\boxed{12.51\%\times\$500}$ e. \$70 f. \$80 g. \$90 h. \$100

Coldspot is considering extending its warranty to cover any replacement refrigerators, and in order to estimate the additional replacement costs, has constructed the following model, where the states are defined as before:

P=transition probabilities

	0	1	2	3
0	0.02	0.98	0	0
1	0.04	0	0.96	0
2	0.07	0	0	0.93
3	0	0	0	1



A =

	3
0	1
1	1
2	1

E=

	0	1	2	row sum
0	1.143	1.12	1.075	3.338
1	0.1225	1.12	1.075	2.318
2	0.08001	0.07841	1.075	1.234

_b_ 9 . Which states are transient, and which are absorbing?
- a. All are transient & none are absorbing
- b. States {0, 1, 2} are transient & {3} is absorbing
- c. All are absorbing & none are transient
- d. States {0, 1, 2} are absorbing & {3} is transient
- e. None of the above

_e_ 10. In addition to the original refrigerator, what is the expected number of new refrigerators each purchaser will own? (That is, how many times will the system return to state 0?) *(Choose nearest value!)*
- a. 0.06
- b. 0.08
- c. 0.10
- d. 0.12
- e. 0.14 $\boxed{e_{00}=1.143}$
- f. 0.16
- g. 0.18
- h. 0.20

_e_ 11. If Coldspot's cost of replacing a refrigerator is $500, what is the expected replacement cost for each refrigerator sold, under this policy*? (Choose nearest value!)*
- a. $30
- b. $40
- c. $50
- d. $60
- e. $70 $\boxed{0.143\times\$500}$
- f. $80
- g. $90
- h. $100

_c_ 12. An absorbing state of a Markov chain is one in which the probability of
- a. moving out of that state is one
- b. moving into that state is one.
- c. moving out of that state is zero.
- d. moving into that state is zero
- e. *NOTA*

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Quiz #9 Solution -- Fall 2001

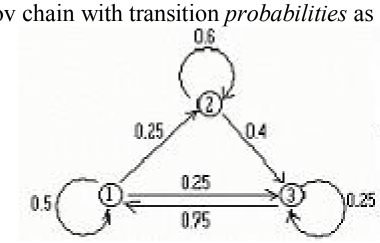
_c_1. Consider P, the matrix of transition probabilities of a *discrete-time* Markov chain. The sum of each...

a. column is 1 | c. row is 1 | e. none of the above
b. column is 0 | d. row is 0

_d__2. To compute the steady state distribution π of a discrete-time Markov chain, one must solve (in addition to sum of components of π equal to 1) the matrix equation (where P^t is the transpose of P):

a. $\pi P = 1$ | c. $P^t \pi = \pi$ | e. $\pi P = 0$
b. $P^t \pi = 1$ | d. $\pi P = \pi$ | f. none of the above

Consider the *discrete-time* Markov chain with transition *probabilities* as shown in the diagram:



$$P=\begin{bmatrix}0.5 & 0.25 & 0.25\\ 0 & 0.6 & 0.4\\ 0.75 & 0 & 0.25\end{bmatrix}$$

_a__ 3. All of the states of the above Markov chain are

a. recurrent | c. absorbing | e. none of the above
b. transient | d. null

4. Check the *two* equations which must be satisfied by the steadystate distribution π of the above Markov chain:

☒ $\pi_1=0.5\pi_1+0.75\pi_3$ | ☐ $\pi_1=0.5\pi_1+0.25\pi_2+0.25\pi_3$ | ☐ $0.5\pi_1+0.25\pi_2+0.25\pi_3 = 1$
☐ $0.5\pi_1 + 0.6\pi_2 + 0.25\pi_3 = 1$ | ☒ $\pi_1 + \pi_2 + \pi_3 = 1$ | ☐ $\pi_3=0.25\pi_1+0.4\pi_2$

❄ ❄ ❄ ❄ ❄ ❄ ❄ ❄ ❄ ❄

_d__5. For a continuous-time Markov chain, consider Λ, the matrix of transition rates. The sum of each...

a. column is 1 | c. row is 1 | e. none of the above
b. column is 0 | d. row is 0

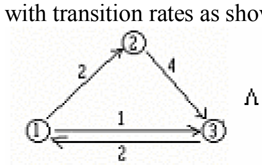
_e__6. To compute the steady state distribution π of a continuous-time Markov chain, one must solve (in addition to sum of components of π equal to 1) the matrix equation (where Λ^t is the transpose of Λ):

a. $\pi \Lambda = 1$ | c. $\Lambda^t \pi = \pi$ | e. $\pi \Lambda = 0$
b. $\Lambda^t \pi = 1$ | d. $\pi \Lambda = \pi$ | f. none of the above

_c__ 7. In the case of every continuous-time Markov chain which is currently in state i, the probability distribution of the time until the next transition occurs is

a. Markov | c. Exponential | e. Poisson
b. Binomial | d. Normal | f. none of the above

Consider the continuous-time Markov chain with transition rates as shown in the diagram, where time units are *hours*:



$$\Lambda=\begin{bmatrix}-3 & 2 & 1\\ 0 & -4 & 4\\ 2 & 0 & -2\end{bmatrix}$$

8. Check the *two* equations which must be satisfied by the steadystate distribution π of the above Markov chain:

☐ $3\pi_1+4\pi_2+2\pi_3 = 1$ | ☐ $\pi_2=2\pi_1-4\pi_2$ | ☐ $2\pi_1+4\pi_2 = 1$
☒ $\pi_1 + \pi_2 + \pi_3 = 1$ | ☒ $2\pi_1 - 4\pi_2 = 0$ | ☐ $\pi_3= \pi_1+4\pi_2$

_f__ 9. The value of λ_{11} in the transition rate matrix is

a. 0 | c. 2/hr | e. 4/hr
b. 1/hr | d. 3/hr | f. none of the above

_c__ 10. The value of λ_{12} in the transition rate matrix is

a. 0 | c. 2/hr | e. 4/hr
b. 1/hr | d. 3/hr | f. none of the above

_a__ 11. The average length of time that the above system spends in state 2 before making a transition is…

a. less than 1 hour ($^1/_4$ hr) | c. 2 hours | e. 4 hours
b. 1 hour | d. 3 hours | f. none of the above

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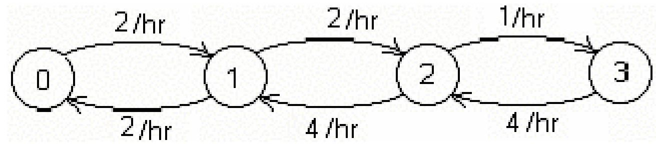
Quiz #10 Solution – Fall 2001

Consider the two-server queue with the birth-death model shown below:



_c_ 1. The steadystate probability that the queue is empty is π_0, where

a. $\frac{1}{\pi_0} = \frac{2\times2\times1}{2\times4\times4} = \frac{1}{8}$

b. $\frac{1}{\pi_0} = 1+\frac{2}{2}+\frac{2}{4}+\frac{1}{4} = \frac{11}{4}$

$\boxed{\text{c. } \frac{1}{\pi_0} = 1+\frac{2}{2}+\frac{2}{2}\times\frac{2}{4}+\frac{2}{2}\times\frac{2}{4}\times\frac{1}{4} = \frac{21}{8}}$

d. $\frac{1}{\pi_0} = \frac{2}{2}+\frac{2}{2}\times\frac{2}{4}+\frac{2}{2}\times\frac{2}{4}\times\frac{1}{4} = \frac{13}{8}$

e. $\frac{1}{\pi_0} = \frac{2}{2}+\frac{2}{4}+\frac{1}{4} = \frac{7}{4}$

f. *None of the above*

_a_ 2. The steadystate probability π_1 that one server is busy is equal to

$\boxed{\text{a. } \pi_0}$ c. $\frac{1}{2}\pi_0$ e. $\frac{1}{4}\pi_0$

b $2\pi_0$ d. $4\pi_0$ f. *None of the above*

_e_ 3. The average time spent by a customer in the system (including the time being served) is usually denoted by

a. λ c. μ $\boxed{\text{e. W}}$

b. N d. L f. *None of the above*

_c_ 4. *If* the average number of customers in the system is 0.9, and the average arrival rate is 1.7 per hour, then the average time spent by a customer in the system is (choose nearest value)

a. 0.25 hr $\boxed{\text{c. 0.5 hr (0.9/1.7 = 0.53)}}$ e. 0.75 hr

b. 1 hr d. 1.25 hr f. >1.5 hr

❒❒❒❒❒❒

Consider a capacity expansion planning problem similar to that in this week's homework assignment. (Costs are expressed in millions of dollars.) As in that homework assignment, at most three plants may be added in a year. The fixed cost for adding one or more plants is any year is **1.5**, and the marginal cost is **5.5** per plant (*same for all years*.) The discount factor to be used for computing present value is 0.9.
The numbers of additional plants needed, by year, are

Version (a):

Year	1	2	3	4	5	6
# add'l plants	1	2	4	5	7	8

Version (b):

Year	1	2	3	4	5	6
# add'l plants	1	2	4	6	7	8

That is, at the end of six years, eight plants (in addition to the current capacity) must be added. The stages are numbered in chronological order, i.e., stage 1 is the beginning stage, and stage 6 is the last stage.
The computational results in stages 5 & 6 are identical in both versions:

```
      ---Stage 6---
s \ x:  0       1   | Min
7  | 9999.99    7.00| 7.00
8  |    0.00 9999.99| 0.00
```

```
          ---Stage 5---
s \ x:  0      1      2      3   | Min
5  | 999.99 999.99  18.80  18.00| 18.00
6  | 999.99  13.30  12.50 999.99| 12.50
7  |   6.30   7.00 999.99 999.99|  6.30
8  |   0.00 999.99 999.99 999.99|  0.00
```

****

Version (a):

---Stage 4---

s \ x:	0	1	2	3	Min
4	999.99	23.20	23.75	23.67	$\boxed{23.20}$
5	16.20	18.25	18.17	18.00	16.20
6	11.25	12.67	12.50	999.99	11.25
7	5.67	7.00	999.99	999.99	5.67
8	0.00	999.99	999.99	999.99	0.00

---Stage 3---

s \ x:	0	1	2	3	Min
2	999.99	999.99	33.38	32.58	32.58
3	999.99	27.88	27.08	28.13	27.08
4	20.88	21.58	22.63	23.10	20.88
5	14.58	17.13	17.60	18.00	14.58
6	10.13	12.10	12.50	999.99	10.13

---Stage 2---

s \ x:	0	1	2	3	Min
1	999.99	36.32	36.87	36.79	36.32
2	29.32	31.37	31.29	31.12	29.32
3	24.37	25.79	25.62	27.11	24.37

---Stage 1---

s \ x:	0	1	2	3	Min
0	999.99	$\boxed{39.69}$	38.89	39.9	38.89

****

Version (b):

---Stage 4---

s \ x:	0	1	2	3	Min
4	999.99	999.99	23.75	23.67	$\boxed{23.67}$
5	999.99	18.25	18.17	18.00	18.00
6	11.25	12.67	12.50	999.99	11.25
7	5.67	7.00	999.99	999.99	5.67
8	0.00	999.99	999.99	999.99	0.00

---Stage 3---

s \ x:	0	1	2	3	Min
2	999.99	999.99	33.80	34.20	33.80
3	999.99	28.30	28.70	28.13	28.13
4	21.30	23.20	22.63	23.10	21.30
5	16.20	17.13	17.60	18.00	16.20
6	10.13	12.10	12.50	999.99	10.13

---Stage 2---

s \ x:	0	1	2	3	Min
1	999.99	37.42	37.81	37.17	37.17
2	30.42	32.31	31.67	32.58	30.42
3	25.31	26.17	27.08	27.11	25.31

---Stage 1---

s \ x:	0	1	2	3	Min
0	999.99	$\boxed{40.46}$	39.88	40.78	39.88

5. One value is missing in the table for **Stage 1** (i.e., the current year, in which 0 plants have already been added, and the decision is to add 1 plant). This value is

   version (a): _39.69__ = $(1.5+5.5) + 0.9 \times 36.32$ ***version (b):*** _40.46__ = $(1.5+5.5) + 0.9 \times 37.17$

6. One value is missing in the table for **Stage 4.** This value is

   version (a): _23.67__ ***version (b):*** _23.67__

Questions #7 – 9 are answered by doing a "forward pass" through the tables:

version (a):

$X_1=2, X_2=0, X_3=3, X_4=0, X_5=3, X_6=0$

version (b):

$X_1=2, X_2=0, X_3=2, X_4=3, X_5=0, X_6=1$

7. The optimal number of plants to add in the first year (1^{st} stage) is

   version (a): _2__ ***version (b):*** _2__

8. The optimal number of plants to add in the second year (2^{nd} stage) is

   version (a): _0__ ***version (b):*** _0__

9. The optimal number of plants to add in the final year (6^{th} stage) is

   version (a): _0__ ***version (b):*** _1__

10. The minimum total present value of the cost of adding the 8 plants is

   version (a): _38.89__ ***version (b):*** _39.88___

56:171 Operations Research
Quiz #11 Solutions – Fall 2001

1. ***Redistricting Problem*** A state is to be allocated **twenty** representatives (Reps) to be sent to the national legislature. There are **nine** districts within the state, whose boundaries are fixed. Every district should be assigned *at least one* representative. The allocation should be done according to the population (Pop) of the districts:

District	1	2	3	4	5	6	7	8	9
Population	47	52	67	41	61	99	16	68	35
Target α_n	1.93	2.14	2.76	1.69	2.51	4.07	0.66	2.80	1.44

The "target allocation" of district i is Reps times Pop[i] divided by the population of the state, but this target is generally non-integer. If each target is rounded to the nearest integer, **21** representatives are required (one more than has been allocated to the state). The objective is the assign the representatives to the districts in such a way that the *maximum absolute deviation from the targets is as small as possible*.

In the DP model, there is a stage for each district, and the state of the system is the number of representatives not yet assigned when we decide upon the allocation to district i. The optimal value function is defined by a forward recursion:

$$\begin{cases} f_n(s) = \underset{x\in\{1,2,3,4\}}{\text{minimum}} \ \max\left\{\left|\alpha_n - x\right|, f_{n+1}(s-x)\right\} \\ f_0(0) = 0 \ \& \ f_0(s) = +\infty \text{ for } s > 0 \end{cases}$$

That is, the optimal value function $f_n(s)$ at stage n with state s is the smallest possible value of the maximum absolute deviations from the targets α of the allocation to districts n, n+1, …. 9 if the total number of representatives available to those districts is given by the state s.

Version a:

District	1	2	3	4	5	6	7	8	9
Population	47	52	67	41	61	99	16	68	35
Target α_n	1.93	2.14	2.76	1.69	2.51	4.07	0.66	2.80	1.44

a. Compute the missing value in the table below for stage **3**. ***Solution***: Target is 2.76, and so, if s=18 & x=2, the value would be $\max\left\{\left|2.76-2\right|, f_4(18-2)\right\} = \max\left\{0.76, 1.20\right\} = 1.20$

b. There is one optimal solution to this problem. Complete the optimal allocations of representatives to districts. ***Solution:***

District	1	2	3	4	5	6	7	8	9
Allocation	2	2	3	2	2	4	1	3	1

```
Stage 9---
 s \ x: 1       2       3       4   | Min
 1  |    0.44 999.99 999.99 999.99| 0.44

Stage 8---
 s \ x: 1       2       3       4   | Min
 4  |    1.80    0.80    0.44 999.99| 0.44

Stage 7---
 s   \ x: 1       2       3       4   | Min
 5  |  0.44    1.34    2.34 999.99| 0.44
Stage 6---
  s \ x: 1       2       3       4   | Min
   9  |  3.07    2.07    1.07    0.44| 0.44

Stage 5---
  s \ x: 1       2       3       4   | Min
 11  |  1.51    0.51    0.80    1.49| 0.51
```

```
Stage 4---
 s \ x: 1       2       3       4   | Min
13  |  0.69    0.51    1.31    2.31| 0.51

Stage 3---
 s \ x: 1       2       3       4   | Min
16  |  1.76    0.76    0.51    1.24| 0.51
18  |  1.76   [1.20]   0.56    1.24| 0.56

Stage 2---
 s \ x: 1       2       3       4   | Min
18  |  1.14    0.51    0.86    1.86| 0.51

Stage 1---
 s \ x: 1       2       3       4   | Min
20  |  0.93    0.51    1.07    2.07| 0.51
```

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Version b:

District	1	2	3	4	5	6	7	8	9
Population	20	57	80	30	15	17	25	76	37
Target α_n	1.12	3.19	4.48	1.68	0.84	0.95	1.40	4.26	2.07

a. Compute the missing value in the table below for stage **3**. ***Solution***: Target is 4.48, and so, if s=18 & x=2, the value would be $\max\left\{\left|4.48-2\right|, f_4\left(18-2\right)\right\} = \max\left\{2.48, 1.32\right\} = 2.48$

b. There is one optimal solution to this problem. Complete the optimal allocations of representatives to districts. ***Solution:***

District	1	2	3	4	5	6	7	8	9
Allocation	1	3	4	2	1	1	2	4	2

```
Stage 9---
 s \ x: 1      2      3      4   | Min
 2  | 99.99   0.07 99.99 99.99| 0.07

Stage 8---
  s \ x: 1      2      3      4   | Min
 6  | 99.99   2.26   1.26   0.26| 0.26

Stage 7---
  s  \ x:1      2      3      4 | Min
  8  | 0.93   0.60   1.60   2.60| 0.60

Stage 6---
   s \ x:1      2      3      4 | Min
  9  | 0.60   1.05   2.05   3.05| 0.60

Stage 5---
  s \ x: 1      2      3      4   | Min
 10  | 0.60   1.16   2.16   3.16| 0.60
```

```
Stage 4---
  s \ x: 1     2      3      4   | Min
 12  | 0.93   0.60   1.32   2.32| 0.60

Stage 3---
  s \ x: 1     2      3      4   | Min
 16  | 3.48   2.48   1.48   0.60| 0.60
 18  | 3.48  [2.48]  1.48   1.05| 1.05

Stage 2---
  s \ x: 1     2      3      4   | Min
 19  | 2.19   1.19   0.60   0.81| 0.60

Stage 1---
  s \ x: 1     2      3      4   | Min
 20 | 0.60   0.88   1.88   2.88 | 0.60
```

❖❖❖❖❖❖❖❖❖❖❖❖❖❖❖❖❖❖❖❖❖

2. ***Stochastic Production Planning.*** The state of the system is the "inventory position", which if positive is the stock on hand, and if negative is the number of backorders which have accumulated, which is observed at the end of the business day. The decision is the number of units to be produced, which is no more than 3. There is a setup cost of $10 if any units are produced, plus $4 per unit. We assume that production is completed in time to meet any demand that occurs the next day. In addition, there is a storage cost of $1 per unit, based upon the end-of-day inventory, and a shortage cost of $15 per unit, based upon any backorders. Finally, at the end of the planning period (5 days), a salvage value of $2 per unit is received for any remaining inventory. If any shortage exists at the end of the planning period, enough must be produced to satisfy that shortage.
A *backward* recursion is used, where $f_n(s)$ is the minimum expected cost of the final n days of the planning period if the initial inventory position is s. Initially, with 5 days remaining (i.e., stage 5) there are 2 units in inventory, so that we wish to compute $f_5(2)$.

Below are the tables used to compute the optimal production policy.

Note: The data is the same as in the homework exercise, except that the probability distribution differs slightly!

Version a:
The demand is a discrete random variable with stationary distribution

D	0	1	2	3	4
P{D}	0.1	0.2	0.2	0.3	0.2

a. What is the missing value in the table for stage 1? _23.00__
   Computation: cost of last day is 1 *(storage)* + 18 *(production)* =19
   Expected remaining cost is $0.1(-2\times3)+0.2(-2\times2)+0.2(-2\times1)+0.3(0)+0.2(15+14) = 4$
b. What is the missing value in the table for stage 5? __96.49__
   Computation: cost of stage 5 is 2 *(storage)* + 18 *(production)* = 20
   Expected remaining cost is $0.1\times f_4(4)+0.2\times f_4(3)+0.2\times f_4(2)+0.3\times f_4(1)+0.2\times f_4(0)$
   $=0.1\times 60.35+0.2\times 68.32+0.2\times 73.17+0.3\times 79.35+0.2\times 91.78= 76.49$
c. What is the optimal production decision at the initial stage (stage 5)? _produce 3_
d. What is the minimum expected cost (total of production, storage, and shortage costs) for the 5-day period? _92.97_
e. Suppose that the demand in the first day (i.e., stage 5) is 1. What is the optimal production decision for day 2 (i.e., stage 4)? __produce 0__

Stage 1

s \ x:	0	1	2	3	Min
1	34.40	32.50	$\boxed{23.00}$	19.60	19.60

Stage 4

s \ x:	0	1	2	3	Min
0	152.80	137.74	110.82	91.78	91.78
1	124.74	107.82	88.78	79.35	79.35
2	94.82	85.78	76.35	73.17	73.17
3	72.78	73.35	70.17	68.32	68.32
4	60.35	67.17	65.32	63.73	60.35

Stage 5

s \ x:	0	1	2	3	Min
2	117.64	106.84	$\boxed{96.50}$	92.97	92.97

Version b:

The demand is a discrete random variable with stationary distribution

D	0	1	2	3	4
P{D}	0.2	0.2	0.2	0.2	0.2

a. What is the missing value in the table for stage 1? __22.40__

   Computation: cost of last day is 1 *(storage)* + 18 (production) = 19

   Expected terminal cost is $0.2(-2\times3)+0.2(-2\times2)+0.2(-2\times1)+0.2(0)+0.2(15+14) = 3.4$

b. What is the missing value in the table for stage 5? __87.94____

   Computation: cost of stage 5 is 2 (storage) + 18 (production) = 20

   Expected remaining cost is $0.2\times f_4(4)+0.2\times f_4(3)+0.2\times f_4(2)+0.2\times f_4(1)+0.2\times f_4(0)$

   $=0.2\times 53.86+0.2\times 62.08+0.2\times 66.84+0.2\times 72.86+0.2\times 84.08= 67.94$

c. What is the optimal production decision at the initial stage (stage 5)? _produce 3_

d. What is the minimum expected cost (total of production, storage, and shortage costs) for the 5-day period? _84.70__

e. Suppose that the demand in the first day (i.e., stage 5) is 2. What is the optimal production decision for day 2 (i.e., stage 4)? _produce 3_

```
Stage 1
    s\ x:0       1      2      3   | Min
   1 | 29.40 29.20 [22.40] 19.00| 19.00

Stage 4
s \ x:0       1      2      3   | Min
0 |127.06 118.01 97.86 84.08| 84.08
1 |105.01  94.86 81.08 72.86| 72.86
2 | 81.86  78.08 69.86 66.84| 66.84
3 | 65.08  66.86 63.84 62.08| 62.08
4 | 53.86  60.84 59.08 58.42| 53.86

Stage 5
  s \ x:0      1      2      3   | Min
  2 |101.33 96.54 [87.94] 84.70| 84.70
```

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Quiz #12 – Version A (P_{win}=45%) – Solution – Fall 2001

Casino Problem Consider the "Casino Problem" as presented in the lectures, but with **six plays** of the game, and the goal being to accumulate at least **five** chips, beginning with **2** chips, where the probability of winning at each play of the game is **only 45%.**
In the DP model with results presented below, the recursion is "forward", i.e., the stages range from n=1 (first play of the game) to n=6 (final play of the game). The state is the number of chips accumulated, and the decision is the number of chips to bet at the current play of the game.

a. Compute the missing number in the table for stage 1. _0.499__
Solution: $0.45\times f_2(4)+0.55\times f_2(2)=0.45\times 0.725+0.55\times 0.314 = 0.499$

b. What is the probability that five chips can be accumulated at the end of six plays of the game? ***Solution:*** _32.6_%

c. How many chips should be bet at the first play of the game?
Solution: __2__

d. If one bets the amount you selected in (c) and the first play of the game is won, what should be the bet at the second play of the game?
Solution: you would then have four chips, and should bet __1__

e. If the first play of the game is lost, what should be the bet at the second play of the game? ***Solution:*** you would have no chips remaining, and must bet none.

---Stage 6---

s \ x:	0	1	2	3	4	5	Max
0	0.000	XXXXXXXXXXXXXXXXXXXXXXXXXXXXX					0.000
1	0.000	0.000	XXXXXXXXXXXXXXXXXXXXXXX				0.000
2	0.000	0.000	0.000	XXXXXXXXXXXXXXXXX			0.000
3	0.000	0.000	**0.450**	**0.450**	XXXXXXXXXXX		0.450
4	0.000	**0.450**	**0.450**	**0.450**	**0.450**	XXXXX	0.450
5	**1.000**	0.450	0.450	0.450	0.450	0.450	1.000

---Stage 5---

s \ x:	0	1	2	3	4	5	Max
0	0.000	XXXXXXXXXXXXXXXXXXXXXXXXXXXXX					0.000
1	0.000	0.000	XXXXXXXXXXXXXXXXXXXXXXX				0.000
2	0.000	**0.203**	**0.203**	XXXXXXXXXXXXXXXXX			0.203
3	0.450	0.203	**0.450**	**0.450**	XXXXXXXXXXX		0.450
4	0.450	**0.698**	0.450	0.450	0.450	XXXXX	0.698
5	**1.000**	0.698	0.698	0.450	0.450	0.450	1.000

---Stage 4---

s \ x:	0	1	2	3	4	5	Max
0	0.000	XXXXXXXXXXXXXXXXXXXXXXXXXXXXX					0.000
1	0.000	**0.091**	XXXXXXXXXXXXXXXXXXXXXXX				0.091
2	0.203	0.203	**0.314**	XXXXXXXXXXXXXXXXX			0.314
3	**0.450**	0.425	0.450	**0.450**	XXXXXXXXXXX		0.450
4	**0.698**	**0.698**	0.561	0.450	0.450	XXXXX	0.698
5	**1.000**	0.834	0.698	0.561	0.450	0.450	1.000

---Stage 3---

s \ x:	0	1	2	3	4	5	Max
0	0.000	XXXXXXXXXXXXXXXXXXXXXXXXXXXXX					0.000
1	0.091	**0.141**	XXXXXXXXXXXXXXXXXXXXXXX				0.141
2	**0.314**	0.253	**0.314**	XXXXXXXXXXXXXXXXX			0.314
3	0.450	0.487	**0.500**	0.450	XXXXXXXXXXX		0.500
4	**0.698**	**0.698**	0.623	0.500	0.450	XXXXX	0.698
5	**1.000**	0.834	0.698	0.623	0.500	0.450	1.000

---Stage 2---

s \ x:	0	1	2	3	4	5	Max
0	0.000	XXXXXXXXXXXXXXXXXXXXXXXXXXXXX					0.000
1	**0.141**	**0.141**	XXXXXXXXXXXXXXXXXXXXXXX				0.141
2	**0.314**	0.303	**0.314**	XXXXXXXXXXXXXXXXX			0.314
3	0.500	0.487	0.528	0.450	XXXXXXXXXXX		0.528
4	0.698	**0.725**	0.623	0.528	0.450	XXXXX	0.725
5	**1.000**	0.834	0.725	0.623	0.528	0.450	1.000

---Stage 1---

s \ x:	0	1	2	3	4	5	Max
0	0.000	XXXXXXXXXXXXXXXXXXXXXXXXXXXXX					0.000
1	0.141	0.141	XXXXXXXXXXXXXXXXXXXXXXX				0.141
2	0.314	0.315	**0.326**	XXXXXXXXXXXXXXXXX			0.326
3	0.528	$\boxed{0.499}$	0.528	0.450	XXXXXXXXXXX		0.528
4	0.725	0.740	0.623	0.528	0.450	XXXXX	0.740
5	1.000	0.849	0.740	0.623	0.528	0.450	1.000

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Quiz #12 – Version B (P_{win}=40%) – Solution – Fall 2001

Casino Problem Consider the "Casino Problem" as presented in the lectures & HW, but with **six plays** of the game, and the goal being to accumulate at least **five** chips, beginning with **2** chips, where the probability of winning at each play of the game is **only 40%.**
*In the DP model with results presented below, the recursion is "**forward**", i.e., the stages range from n=1 (first play of the game) to n=6 (final play of the game). The state is the number of chips accumulated, and the decision is the number of chips to bet at the current play of the game.*

a. Compute the missing number in the table for stage 1. _0.419__
Solution: $0.4 \times f_2(4) + 0.6 \times f_2(2) = 0.4 \times 0.663 + 0.6 \times 0.256 = 0.419$

b. What is the probability that five chips can be accumulated at the end of six plays of the game? ***Solution:*** _26.5_%

c. How many chips should be bet at the first play of the game?
Solution: _2____

d. If one bets the amount you selected in (c) and the first play of the game is won, what should be the bet at the second play of the game?
Solution: One would then have 4 chips, and should bet 1

e. If the first play of the game is lost, what should be the bet at the second play of the game? ***Solution:*** you would have no chips remaining, and must bet none.

---Stage 6---

s \ x:	0	1	2	3	4	5	Max
0	0.000	XXXXXXXXXXXXXXXXXXXXXXXXXXXXX					0.000
1	0.000	0.000	XXXXXXXXXXXXXXXXXXXXXXX				0.000
2	0.000	0.000	0.000	XXXXXXXXXXXXXXXXX			0.000
3	0.000	0.000	**0.400**	**0.400**	XXXXXXXXXXX		0.400
4	0.000	**0.400**	**0.400**	**0.400**	**0.400**	XXXXX	0.400
5	**1.000**	0.400	0.400	0.400	0.400	0.400	1.000

---Stage 5---

s \ x:	0	1	2	3	4	5	Max
0	0.000	XXXXXXXXXXXXXXXXXXXXXXXXXXXXX					0.000
1	0.000	0.000	XXXXXXXXXXXXXXXXXXXXXXX				0.000
2	0.000	**0.160**	0.160	XXXXXXXXXXXXXXXXX			0.160
3	**0.400**	0.160	**0.400**	**0.400**	XXXXXXXXXXX		0.400
4	0.400	**0.640**	0.400	0.400	0.400	XXXXX	0.640
5	**1.000**	0.640	0.640	0.400	0.400	0.400	1.000

---Stage 4---

s \ x:	0	1	2	3	4	5	Max
0	0.000	XXXXXXXXXXXXXXXXXXXXXXXXXXXXX					0.000
1	0.000	0.064	XXXXXXXXXXXXXXXXXXXXXXX				0.064
2	0.160	0.160	**0.256**	XXXXXXXXXXXXXXXXX			0.256
3	**0.400**	0.352	**0.400**	**0.400**	XXXXXXXXXXX		0.400
4	**0.640**	**0.640**	0.496	0.400	0.400	XXXXX	0.640
5	**1.000**	0.784	0.640	0.496	0.400	0.400	1.000

---Stage 3---

s \ x:	0	1	2	3	4	5	Max
0	0.000	XXXXXXXXXXXXXXXXXXXXXXXXXXXXX					0.000
1	0.064	**0.102**	XXXXXXXXXXXXXXXXXXXXXXX				0.102
2	**0.256**	0.198	**0.256**	XXXXXXXXXXXXXXXXX			0.256
3	0.400	0.410	**0.438**	0.400	XXXXXXXXXXX		0.438
4	**0.640**	**0.640**	0.554	0.438	0.400	XXXXX	0.640
5	**1.000**	0.784	0.640	0.554	0.438	0.400	1.000

---Stage 2---

s \ x:	0	1	2	3	4	5	Max
0	0.000	XXXXXXXXXXXXXXXXXXXXXXXXXXXXX					0.000
1	**0.102**	**0.102**	XXXXXXXXXXXXXXXXXXXXXXX				0.102
2	**0.256**	0.237	**0.256**	XXXXXXXXXXXXXXXXX			0.256
3	0.438	0.410	**0.461**	0.400	XXXXXXXXXXX		0.461
4	0.640	**0.663**	0.554	0.461	0.400	XXXXX	0.663
5	**1.000**	0.784	0.663	0.554	0.461	0.400	1.000

---Stage 1---

s \ x:	0	1	2	3	4	5	Max
0	0.000	XXXXXXXXXXXXXXXXXXXXXXXXXXXXX					0.000
1	0.102	0.102	XXXXXXXXXXXXXXXXXXXXXXX				0.102
2	0.256	0.246	**0.265**	XXXXXXXXXXXXXXXXX			0.265
3	0.461	$\boxed{0.419}$	0.461	0.400	XXXXXXXXXXX		0.461
4	0.663	0.677	0.554	0.461	0.400	XXXXX	0.677
5	1.000	0.798	0.677	0.554	0.461	0.400	1.000