Bounds for Stop-Loss Premium Under Restrictions on the Chi-Square Statistic

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Abstract

Stop-loss reinsurance is one type of reinsurance contract that has attracted recent attention. In the simplest form of this contract, a reinsurer agrees to pay all losses of the insurer in excess of an agreed limit. This paper concerns the computation of bounds on the stop–loss premium when the loss distribution is unknown, but information about past claim experience is available in the form of frequencies of claims in each of a set of intervals. We use a dual approach to calculate the bounds on the premium by placing limits on the chi–square statistic or modified chi-square statistic as measures of the proximity of the loss distribution to the empirical distribution. This approach requires optimization with respect to a single dual variable if the chi-square statistic is

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restricted. On the other hand, when the modified chi-square statistic is restricted, the bounds can be given in closed form.

Key Words: Lagrangian dual, chi–square goodness–of–fit, Pearson's goodness–of–fit, chi–square statistic, modified chi–square statistic, stop–loss premium.

1 Introduction

An insurance company seeks to reduce the probability of suffering catastrophic losses by means of reinsurance. If possible, the company would like to reduce the possibility of these losses to zero. When seeking reinsurance, the company must choose between high profits and relatively high risk on the one hand, and modest profits and high security on the other. To obtain a degree of security, the company has to pay by buying this security on the reinsurance market. The degree of security the company wishes to buy at prevailing prices will depend on the company's objectives and overall policies.

If excess-of-loss insurance or stop-loss reinsurance with deductible t is purchased, the amount paid by the reinsurer to the ceding insurer is

$$I_t = \begin{cases} 0 & \text{if } X \leq t \\ X - t & \text{if } X > t, \end{cases} \tag{1.1}$$

where X, a random variable, is the aggregate claim. This class of policies is characterized by the fact that the claim payments do not start until the loss exceeds the deductible amount, t (the *retention*). The amount of claims retained by the ceding insurer is

$$X - I_t = \begin{cases} X & \text{if } X \leq t \\ t & \text{if } X > t. \end{cases} \tag{1.2}$$

The amount retained, therefore, is bounded by t, explaining the name "stop–loss contract", and (referring to t) the "retention".

When the deductible is t, the *net stop–loss reinsurance premium* $E_F[I_t]$ is

$$E_F[I_t] = \int_t^{\infty} (x - t)\, dF(x), \tag{1.3}$$

where F is the claim distribution. In practice, however, the company will have only incomplete information about this distribution; perhaps only a mean and variance calculated from past claims experience, or perhaps an empirical distribution of claim frequencies in a set of intervals $[a_{j-1}, a_j)$ $j = 1, ...n$. We denote by $\mathcal{F}$ the family of distribution functions consistent with the known properties of the claim distribution. In trying to transfer their risk, the company is interested in placing an upper bound on $E_F[I_t]$. This is the "worst case" stop–loss premium consistent with their information about the claim distribution function. The company may also wish to bound the stop-loss premium from below. This determines the "best case" stop–loss premium consistent with their information about the claim distribution function.

In this paper, we consider computing upper and lower bounds on the *stop–loss premium* with deductible t, when either the *chi–square statistic* or the *modified chi–square statistic*, which measure the discrepancy between the expected frequency and observed claim frequency, have been assigned upper limits. In both cases, we apply the *Lagrangian relaxation method*. It will be shown in Section 2 that the extremal distributions yielding the maximum and minimum stop–loss premiums are discrete.

In Section 3, chi–square and modified chi–square statistics will be reviewed along with the empirical distribution of claims. Bounds will be derived in Section 4 and 5 and an example will be provided to illustrate the computation. Proofs of all results are presented in the Appendix.

2 The Extremal Distributions Yielding Maximum and Minimum Stop–Loss Premiums

Several bounds on the stop–loss premium $E_F[I_t]$ have appeared in the literature for situations in which only partial information is known about the claim distribution F, e.g., the first two moments. Bowers [1, Bowers,1969] showed that, when mean μ and variance σ^2 are known,

$$\max_F E_F[I_t] = \frac{\sigma}{2} \frac{1}{K + \sqrt{1 + K^2}}, \tag{2.1}$$

where $K = \frac{t-\mu}{\sigma}$. This maximum is attained by the discrete extremal distribution

$$\begin{aligned} P[X = \mu + \sigma \left(K - \sqrt{1 + K^2}\right)] &= \frac{\left(K + \sqrt{1 + K^2}\right)^2}{1 + \left(K + \sqrt{1 + K^2}\right)^2} \\ P[X = \mu + \sigma \left(K + \sqrt{1 + K^2}\right)] &= \frac{1}{1 + \left(K + \sqrt{1 + K^2}\right)^2}. \end{aligned} \tag{2.2}$$

Kemperman [4, Kemperman,1987] provided closed–form upper and/or lower bounds on the stop-loss premium $E_F[I_t]$ with various restrictions on the claim distribution F. Cox [2, Cox,1991] extended this result to the case in which, in addition to μ and σ^2 being known, an upper limit b is placed on the liability of the reinsurer. Under these conditions, the upper and lower bounds on $E_F[I_t]$ are, respectively,

$$\max_F E_F[I_t] = \begin{cases} \frac{\mu\left(\sigma^2+\mu^2-t\mu\right)}{\sigma^2+\mu^2} & 0 \le t \le \frac{\sigma^2+\mu^2}{2\mu} \\ \frac{\mu-t+\sqrt{(\mu-t)^2+\sigma^2}}{2} & \frac{\sigma^2+\mu^2}{2\mu} < t \le \frac{b^2-\sigma^2-\mu^2}{2(b-\mu)} \\ \frac{(b-t)\sigma^2}{(b-\mu)^2+\sigma^2} & \frac{b^2-\sigma^2-\mu^2}{2(b-\mu)} < t \le b \end{cases} \tag{2.3}$$

and

$$\min_F E_F[I_t] = \begin{cases} \mu - t & 0 \le t \le \mu - \frac{\sigma^2}{b-\mu} \\ \frac{\sigma^2+\mu^2-t\mu}{b} & \mu - \frac{\sigma^2}{b-\mu} < t \le \mu + \frac{\sigma^2}{\mu} \\ 0 & \frac{\sigma^2}{\mu} < t \le b. \end{cases} \tag{2.4}$$

In these cases, the extremal distributions are also discrete, with probability massed at two points.

In this paper we concern ourselves with the family of distribution functions $\mathcal{F}$ consisting of distributions for which $P\{a_{i-1} \le X < a_i\} = q_i$, $i = 1, 2, \ldots, n$. X is the loss, a random variable, and the a_i's are nonnegative real numbers which form an increasing sequence, i.e. $a_0 < a_1 < a_2 < \ldots < a_n$. The set of probabilities $q_i, i = 1, 2, \ldots, n$, form a probability function, that is $0 \le q_i \le 1$, and $\sum_{i=1}^n q_i = 1$.

The following two lemmas were presented and proved in our earlier paper [5, Xu, Bricker, and Kortanek, 1998]. According to these lemmas, the extremal distribution function will have probability massed at n points. Furthermore, the number of points at which the probability is massed will be the same as the number of intervals determining the empirical claims frequency distribution.

Lemma 2.1 *The distribution function F in the family $\mathcal{F}$ which maximizes the expectation $E_F(I_t)$ given by (1.3) is discrete, with probability q_i massed at a_i, the right endpoint of interval $[a_{i-1}, a_i)$, i = 1,2,...,n.*

Lemma 2.2 *The distribution function F in the family $\mathcal{F}$ which minimizes the expectation $E_F(I_t)$ given by (1.3) is discrete, with probability q_i massed at a_{i-1}, the left endpoint of* *interval* $[a_{i-1}, a_i)$, i = *1,2,...,n.*

3 Empirical Observations and Chi–Square or Modified Chi–Square Statistics

For this paper we assume that empirical observations based on past claim experience are available. These observations take the form of observed frequencies in the intervals $[a_{i-1}, a_i)$ for $i = 1, 2, ..., n$. We assume that this interval partition has at least one observation in each interval. We have shown in the previous section that under these assumptions, the search for extremal distributions in $\mathcal{F}$ can be restricted to those having mass only on the endpoints of the given intervals.

The *chi–square statistic*, *chi–square goodness–of–fit*, or *Pearson's goodness–of–fit* is defined as the statistic

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - Nq_i)^2}{Nq_i} \tag{3.1}$$

where O_i is the observed number of losses in the interval $[a_{i-1}, a_i)$ for $i = 1, 2, ..., n$. N is the total number of observations, and $Nq_i = N[F(a_i) - F(a_{i-1})]$ is the expected number of losses in group i, the interval $[a_{i-1}, a_i)$. These classes could be those of the fitted ogive, which is a continuous piecewise–linear function. The test statistic, χ^2, is approximately distributed as a *chi–square* distribution with a number of degrees of freedom one less that the number of groups. Since the Nq_i in the denominator may be difficult to evaluate, statisticians sometimes use the *modified chi–square* statistic which has the same asymptotic properties as the chi–square statistic. This modified chi–square statistic [3, Hogg and Klugman,1984] is defined to be

$$\chi'^2 = \sum_{i=1}^{n} \frac{(O_i - Nq_i)^2}{O_i}. \tag{3.2}$$

We will imposes bounds on either χ^2 or χ'^2 in this paper as a constraint. Bounds on the stop–loss premium will be calculated subject to either the restriction $\sum_i \left(\frac{(Nq_i - O_i)^2}{Nq_i}\right) \leq c$, or $\sum_i \left(\frac{(Nq_i - O_i)^2}{O_i}\right) \leq c$, for some $c > 0$. (Details are in Sections 4 and 5.)

Also note that the bound on the chi-square statistic can be rewritten as

$$\sum_{i=1}^{n} \frac{O_i^2}{q_i} \leq B, \tag{3.3}$$

where $B = N(C + N)$. The bound on the modified chi-square statistic can be rewritten

$$\sum_{i=1}^{n} \frac{q_i^2}{O_i} \leq B', \tag{3.4}$$

where $B' = \frac{C+N}{N^2}$. Note that the functions of q_i on the left sides of (3.3) and (3.4) are both convex, so that the inequalities above define convex feasible regions in the non–negative orthant of R^n. This property is very important from a computational point of view.

4 Restricting the Chi–square Statistic

We use a Lagrangian approach to compute the bounds on the stop-loss premium subject to the restriction $\chi^2 \leq c$. We define a Lagrangian dual objective function as the optimum of a Lagrangian function for given values of the Lagrangian multipliers of the relaxed constraints. The Lagrangian dual problem can be reduced to an optimization with respect to a single dual variable.

We wish to compute upper and lower bounds on the stop–loss premium when the chi-square statistic of the loss distribution with respect to the observed losses is bounded by a constant $c > 0$.

According to Lemma 2.1 and Lemma 2.2, the extremal distributions which yield the upper bound and the lower bound must be discrete distributions $x = (x_1, x_2, ..., x_n)$, where

$$x_i = P\{X = a_i\} \text{ and } x_i = P\{X = a_{i-1}\} \quad i = 1, 2, ..., n,$$

for upper and lower bounds respectively. Therefore, we restrict our search to discrete distributions of this form, and find bounds on the stop–loss premium by solving the following optimization problem,

$$\begin{aligned} \min_x \sum_{i=1}^{n} \ & \alpha_i x_i \\ \text{subject to } \frac{1}{B} \sum_{i=1}^{n} \frac{O_i^2}{x_i} & \leq 1 \\ \sum_{i=1}^{n} x_i & = 1 \\ x_i \geq 0 \ \ i & = 1, 2, ..., n, \end{aligned} \tag{4.1}$$

where

$$\alpha_i = \begin{cases} -(a_i - t)_+ & \text{for the upper bounding case} \\ (a_{i-1} - t)_+ & \text{for the lower bounding case,} \end{cases} \tag{4.2}$$

for $i = 1, 2, ..., n$. The function $(x - t)_+$ is defined as $\max_x(x - t, 0)$. Note that $x_i > 0$, for any $1 \leq i \leq n$, since if there is an i, $1 \leq i \leq n$, such that $x_i = 0$, then we must have $O_i = 0$, so that $\frac{(O_i - N x_i)^2}{N x_i} = 0$. Otherwise the constraint (3.1) will be violated. If $O_i = 0$, there are no observations in the interval $[a_{i-1}, a_i)$ and readjustment of the endpoints is needed. Finally note that the upper bounding objective is the negative of the objective of (4.1).

Our primal problem is to minimize a convex (in fact linear) objective function over a convex region. Introducing the Lagrangian multipliers μ and λ for the constraints above, we obtain the Lagrangian function,

$$L^\kappa(x,\mu,\lambda) = \sum_{i=1}^{n} \alpha_i x_i + \mu \left(\sum_{i=1}^{n} (O_i^2/B) x_i^{-1} - 1 \right) + \lambda \left(\sum_{i=1}^{n} x_i - 1 \right), \tag{4.3}$$

where $x = (x_1, x_2, ..., x_n)^t > 0,\ \ \lambda \in R\ \ \mu \geq 0$, and

$$\kappa = \begin{cases} U & \text{for the upper bounding case,} \\ L & \text{for the upper bounding case.} \end{cases} \tag{4.4}$$

The dual optimization will be

$$\max_{\mu \geq 0, \lambda} h^\kappa(\mu,\lambda) \tag{4.5}$$

with the dual objective function h^κ defined as the value of the Lagrangian relaxation

$$h^\kappa(\mu,\lambda) = \min_{x>0} L^\kappa(x,\mu,\lambda). \tag{4.6}$$

The following lemma, whose proof appears in the Appendix, provides us a closed-form expression for the dual objective function.

Lemma 4.1 *The Lagrangian dual objective function (of problem (4.1)) is*

$$h^\kappa(\mu,\lambda) = \begin{cases} 2\sqrt{\frac{\mu}{B}} \sum_{i=1}^{n} O_i \sqrt{\lambda + \alpha_i} - (\lambda + \mu) & \textit{for } \lambda \in G^\kappa \\ -\infty & \textit{for } \lambda \in \overline{G^\kappa}, \end{cases} \tag{4.7}$$

where $\kappa \in \{U, L\}$, $\overline{G^\kappa}$ is the complement of G^κ, and G^κ is defined as the following :

$$G^\kappa = \begin{cases} (a_n - t, +\infty) & \text{for } \kappa = U \\ [0, +\infty) & \text{for } \kappa = L. \end{cases} \tag{4.8}$$

To maximize the Lagrangian dual objective function $h^\kappa(\mu, \lambda)$ in (4.7), we may restrict our search to $\lambda \in G^\kappa$. The partial derivatives of $h^\kappa(\mu, \lambda)$ with respect to μ and λ are respectively,

$$\frac{\partial h^\kappa(\mu, \lambda)}{\partial \mu} = \frac{1}{\sqrt{\mu B}} \sum_{i=1}^{n} O_i \sqrt{\lambda + \alpha_i} - 1 \tag{4.9}$$

$$\frac{\partial h^\kappa(\mu, \lambda)}{\partial \lambda} = \sqrt{\frac{\mu}{B}} \sum_{i=1}^{n} \frac{O_i}{\sqrt{\lambda + \alpha_i}} - 1. \tag{4.10}$$

Setting $\frac{\partial h^\kappa(\mu,\lambda)}{\partial \mu} = 0$, and solving for μ yields

$$\mu = \frac{1}{B} \left(\sum_{i=1}^{n} O_i \sqrt{\lambda + \alpha_i} \right)^2, \tag{4.11}$$

which is nonnegative, i.e., feasible, for $\lambda \in G^\kappa$. Substituting (4.11) into (4.7) to eliminate the dual variable μ, we obtain

$$h^\kappa(\lambda) = \begin{cases} \frac{1}{B} \left(\sum_{i=1}^{n} O_i \sqrt{\lambda + \alpha_i} \right)^2 - \lambda & \text{for } \lambda \in G^\kappa \\ -\infty & \text{for } \lambda \in \overline{G^\kappa}. \end{cases} \tag{4.12}$$

In terms of the single variable λ, the expressions for the dual objective function for the upper bound (negative of $h^\kappa(\lambda)$) and lower bound will be as follows respectively,

$$-h^U(\lambda) = -\frac{1}{B} \left(\sqrt{\lambda} \sum_{i<k} O_i + \sum i \geq k O_i \sqrt{\lambda - (a_i - t)} \right)^2 + \lambda \quad \text{for } \lambda > a_n - t \tag{4.13}$$

$$h^L(\lambda) = \frac{1}{B} \left(\sqrt{\lambda} \sum_{i \leq k} O_i + \sum i > k O_i \sqrt{\lambda + (a_{i-1} - t)} \right)^2 - \lambda \qquad \text{for } \lambda \geq 0. \tag{4.14}$$

The Lagrangian dual objective function we have obtained above is convex in the one variable λ, so that the maximization of the Lagrangian dual objective is an optimization problem with no constraint except for lower bounds. Lower bounds of $a_n - t$ for the upper bounding problem and 0 for the upper bounding problem on the dual variable λ. Therefore, we want to solve the optimization problem

$$\max_{\lambda \in G^\kappa} h^\kappa(\lambda). \tag{4.15}$$

Since our primal problem is to minimize a convex objective function subject to convex constraints (as we mentioned earlier), there is no duality gap between primal and dual problems. The optimal value of the dual problem (4.15) will be the optimal value of the primal problem (4.1).

The dual problem (4.15) can be easily solved by a one–dimensional search method (e.g., Newton's method) modified to enforce the lower bound on the dual variable λ_κ for $\kappa = U$ or L.

If λ_κ^* for $\kappa = U$ or L are the optimal values of the dual variables, then the optimal primal solutions, i.e., the extremal distributions, will be

$$x_i^U = \begin{cases} O_i\sqrt{\frac{\mu_U^*}{B\lambda_U^*}} & i < k \\ O_i\sqrt{\frac{\mu_U^*}{B(\lambda_U^* - (a_i - t))}} & i \geq k, \end{cases} \tag{4.16}$$

for the upper bounding problem, where μ_U^* is given by (4.11) when $\kappa = U$ as shown in the proof of Lemma 4.1, and where k is the smallest index such that $a_k > t$. Similarly for lower bounding we have,

$$x_i^L = \begin{cases} O_i\sqrt{\frac{\mu_L^*}{B\lambda_L^*}} & i \leq k \\ O_i\sqrt{\frac{\mu_L^*}{B(\lambda_L^* + (a_{i-1} - t))}} & i > k, \end{cases} \tag{4.17}$$

where μ_L^* is given by equation (4.11) when $\kappa = L$. (The derivation of (4.17) appears in the proof of Lemma 4.1.)

5 Restricting the Modified Chi-Square Statistic

We will next consider the upper and lower bounds on the stop–loss premium over all distributions whose modified chi–square statistic does not exceed a positive quantity c. As in Section 4, we will use the Lagrangian dual approach as a computational tool.

The Modified Chi–square statistic of a discrete distribution x is defined to be

$$\chi'^2(x) = \sum_{i=1}^{n} \frac{(Nx_i - O_i)^2}{O_i}. \tag{5.1}$$

Bounds on the stop–loss premium will be provided by solving the optimization problem,

$$\begin{aligned} & \min \quad \sum_{i=1}^{n} \alpha_i x_i \\ \text{subject to} \quad & \sum_{i=1}^{n} \frac{(Nx_i - O_i)^2}{O_i} \leq c \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, \end{aligned} \tag{5.2}$$

where α_i is defined the same as (4.2) in Section 4. Note that the objective function of (5.2) is the same as (4.1), so the optimal objective value of the upper bounding problem is the negative of min $\sum_{i=1}^{n} -(a_i - t)_+ x_i$.

The Lagrangian function of the problem (5.2) will be

$$L^\kappa(x, \mu, \lambda) = \sum_{i=1}^{n} \alpha_i x_i + \mu \left(\sum_{i=1}^{n} \frac{(Nx_i - O_i)^2}{O_i} - c \right) + \lambda \left(\sum_{i=1}^{n} x_i - 1 \right), \tag{5.3}$$

where $x \geq 0$, $\mu \geq 0$, $\lambda \in R$, and $\kappa \in \{U, L\}$ is defined the same as (4.4) in Section 4.

The Lagrangian dual of problem (5.2) will be

$$\max_{\mu \geq 0, \lambda} \quad h^\kappa(\mu, \lambda) \tag{5.4}$$

where $h^\kappa(\mu, \lambda) = \min_{x \geq 0} L^\kappa(x, \mu, \lambda)$.

A closed–form expression for $h^\kappa(\mu, \lambda)$ is provided by the lemma below.

Lemma 5.1 *The Lagrangian dual objective function $h^\kappa(\mu, \lambda)$ will be*

$$\begin{aligned} h^\kappa(\mu, \lambda) &= \frac{-\lambda}{2N^2\mu}\sum_{i=1}^{n} O_i\alpha_i + \frac{1}{N}\sum_{i=1}^{n} O_i\alpha_i - \frac{1}{4N^2\mu}\sum_{i=1}^{n} O_i\alpha_i^2 - (\mu c + \frac{\lambda^2}{4N\mu}) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for } \lambda \in G^\kappa \qquad (5.5) \\ &= -\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \textit{otherwise}, \qquad (5.6) \end{aligned}$$

where G^κ is a feasible set for λ, and is given as follows

$$G^\kappa = \begin{cases} (-\infty, 2N\mu] & \textit{for } \kappa = U \\ (-\infty, 2N\mu - (a_{n-1} - t)] & \textit{for } \kappa = L. \end{cases} \tag{5.7}$$

We can obtain the required lower bound on the stop–loss premium by findint the optimum of the following Lagrangian dual problem, namely

$$\max_{\mu \geq 0, \lambda \in G^\kappa} \quad h^\kappa(\mu, \lambda). \tag{5.8}$$

The partial derivatives of h^κ with respect to μ and λ are given by

$$\begin{aligned} \frac{\partial h^\kappa(\mu, \lambda)}{\mu} &= \frac{1}{4N^2\mu^2}\sum_{i=1}^{n} O_i\alpha_i^2 - c + \frac{\lambda}{2N^2\mu^2}\sum_{i=1}^{n} O_i\alpha_i + \frac{\lambda^2}{4N\mu^2} - c \\ \frac{\partial h^\kappa(\mu, \lambda)}{\lambda} &= -\frac{1}{2N^2\mu}\sum_{i=1}^{n} O_i\alpha_i - \frac{\lambda}{2N\mu}. \end{aligned} \tag{5.9}$$

Setting the partial derivative (5.9b) equal to zero yields the following stationary point for λ

$$\lambda = -\frac{1}{N}\sum_{i=1}^{n} O_i\alpha_i.$$

Substituting this expression into (5.9a) and setting the resultant expression equal to zero yields

$$\begin{aligned}
\mu^2 &= \frac{1}{4cN^2}\left[\lambda^2 N + 2\lambda\sum_{i=1}^{n} O_i\alpha_i + \sum_{i=1}^{n} O_i\alpha_i^2\right] \\
&= \frac{1}{4cN^2}\left[-\frac{1}{N}\left(\sum_{i=1}^{n} O_i\alpha_i\right)^2 + \sum_{i=1}^{n} O_i\alpha_i^2\right].
\end{aligned}$$

Thus we obtain the stationary point (μ, λ),

$$\begin{aligned}
\mu &= \frac{1}{2N\sqrt{c}}\sqrt{-\frac{1}{N}\left(\sum_{i=1}^{n} O_i\alpha_i\right)^2 + \sum_{i=1}^{n} O_i\alpha_i^2} \\
\lambda &= -\frac{1}{N}\sum_{i=1}^{n} O_i\alpha_i.
\end{aligned} \tag{5.10}$$

Lemma 5.2 *The optimal dual objective value of the problem (5.2) will be*

$$h^{\kappa^*} = \frac{1}{N}\left(\sum_{i=1}^{n} O_i\alpha_i\right) - \frac{\sqrt{c}}{N}\sqrt{\sum_{i=1}^{n} \alpha_i^2 O_i - \frac{1}{N}\left(\sum_{i=1}^{n} \alpha_i O_i\right)^2}. \tag{5.11}$$

Expressing the optimal dual objective values for the upper bound (negative of h^{U*}) and the lower bound separately, we have,

$$-h^{U*} = \frac{1}{N}\sum_{i\geq k}(a_i - t)O_i + \frac{\sqrt{c}}{N}\sqrt{\sum_{i\geq k}(a_i - t)^2 O_i - \frac{1}{N}\left(\sum_{i\geq k}(a_i - t)O_i\right)^2}, \tag{5.12}$$

$$h^{L^*} = \frac{1}{N}\left(\sum_{i>k} O_i(a_{i-1} - t)\right) - \frac{\sqrt{c}}{N}\sqrt{\sum_{i>k}(a_{i-1} - t)^2 O_i - \frac{1}{N}\left(\sum_{i>k}(a_{i-1} - t)O_i\right)^2}. \tag{5.13}$$

While the bounds on the stop–loss premium with a restriction on the regular chi–square statistic may be computed numerically by an iterative one–dimensional search procedure,

Amount of loss	Number of losses
0–250	71
250–350	327
350–400	167
400–450	123
450–500	97
500–600	128
600–700	103
700–800	67
800–1000	68
1000–1500	25

Table 6.1: Family Dental Coverage

as shown in Section 4, we have shown in this section that when the *modified* chi–square statistic is used, a closed–form expression is obtained for both upper and lower bounds, thus eliminating the need for numerical optmization.

6 Example

In this section we present a small example as an illustration of the results presented in earlier sections. The data we consider in Table 6.1 represents losses on individual family dental claims. We will consider aggregate losses on dental coverage.

The sample mean and the sample standard deviation for these losses are $\overline{x} = 465.986$ and $s = 223.1346$, respectively.

The endpoints of the claim intervals which determine the a_i's, are 0, 250, 350, 400, 450, 500, 600, 700, 800, 1000, 1500 for $i = 0, 1, \ldots, 9$, respectively. The observed relative frequencies p are

$$(0.060,\ 0.278,\ 0.142,\ 0.105,\ 0.082,\ 0.109,\ 0.088,\ 0.057,\ 0.058,\ 0.021)$$

. The deductible t and the upper limit c on the chi–square statistic will be arbitrarily selected to be 750 and 3.325, respectively. The upper bound on the stop–loss premium can be found by solving the optimization problem

$$\max \quad 50x_8 + 250x_9 + 750x_{10} \tag{6.1}$$

subject to

$$\sum_{i=1}^{10} \left(O_i^2/1386886.2\right) x_i^{-1} \leq 1 \tag{6.2}$$

$$\sum_{i=1}^{10} x_i = 1 \tag{6.3}$$

$$x_i \geq 0\,, \quad i = 1, 2, \ldots, 10 \tag{6.4}$$

where O_i is the observed frequency in ith interval, i.e.,

$$O = (71,\ 327,\ 167,\ 123,\ 97,\ 128,\ 103,\ 67,\ 68,\ 25)\,.$$

Equivalently, this upper bound may be found by solving the dual problem

$$\min_{\lambda > 750} \quad \lambda - \frac{1}{1604}\left[1016\sqrt{\lambda} + 67\sqrt{\lambda - 50} + 68\sqrt{\lambda - 250} + 25\sqrt{\lambda - 750}\right]^2. \tag{6.5}$$

Figure 6.1 shows this convex dual objective function. The dual problem is computationally far easier to solve than the primal. A one-dimensional numerical optimization method yields the objective value 40.568363 for $\lambda = 1511.6387$.

The extremal distribution that yields the maximum stop–loss premium may be found by using formula (4.16) in section 4 with λ_U^* equal to 1511.6387 and μ_U^* equal to 1471.1. Using these values gives

$$P^U = (0.05947448,\ 0.27391768,\ 0.13989068,\ 0.10303325,\ 0.08125387,\ 0.10722160,$$
$$0.08627988,\ 0.05707568,\ 0.06235018,\ 0.02950270),$$

which is shown in Figure 6.2. Here we plot the histogram of the upper extremal distribution and the frequency histogram.

Similarly the lower bound on the stop–loss premium may be found by solving

$$\min \quad 50x_9 + 250x_{10}$$

subject to (6.2), (6.3), and (6.4), or, equivalently by solving the Lagrangian dual problem, namely

$$\max_{\lambda \geq 0} \quad -\lambda + \frac{1}{1604}\left[1083\sqrt{\lambda} + 68\sqrt{\lambda+50} + 25\sqrt{\lambda+250}\right]^2. \tag{6.6}$$

This concave dual objective function is shown in Figure 6.4. Its maximum value is 6.4885016 , achieved at $\lambda = 254.14634$. Again, the extremal probability distribution producing this premium is found from formula (4.17), in section 4, where $\lambda_L^* = 254.14634$ and $\mu_L^* = 260.6348$. This distribution is

$$P^L = (0.06105374,\ 0.28119115,\ 0.14360527,\ 0.10576915,\ 0.08341144,\ 0.110068710,$$

$0.08857091,\ 0.05761409,\ 0.05345194,\ 0.01526362).$

See the histogram of the lower extremal distribution together with the observed frequency histogram in Figure 6.3.

Figure 6.5 illustrates the variation of the upper and lower bounds, again for deductible $t = 750$, as the upper bound on the chi–square statistic is relaxed.

We envision that one use of our methodology will be in the preparation of curves such as that shown in Figure 6.6 to be used by the risk manager in choosing the deductible t. Here, the maximum and minimum stop–loss premium are shown for varying values of the deductible t, using $c = 3.325$.

The modified chi–square model in section 5 with $t = 750$ and $c = 3.325$ yields the upper and lower bounds on the premium 39.6753 and 6.2120, given by (5.12) and (5.13), respectively. Also, the optimal dual variables, i.e., λ's in

(5.10), are $\lambda_U = -33.2483$ and $\lambda_L = -8.2058$. Equation (0.12) yields the primal solutions

$$x^{U^*} = (0.0595,\ 0.2681,\ 0.1399,\ 0.1031,\ 0.0813,\ 0.1073,\ 0.0863,\ 0.0574,\ 0.0633,\ 0.0280)$$

$$x^{L^*} = (0.0611,\ 0.2754,\ 0.1437,\ 0.1059,\ 0.0835,\ 0.1102,\ 0.0886,\ 0.0577,\ 0.0543,\ 0.0137),$$

i.e. the extremal distributions attaining the upper and lower bounds, respectively.

As the deductible t is varied, the variation of the bounds on the premium is shown in Figure 6.7. Here, the maximum and minimum stop–loss premium is shown for varying values of the deductible t, using $c = 3.325$ as upper limit of the modified chi–square statistic.

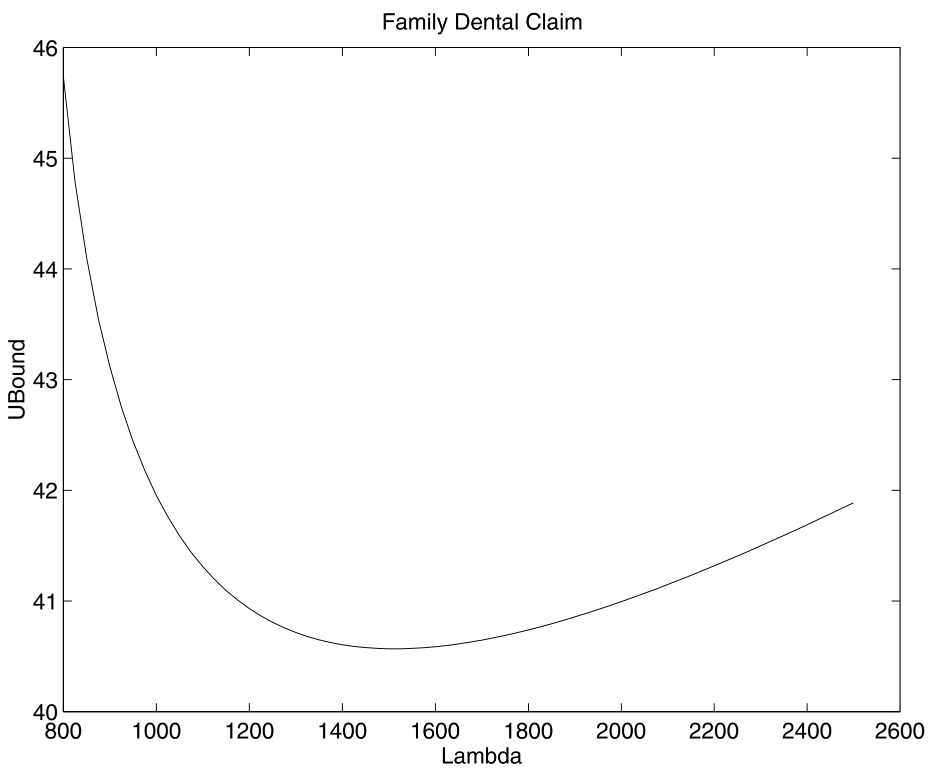

Figure 6.1: Example dual objective function yielding upper bound on premium

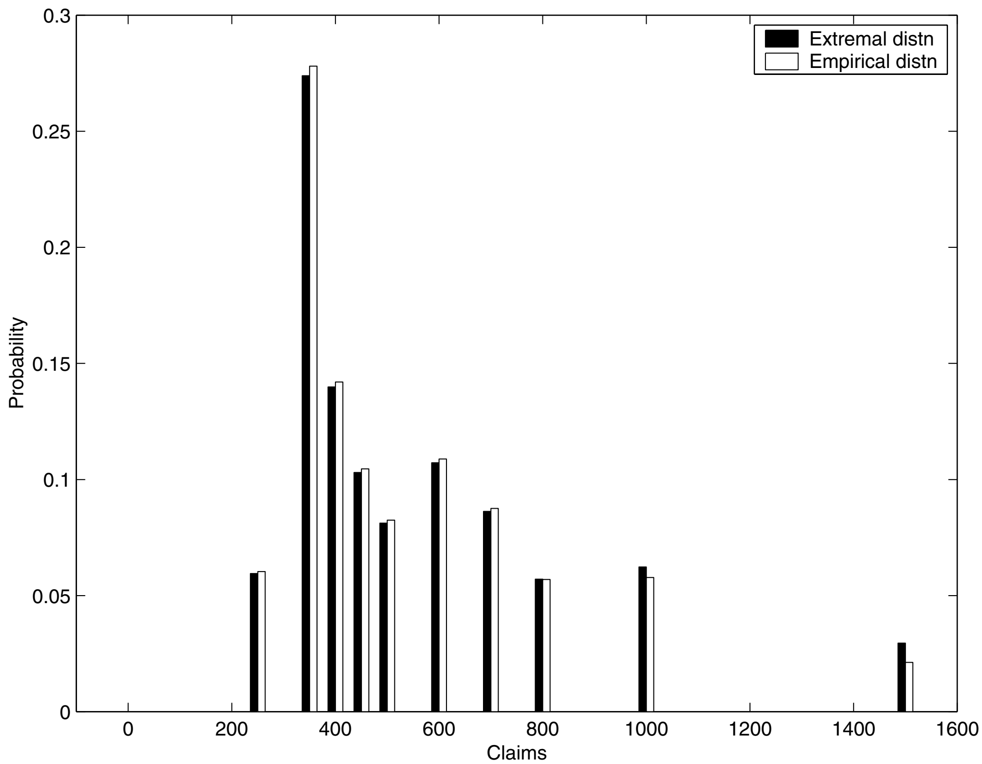

Figure 6.2: The extremal distributions of p^U and $\overline{p}$

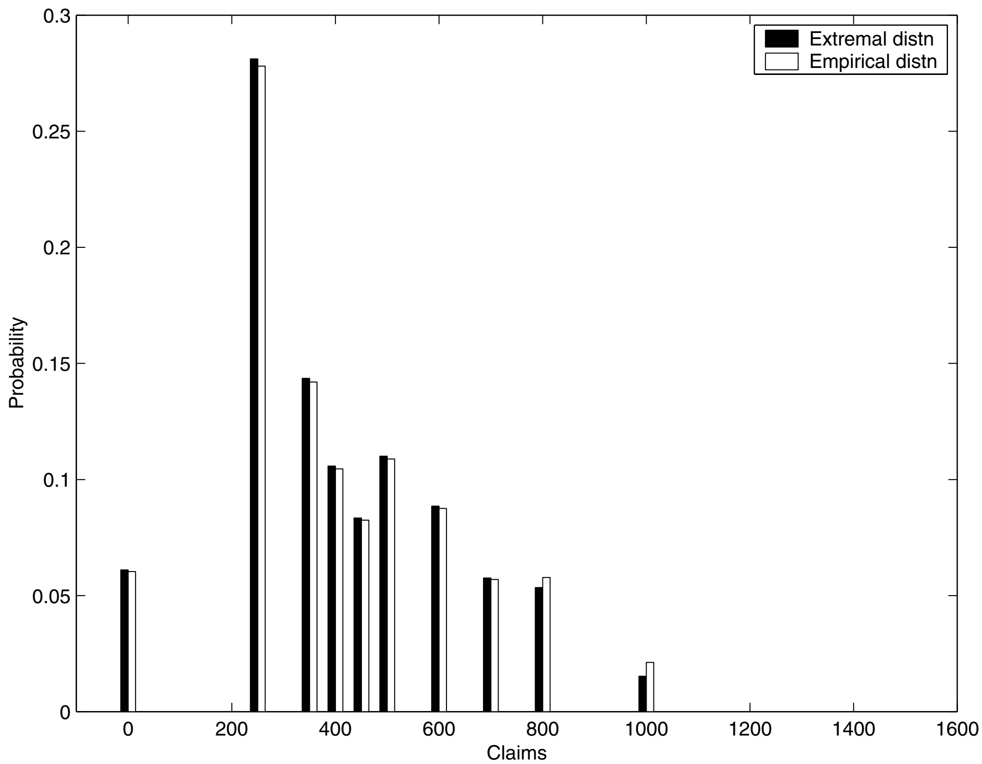

Figure 6.3: The extremal distributions of p^L and $\overline{p}$

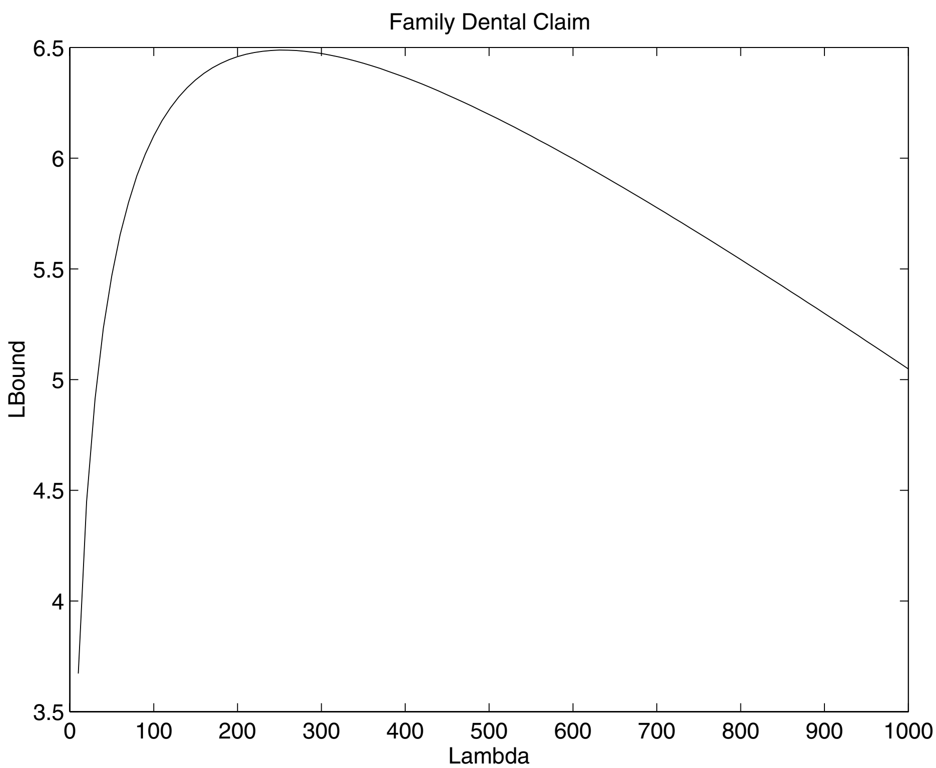

Figure 6.4: Dual objective function yielding lower bound on premium

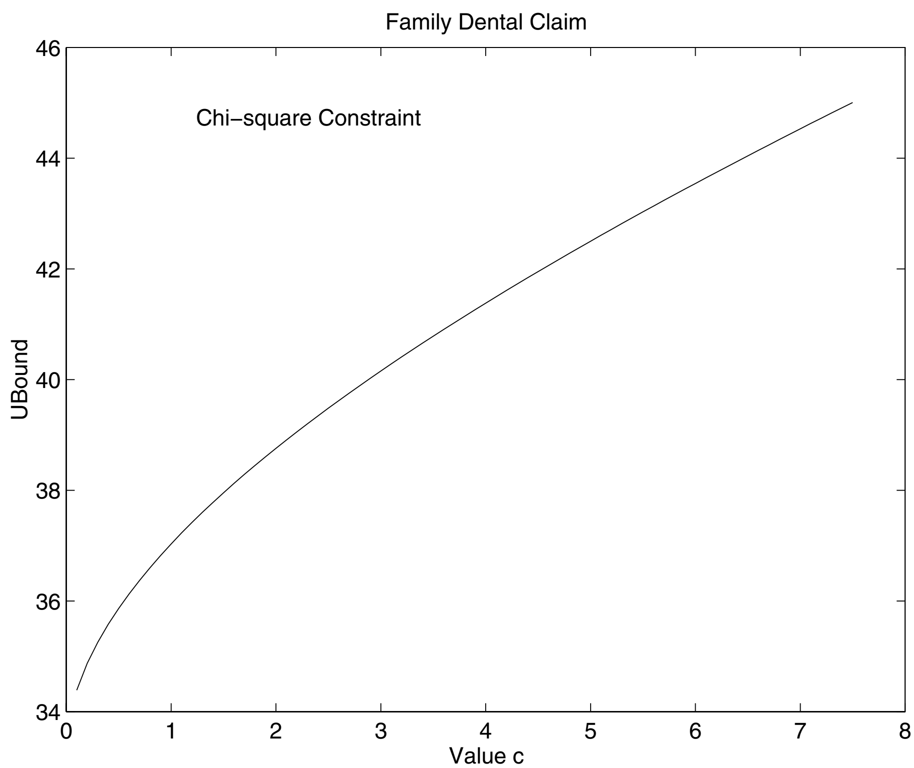

Figure 6.5: Upper bound on premium vs. maximum chi–square c for $t = 750$

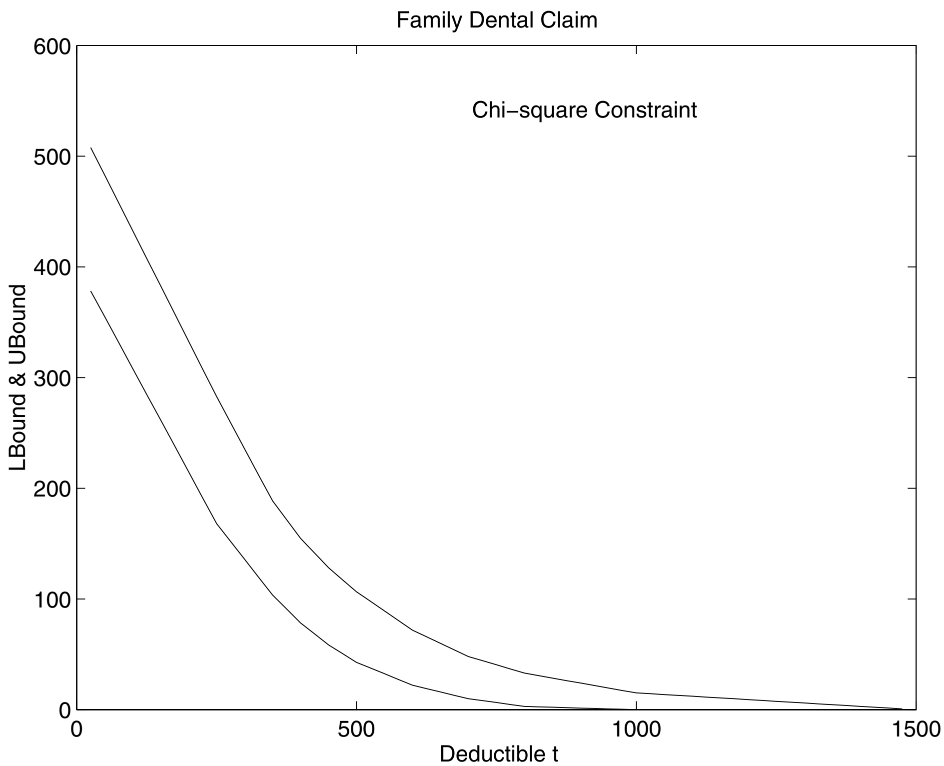

Figure 6.6: Upper and lower bounds of premium vs the deductible t for maximum chi-square statistics $c = 3.325$

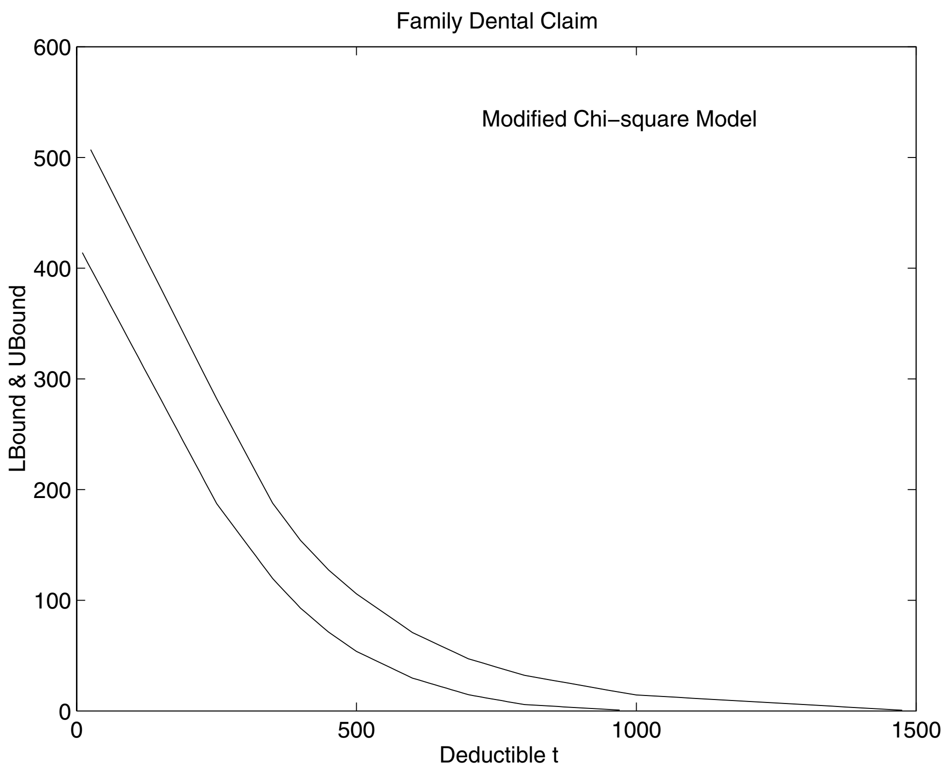

Figure 6.7: Upper and lower bounds of premium vs the deductible t for maximum modified chi–square statistics $c = 3.325$

Appendix

Proof (of the inequality(3.3)): since

$$\begin{aligned}
& \sum_{i=1}^{n} \frac{(Nq_i - O_i)^2}{Nq_i} \\
= & \sum_{i=1}^{n} \frac{N^2 q_i^2 - 2NO_i q_i + O_i^2}{Nq_i} \\
= & \sum_{i=1}^{n} \left(Nq_i - 2O_i + \frac{O_i^2}{Nq_i} \right) \\
= & N \sum_{i=1}^{n} q_i - 2 \sum_{i=1}^{n} O_i + \sum_{i=1}^{n} \frac{O_i^2}{Nq_i} \\
= & N - 2N + \sum_{i=1}^{n} \frac{O_i^2}{Nq_i} \\
= & -N + \sum_{i=1}^{n} \frac{O_i^2}{Nq_i}.
\end{aligned}$$

The last equation is obtained using the fact that $\sum_{i=1}^{n} q_i = 1$ and $\sum_{i=1}^{n} O_i = N$. Letting $B = (N + c)N$, the constraint (3.3) is obtained. □

Proof (of Lemma 4.1): The Lagrangian dual objective function (of problem (4.1)) will be

$$h^{\kappa}(\mu, \lambda) = \min_{x \geq 0} L^{\kappa}(x, \mu, \lambda). \tag{0.1}$$

Recall L^{κ} in (4.3),

$$L^{\kappa}(x, \mu, \lambda) = \sum_{i=1}^{n} \alpha_i x_i + \mu \left(\sum_{i=1}^{n} (O_i^2/B) x_i^{-1} - 1 \right) + \lambda \left(\sum_{i=1}^{n} x_i - 1 \right).$$

The Lagrangian function L^{κ} has partial derivatives

$$\frac{\partial L^{\kappa}}{\partial x_i} = \lambda + \alpha_i - \frac{\mu O_i^2}{B} x_i^{-2} \tag{0.2}$$

From (0.2), we have

$$\overline{x_i} = O_i \sqrt{\frac{\mu}{B(\lambda + \alpha_i)}} \quad \text{for } i = 1, 2, \ldots, n \ . \tag{0.3}$$

For x_i to be feasible, we must have $\lambda > -\alpha_i$, for all i. The stationary points only when $\lambda \in G^\kappa$, where $\kappa \in \{U(\text{ Upper bound case}), L(\text{ Lower bound case })\}$, and

$$G^\kappa = \begin{cases} (a_n - t, +\infty) & \text{if } \kappa = U \\ (0, +\infty) & \text{if } \kappa = L \end{cases} \tag{0.4}$$

If $\kappa = U$, $\lambda \leq (a_n - t)$, no stationary point exists, recall in (4.3) whem $\kappa = U$ and $\alpha_i = -(a_i - t)_+$,

$$L^U(x, \mu, \lambda) = \frac{\mu}{B} \sum_{i=1}^{n} O_i^2 x_i^{-1} + \sum_{i<k} \lambda x_i + \sum_{i \geq k} (\lambda - (a_i - t)) x_i - (\mu + \lambda) \tag{0.5}$$

where k is the smallest integer such that $a_k \geq t$. If $\lambda - (a_n - t) \leq 0$ then $L^U(x, \mu, \lambda)$ is unbounded below for sufficiently large x_n, i.e.,$h^U(\mu, \lambda) = -\infty$ if $\lambda - (a_n - t) \leq 0$.)

Therefore, when the dual problem

$$\max_{\mu,\lambda} h^U(\mu, \lambda)$$

is solved, we can restrict our search to λ larger than $(a_n - t)$, i.e., for which

$$h^U(\mu, \lambda) > -\infty.$$

Consider now the lower bounding problem, that is when $\kappa = L$. Recall in (4.3) when $\kappa = L$ and $\alpha_i = (a_{i-1} - t)_+$,

$$L^L(x, \mu, \lambda) = \sum_{i \leq k} \lambda x_i + \sum_{i>k} [\lambda + (a_{i-1} - t)] x_i + \mu \sum_i (O_i^2/B) x_i^{-1} - (\lambda + \mu). \tag{0.6}$$

Consider first $\lambda < 0$. In this case, it is clear from (0.6) that $L^L(x, \mu, \lambda)$ is unbounded below as x_1 increase, i.e.

$$h^L(\mu, \lambda) = \min_{x \geq 0} L^L(x, \mu, \lambda) = -\infty \quad \lambda < 0.$$

Secondly, consider the case $\lambda = 0$, i.e.

$$\min_{x \geq 0} L^L(x, \mu, 0)$$

where

$$L^L(x, \mu, 0) = \sum_{i \leq k} \frac{\mu O_i^2}{B x_i} + \sum_{i > k} \left[(a_{i-1} - t) x_i + \frac{\mu O_i^2}{B x_i} \right] - \mu.$$

For $i > k$,

$$\frac{\partial L^L(x, \mu, 0)}{\partial x_i} = \frac{d}{dx_i} \left\{ [a_{i-1} - t] x_i + \left(\mu O_i^2 / B \right) x_i^{-1} \right\} = (a_{i-1} - t) - \mu O_i^2 / B x_i^{-2} = 0$$

$$\Longrightarrow x_i^2 = \frac{\mu O_i^2}{B(a_{i-1} - t)}, \quad x_i = O_i \sqrt{\frac{\mu}{B(a_{i-1} - t)}} \quad \text{for } i > k.$$

For $i \leq k$, however,

$$\frac{\partial L^L(x, \mu, 0)}{\partial x_i} = \frac{d}{dx_i} \left[\mu O_i^2 / B x_i^{-1} \right] = -\mu O_i^2 / B x_i^{-2} \neq 0 \;\; \forall x_i,$$

i.e., $L^L(x, \mu, \lambda)$ doesn't have a stationary point, but terms in $\frac{1}{x_i} \to 0$ as $x_i \to \infty$ for $i \leq k$.

And so we have, for the case $\lambda = 0$,

$$h^L(\mu, 0) = \sum_{i > k} (a_{i-1} - t) O_i \sqrt{\frac{\mu}{B(a_{i-1} - t)}} + \mu \sum_{i > k} (O_i^2 / B) \frac{1}{O_i \sqrt{\frac{B(a_{i-1} - t)}{\mu}}} - \mu,$$

which simplifies to

$$h^L(\mu, 0) = 2 \sqrt{\frac{\mu}{B}} \sum_{i > k} O_i \sqrt{a_{i-1} - t} - \mu. \tag{0.7}$$

To find the stationary point for $h^L(\mu, 0)$, we find its partial derivative with respect to μ, which is

$$\frac{\partial h^L(\mu, 0)}{\partial \mu} = \frac{1}{\sqrt{\mu B}} \sum_{i > k} O_i \sqrt{a_{i-1} - t} - 1.$$

Setting $\frac{\partial h^L(\mu,0)}{\partial \mu} = 0$, and solving for

μ yields

$$\mu = \frac{1}{B}\left(\sum_{i>k} O_i \sqrt{a_{i-1} - t}\right)^2. \tag{0.8}$$

That is

$$\begin{aligned} \max_{\mu \geq 0} h^L(\mu, 0) &= \frac{2}{\sqrt{B}}\left(\sum_{i>k} O_i \sqrt{a_{i-1} - t}\right)\frac{1}{\sqrt{B}}\left(\sum_{i>k} O_i \sqrt{a_{i-1} - t}\right) - \frac{1}{B}\left(\sum_{i>k} O_i \sqrt{a_{i-1} - t}\right)^2 \\ &= \frac{1}{B}\left(\sum_{i>k} O_i \sqrt{a_{i-1} - t}\right)^2. \end{aligned} \tag{0.9}$$

Finally, we consider $\lambda \in G^\kappa$, which is shown on (0.4) above, $\overline{x_i}$ in expression of (0.3) is valid.

Therefore, in the Lagrangian dual problem

$$\max_{\mu \geq 0, \lambda} h^\kappa(\mu, \lambda)$$

we may restrict λ to values for which $h^L(\mu, \lambda) > -\infty$, i.e.,

$$\max_{\mu \geq 0, \lambda \in G^\kappa} h^\kappa(\mu, \lambda).$$

Now let's examine more closely our Lagrangian dual objective function when $\lambda \in G^\kappa$:

$$\begin{aligned} h^\kappa(\mu, \lambda) &= \sum_{i=1}^{n} (\lambda + \alpha_i)\, O_i \sqrt{\frac{\mu}{B(\lambda + \alpha_i)}} + \frac{\mu}{B}\sum_{i=1}^{n} O_i^2 \frac{1}{O_i \sqrt{\frac{\mu}{B(\lambda + (a_{i-1} - t))}}} - (\lambda + \mu) \\ &= \sqrt{\frac{\mu}{B}}\sum_{i=1}^{n} O_i \sqrt{\lambda + \alpha_i} + \sqrt{\frac{\mu}{B}}\sum_{i=1}^{n} O_i \sqrt{\lambda + \alpha_i} - (\mu + \lambda), \end{aligned}$$

which simplifies to

$$h^\kappa(\mu, \lambda) = 2\sqrt{\frac{\mu}{B}}\sum_{i=1}^{n} O_i \sqrt{\lambda + \alpha_i} - (\lambda + \mu), \qquad \text{for } \lambda \in G^\kappa,\ \mu \geq 0. \tag{0.10}$$

where $G^\kappa = \begin{cases} (a_n - t, +\infty) & \text{for } \kappa = U \\ [0, +\infty) & \text{for } \kappa = L\ . \end{cases}$

□

Proof (of Lemma 5.1): Note that L^κ is quadratic in x, that is, we consider the stationary point of the convex quadratic function L^κ, and so the partial derivatives are linear:

The partial derivative of L^κ with respect to x_i is,

$$\frac{\partial L^\kappa}{\partial x_i} = \frac{2\mu N^2}{O_i} x_i + (\alpha_i + \lambda - 2N\mu), \tag{0.11}$$

from which we obtain the stationary point,

$$x_i = \frac{[2N\mu - (\lambda + \alpha_i)]\, O_i}{2N^2\mu} \quad \text{for } i = 1, 2, \ldots, n \tag{0.12}$$

Since x is restricted to nonnegative values, the stationary point (0.12) is feasible if and only if λ and μ have appropriate values, namely $\lambda \leq 2N\mu - \alpha_i$, for all i, that implies that $\lambda \in G^\kappa$, where $G^\kappa \subseteq R$ are the feasible regions , where $\kappa \in \{U, L\}$, for λ for the upper bounding and lower bounding problems respectively. The nonnegativity of x_i give us

$$G^\kappa = \begin{cases} (-\infty, 2N\mu] & \text{if } \kappa = U \\ (-\infty, 2N\mu - (a_{n-1} - t)] & \text{if } \kappa = L. \end{cases}$$

The KKT conditions for optimality (since x is restricted to be nonnegative) includes the "complementary slackness" conditions

$$x_i \frac{\partial L^\kappa}{\partial x_i} = 0, \quad \text{for } i = 1, 2, ...n,$$

which is satisfied if stationary point (0.12) is nonnegative. Otherwise, the optimum must satisfy $x_i = 0$ and $\frac{\partial L^\kappa}{\partial x_i} \geq 0$. But when $x_i = 0$ and $\lambda \in \overline{G^\kappa}$, the values of $\frac{\partial L^\kappa}{\partial x_i}$ are

$$\frac{\partial L^\kappa}{\partial x_i} = \lambda - 2\mu N + \alpha_i \quad \text{for } i = 1, \ldots, n. \tag{0.13}$$

We can see that $\frac{\partial L^\kappa}{\partial x_i}$ is negative. for some i, for instance, $i < n-1$ if $\kappa = L$, and $i > k$ if $\kappa = U$, x_i cannot be optimal while $\frac{\partial L^\kappa}{\partial x_i}$ is negative. If $\lambda < 2\mu N - (a_{n-1} - t)$, then, $h^\kappa(\mu, \lambda) = -\infty$. Hence the search in (5.4) may be restricted to

$$\max_{\mu \geq 0, \lambda \in G^\kappa} \quad h^L(\mu, \lambda),$$

i.e., the case in which the optimizing x in

$$L^\kappa(x, \mu, \lambda) = \sum_{i=1}^n \alpha_i x_i + \mu \left(\sum_{i=1}^n \frac{(N x_i - O_i)^2}{O_i} - c \right) + \lambda \left(\sum_{i=1}^n x_i - 1 \right) \tag{0.14}$$

is a stationary point.

Substituting (0.12) into $L^\kappa(x, \mu, \lambda)$ yields

$$\begin{aligned}
h^\kappa(\mu, \lambda) &= \sum_{i=1}^n \frac{2N\mu - \lambda - (a_{i-1} - t)}{2N^2\mu} O_i \alpha_i + \mu \sum_{i=1}^n \frac{\left(N \frac{2N\mu - \lambda - \alpha_i}{2N^2\mu} O_i - O_i \right)^2}{O_i} \\
&+ \lambda \sum_{i=1}^n \frac{2N\mu - \lambda - \alpha_i}{2N^2\mu} O_i - \lambda - \mu c \\
&= \frac{1}{4N^2\mu} \sum_{i=1}^n (2N\mu - \lambda - \alpha_i)^2 O_i \\
&- \frac{1}{2N^2\mu} \sum_{i=1}^n (2N\mu - \lambda - \alpha_i)^2 O_i - (\mu c + \lambda) \\
&= -\frac{1}{4N^2\mu} \sum_{i=1}^n (2N\mu - \lambda - \alpha_i)^2 O_i - (\mu c + \lambda) \\
&= -\frac{1}{4N^2\mu} \sum_{i=1}^n O_i \left(4N^2\mu^2 + \alpha_i + \lambda^2 - 4\mu N \alpha_i - 4\mu N \lambda + 2\alpha_i \lambda \right) - (\mu c + \lambda) \\
&= -\frac{1}{4N^2\mu} \sum_{i=1}^n O_i \left(\alpha_i^2 - 4N\mu\alpha_i + 2\lambda\alpha_i \right) - \left(\mu c + \frac{\lambda^2}{4N\mu} \right)
\end{aligned}$$

Thus we obtain

$$h^{\kappa}(\mu,\lambda) = -\frac{1}{4N^2\mu}\sum_{i=1}^{n} O_i\alpha_i^2 + \frac{-\lambda}{2N^2\mu}\sum_{i=1}^{n} O_i\alpha_i + \frac{1}{N}\sum_{i=1}^{n} O_i\alpha_i - \mu c - \frac{\lambda^2}{4N\mu}. \tag{0.15}$$

□

Proof (of Lemma 5.2):

Recall that the Lagrangian dual objective function $h^{\kappa}(\mu,\lambda)$ in Lemma 5.1 is:

$$h^{\kappa}(\mu,\lambda) = -\frac{1}{4N^2\mu}\sum_{i=1}^{n} O_i\alpha_i^2 + \frac{-\lambda}{2N^2\mu}\sum_{i=1}^{n} O_i\alpha_i + \frac{1}{N}\sum_{i=1}^{n} O_i\alpha_i - \mu c - \frac{\lambda^2}{4N\mu} \tag{0.16}$$

and the stationary point of (μ,λ) in the equations of (5.10):

$$\begin{aligned} \mu &= \frac{1}{2N\sqrt{c}}\sqrt{-\frac{1}{N}\left(\sum_{i=1}^{n} O_i\alpha_i\right)^2 + \sum_{i=1}^{n} O_i\alpha_i^2} \\ \lambda &= \frac{-1}{N}\sum_{i=1}^{n} O_i\alpha_i. \end{aligned} \tag{0.17}$$

Substituting (μ,λ) of (0.17) into $h^{\kappa}(\mu,\lambda)$ of equation (0.16), we have,

$$\begin{aligned}
&\max_{\mu\geq 0,\lambda\in G^{\kappa}} \quad h^{\kappa}(\mu,\lambda) = -\sum_{i=1}^{n}\frac{1}{2N^2}\alpha_i O_i\left(\frac{-1}{N}\sum_{i=1}^{n}\alpha_i O_i\right)\frac{2N\sqrt{c}}{\sqrt{\sum_{i=1}^{n} O_i\alpha_i^2 - \frac{1}{N}\left(\sum_{i=1}^{n}\alpha_i O_i\right)^2}} \\
&+\frac{1}{N}\sum_{i=1}^{n}\alpha_i O_i - \left(\frac{-1}{N}\sum_{i=1}^{n}\alpha_i O_i\right)\frac{1}{2N^2}\left(\sum_{i=1}^{n}\alpha_i O_i\right)\frac{2N\sqrt{c}}{\sqrt{\sum_{i=1}^{n} O_i\alpha_i^2 - \frac{1}{N}\left(\sum_{i=1}^{n}\alpha_i O_i\right)^2}} \\
&-\sqrt{c}\frac{\sqrt{\sum_{i=1}^{n} O_i\alpha_i^2 - \frac{1}{N}\left(\sum_{i=1}^{n}\alpha_i O_i\right)^2}}{2N\sqrt{c}} - \frac{1}{N^2}\left(\sum_{i=1}^{n}\alpha_i O_i\right)^2\frac{1}{4N}\frac{2N\sqrt{c}}{\sqrt{\sum_{i=1}^{n} O_i\alpha_i^2 - \frac{1}{N}\left(\sum_{i=1}^{n}\alpha_i O_i\right)^2}} \\
= \; & -\frac{1}{2N}\frac{\sqrt{c}\left(\sum_{i=1}^{n} O_i\alpha_i^2\right)}{\sqrt{\sum_{i=1}^{n} O_i\alpha_i^2 - \frac{1}{N}\left(\sum_{i=1}^{n}\alpha_i O_i\right)^2}} + \frac{1}{N}\sum_{i=1}^{n}\alpha_i O_i \\
&+\frac{\sqrt{c}\left(\sum_{i=1}^{n} O_i\alpha_i\right)^2}{N^2\sqrt{\sum_{i=1}^{n} O_i\alpha_i^2 - \frac{1}{N}\left(\sum_{i=1}^{n}\alpha_i O_i\right)^2}} - \frac{\sqrt{c}}{2N}\sqrt{\sum_{i=1}^{n} O_i\alpha_i^2 - \frac{1}{N}\left(\sum_{i=1}^{n}\alpha_i O_i\right)^2} \\
= \; & \frac{1}{N}\sum_{i=1}^{n}\alpha_i O_i + \sqrt{c}\frac{-N\sum_{i=1}^{n} O_i\alpha_i^2 + \left(\sum_{i=1}^{n}\alpha_i O_i\right)^2}{N^2\sqrt{\sum_{i=1}^{n} O_i\alpha_i^2 - \frac{1}{N}\left(\sum_{i=1}^{n}\alpha_i O_i\right)^2}}
\end{aligned}$$

We obtain the following,

$$h^{\kappa *} = \frac{1}{N}\sum_{i=1}^{n} \alpha_i O_i - \frac{\sqrt{c}}{N}\sqrt{\sum_{i=1}^{n} O_i \alpha_i^2 - \frac{1}{N}\left(\sum_{i=1}^{n} \alpha_i O_i\right)^2} \tag{0.18}$$

□

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