QUASI-CONJUGACY, QUASI-SUBGRADIENTS, AND SURROGATE DUALITY IN NONCONVEX PROGRAMMING

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Presented at the ORSA/TIMS Annual Meeting,
November 1980, in Colorado Springs.

Conjugate functions have played an important role in the theory of convex programming. (For example, see [4].) An analogous role in quasi-convex programming is played by quasi-conjugate functions. Conjugates relate to epigraph supports, whereas quasi-conjugates relate to level set supports and barriers; conjugate functions provide a basis for Lagrangian duality, whereas quasi-conjugate functions provide a basis for surrogate duality. In this paper, we shall briefly survey the existing theory of quasi-conjugacy and surrogate duality as developed by Greenberg and Pierskalla ([2] and [3]) as it relates to nonconvex programming, interpreting it geometrically, and shall then add several extensions to this theory.

QUASI-CONJUGATES

A *hyperplane* in E^n is a set, with parameters $u \in E^n$, $u \neq 0$, and $c \in E^1$, of the form

$$H_u^c = \{x \in E^n : (u,x) = c\} \tag{1.1}$$

where (u,x) and ux will interchangeably denote the inner product of u and x. The parameter u determines the orientation of H_u^c and may be referred to as its *direction vector*. In particular, the hyperplane with direction vector u passing through the fixed point x^0 is $H_u^{ux^0}$. (See Figure 1.)

A hyperplane H_u^c determines two closed halfspaces, one of which we will denote by

$$\boldsymbol{H}_u^c = \{x \in E^n : (u,x) \geq c\} \tag{1.2}$$

If f is a function from E^n into the (extended) real line, $E^1 = [-\infty, +\infty]$, i.e., a functional, then we denote its *c-level-sets* by

$$L_c f = \{x \in E^n : f(x) \leq c\} \tag{1.3}$$

and

$$L_c^o f = \{x \in E^n : f(x) < c\} \tag{1.4}$$

Figure 2 denotes $L_c f$ for a case in which n=1.

Our interest in level sets results chiefly from the fact that quasi-convex functions may be defined to be functions all of whose level sets are convex.

For many functions, $L_c f = \text{cl}\, L_c^o f$ (the closure of $L_c^0 f$), but such is not always the case, as demonstrated by Figure 3. Here $L_c^o f \subset L_c f$, but it is neither the case that $\text{cl}\, L_c^o f = L_c f$, nor even that $\text{cl}\, L_c^o f \subset L_c f$. Note that f is not lower semi-continuous, nor explicitly quasi-convex (because of the "flat" spot in the graph).

Figure 4 depicts a c-level-set of a function defined on E^2. The boundaries of level sets are simply the contour curves of the function. Given a point $x \in E^n$, a level set of particular interest is $L_{f(x)} f$, depicted in Figures 5 a&b. In Figure 5b we note that x need not be a boundary point of $L_{f(x)} f$.

We next define the *z-quasi-conjugate* function $f_z^+ : E^n \to E^1$ where $z \in E^1$ and

$$f_z^+(u) = z - \inf\left\{ f(x) : (u,x) \geq z \right\} \tag{1.5}$$

Note that it is helpful to consider f_z^+ as a function of direction vectors, i.e.,

$$f_z^+(u) = z - \inf\left\{ f(x) : x \in \boldsymbol{H}_u^z \right\}. \tag{1.6}$$

If f is a quasi-convex function, as in Figure 6, and H_u^z is a supporting hyperplane for some level set $L_c f$, then $f_z^+(u) = z - c$, provided that the global minimum point x*=argmin $f(x)$ does not lie in $\boldsymbol{H}_u^z$ (in which case $f_z^+(u) = z - f(x^*)$).

One important property which should be noted is that f_z^+ is quasi-convex (without assuming any properties of f).

We now consider the *second z-quasi-conjugate* $\left(f_z^+\right)_z^+(x)$, defined in the obvious way as the z-quasi-conjugate of f_z^+ , and define the *normalized second quasi-conjugate* of f as

$$f^{++}(x) = \sup_{z \in E^1} \left(f_z^+\right)_z^+(x) \tag{1.7}$$

Example:

Let $f(x) = -e^{-x^2} = -\exp\left(-x^2\right)$, for $x \in E^1$ (see Figure 7a). Note that f is quasi-convex. Then

$$f_z^+(u) = z - \inf\{f(x) : ux \geq z\}$$
$$= z - \inf\{-\exp(-x^2) : ux \geq z\}$$
$$= \begin{cases} z + \exp\left(-z^2/u^2\right) \text{ if } z > 0 \text{ and } u \neq 0 \\ -\infty \text{ if } z>0 \text{ and } u = 0 \\ z+1 \text{ if } z \leq 0 \end{cases}$$

The second z-quasi-conjugate is

$$\left(f_z^+\right)_Z^+ = z - \inf\left\{f_z^+(u) : ux \geq z\right\}$$
$$= \begin{cases} z - (z+1) \text{ if } z \leq 0 \\ z - \infty \text{ if } z > 0 \ \& \ x = 0 \\ z - \inf\left\{z + \exp\left(-z^2/u^2\right) : z/u \leq 0\right\} \text{ if } z > 0 \ \& \ x < 0 \\ z - \inf\left\{z + \exp\left(-z^2/u^2\right) : z/u \geq x\right\} \text{ if } z > 0 \ \& \ x < 0 \end{cases}$$
$$= \begin{cases} -1 \text{ if } z \leq 0 \\ -\infty \text{ if } z > 0 \ \& \ x = 0 \\ -\exp\left(-x^2\right) \text{ if } x > 0 \ \& \ x \neq 0 \end{cases}$$

And hence

$$f^{++}(x) = \sup\left(f_z^+\right)_z^+(x)$$
$$= \begin{cases} \text{Max } \{-1, -\infty\} \text{ if } x = 0 \\ \text{Max } \{-1, -\exp(-x^2)\} \text{ if } x \neq 0 \end{cases}$$
$$= -\exp\left(-x^2\right)$$
$$= f(x)$$

(See Figure 7b.)

The function f^{++} has several important properties:

PROPERTY (i): (see [2]): f^{++} is quasi-convex, and

$$f(x) \geq f^{++}(x) \geq f^{\vee\vee}(x),$$

where $f^{\vee\vee}$ is the second (convex) conjugate. That is, f^{++} provides a quasi-convex approximation to f, from below, which is better than the convex approximation provided by $f^{\vee\vee}$.

PROPERTY (ii) (see [2]):

$$f^{++}(x) = \sup_u \inf_w \{ f(w) : (u,w) \geq (u,x) \} \tag{1.8}$$

$$= \sup_u \inf_w \{ f(w) : w \in \boldsymbol{H}_u^{ux} \}$$

This relaxation is easier to interpret geometrically than the definition of f^{++}. In Figure 8, we see depicted the three hyperplanes through x with the direction vectors u^2, u^1, and u^0, together with the points

$$\hat{w}^i = \operatorname*{argmin}_w \{ f(w) : w \in H_{u^i}^{u^i x} \}$$

It is clear that for the function depicted, rotating a hyperplane clockwise from $H_{u^2}^{u^2 x}$ through $H_{u^1}^{u^1 x}$ to $H_{u^0}^{u^0 x}$ (which supports the contour curve through x of the function f) produces a maximizing sequence $\{\hat{w}^i\}$ converging to x, and $f^{++}(x) = f(x)$.

Figure 9 depicts a function which (unlike that in Figure 8) is *not* quasi-convex. Again, rotating a hyperplane clockwise from $H_{u^2}^{u^2 x}$ to $H_{u^0}^{u^0 x}$ produces a maximizing sequence $\{\hat{w}^i\}$, which does not, however, converge to x. Moreover,

$$f(w^{*0}) = f(w^0) = f^{++}(x) < f(x).$$

Furthermore, it is shown in [2] that if f is an *isotonic* function, i.e.,

$$w \geq v \Rightarrow f(w) \geq f(v),$$

the optimal u in equation (1.8) has the property $u \in E_+^n$, i.e., u≥0. Hence, if f is isotonic,

$$f^{++}(x) = \sup_{u \geq c} \inf_w \{ f(w) : uw \geq ux \} \tag{1.9}$$

PROPERTY (iii): If $L_c f$ is compact for all c, then

$$L_c f^{++} = \operatorname{conv} L_c f$$

for all c (cf. [3].) More generally, for all c,

$$L_c^o f^{++} \subset \operatorname{cl} \operatorname{conv} L_c f$$

and

$$L_c f^{++} \supset \operatorname{conv} L_c f. \tag{1.10}$$

Proof: The proof of (1.10) is a trivial result of property (i). Let $x \notin \text{cl conv } L_c f$. Then x may be separated from $\text{cl conv } L_c f$, i.e., there is a y such that $xy > wy$ for all $w \in \text{cl conv } L_c f$. By Property (ii),

$$f^{++}(x) = \sup_u \inf_w \{ f(w) : wu \geq xu \}$$

and so, in particular,

$$f^{++}(x) \geq \mathrm{K}$$

where

$$K = \inf_w \{ f(w) : wy \geq xy \}.$$

Now, given $d > 0$, there must exist w_d such that $w_d y \geq xy$ and $f(w_d) < K + d$. But $w_d y \geq xy$ implies that $w_d \notin \text{conv } L_c f$ and hence $w_d \notin L_c f$, i.e., $f(w_d) > c$. Therefore, we have, for all $d > 0$,

$$c < f(w_d) < K + d \leq f^{++}(x) + d$$

or simply $c - d < f^{++}(x)$ for all $d > 0$. Therefore, $c \leq f^{++}(x)$ and so $x \notin L_c^o f^{++}$, proving that

$$L_c^o f^{++} \subset \text{cl conv } L_c f.$$

We are now in a position to introduce the concept of surrogate mathematical programming.

Surrogate Mathematical Programming

Consider the family of mathematical programs obtained by parameterizing the constraint right-hand-side vector and whose optimal value F:E^m is defined by

$$F(b) = \inf\{ f(x) : g(x) \ge b, x \in S\} \tag{2.1}$$

where $f: S \to E^1$, $S \subset E^n$, $g: S \to E^m$, and $b \in E^m$. (If the problem is infeasible, then we define $F(b) = +\infty$.)

Note that if $b^1 \ge b^2$, then

$$\{x : g(x) \ge b^1, x \in S\} \subset \{x : g(x) \ge b^2, x \in S\},$$

and so

$$\inf\{x : g(x) \ge b^1, x \in S\} \ge \inf\{x : g(x) \ge b^2, x \in S\},$$

i.e., $F(b^1) \ge F(b^2)$. Thus F is isotonic.

A *surrogate* problem, parameterized by b and the surrogate multiplier vector $u \in E_+^m$, is defined to be that of computing

$$S(u,b) = \inf\{ f(x) : ug(x) \ge ub, x \in S\}. \tag{2.2}$$

This is equivalent to

$$S(u,b) = \inf\{F(b) : ub \ge ub, x \in S\}. \tag{2.3}$$

We further define the *surrogate dual* problem to be that of computing

$$\begin{aligned} \hat{S}(b) &= \sup_{u \ge 0} S(b,u) \\ &= \sup_{u\ge 0} \inf_x \{ f(x) : ug(x) \ge ub, x \in S\} \\ &= \sup_{u\ge 0} \inf \{F(b) : ub \ge ub\}. \end{aligned} \tag{2.4}$$

Without affecting the supremum we may perform the outer optimization over the subset of surrogate multipliers

$$\boldsymbol{U} = \left\{ u \in E_+^m : \sum_{i=1}^m u_i = 1 \right\}$$

which is both convex and compact. Any direction in E_+^m has a representative vector in $\boldsymbol{U}$. We may then write

$$\hat{S}(b) = \sup_{u \in \boldsymbol{U}} S(b, u).$$

Comparison of (2.4) with equation (1.9) shows that, since F is isotonic,

$$\hat{S}(b) = F^{++}(b). \tag{2.5}$$

We know that $F^{++}(b) \leq F(b)$, and we are naturally interested in knowing under what conditions equality holds. That is, when does there exist a $u \geq 0$ such that solving the surrogate problem $S(b, u)$ solves our original problem, and $S(u, b) = F(b)$? If such a u does not exist, b is said to lie in a *surrogate gap*. The point b^0 is in such a gap in Figure 11, where

$$F^{++}(b) = F(b^1) < F(b^0).$$

This figure also illustrates one of the results stated in [2]. Suppose, for some $u^* \geq 0$, b^0 is a convex combination of points in the set $\operatorname{argmin}\{F(b) : ub \geq ub^0\}$. Then either some solution x of the surrogate problem $S(b^0, u^*)$ is a solution of $F(b^0)$, or else b^0 is in a surrogate gap.

The quasi-subgradient, to be introduced next, will help to characterize the surrogate gaps of a mathematical program.

Quasi-Subgradients

The conjugate inequality [4], namely,

$$(x,y) \le f(x) + f^{\vee}(y),$$

with equality if and only if $y \in \partial f(x)$, where $\partial f(x)$ is the subgradient set of f at the point x, and $f^{\vee}$ is the convex conjugate of the function f has an analogue in quasi-conjugate theory. It is easy to derive the result

$$(u,x) \le f(x) + f^{+}_{ux}(u) \tag{3.1}$$

and we shall define $\partial^{+} f(x)$, the set of quasi-subgradients of f at x, to be those vectors u such that equality holds in (3.1), i.e.,

$$(u,x) \le f(x) + f^{+}_{ux}(u),$$

with equality if and only if $u \in \partial^{+} f(x)$.

Equivalently,

$$\begin{aligned}\partial^{+} f(x) &= \left\{u : (u,w) \ge (u,x) \Rightarrow f(w) \ge f(x)\right\} \\ &= \left\{u : w \in \boldsymbol{H}_{u}^{ux} \Rightarrow f(w) \ge f(x)\right\} \\ &= \left\{u : f(w) < f(x) \Rightarrow w \notin \boldsymbol{H}_{u}^{ux}\right\} \\ &= \left\{u : L^{o}_{f(x)} f \cap \boldsymbol{H}_{u}^{ux} = \Phi\right\}.\end{aligned}$$

That is, u is a quasi-subgradient of f at x if $L^{o}_{f(x)} f$ lies entirely on one side of the hyperplane through x with direction vector u, or equivalently, H_{u}^{ux} is a non-intersecting barrier of $L^{o}_{f(x)} f$. (H_{u}^{z} is a barrier for a set S if

$$\sup_{x \in S} (u,x) \le z.)$$

In many cases (e.g., as we shall see, when f is continuous and convex or explicitly quasi-convex), there is a one-to-one correspondence between quasi-subgradients and level set supports (see Figure 12). (This assumes, of course, that the vectors in $\partial^{+} f(x)$ are normalized in some manner, since any multiple of a quasi-subgradient is also a quasi-subgradient.) However, Figure 13 depicts quasi-subgradients which do not produce corresponding level set supports. Any u which is a convex combination of u^0 and u^1 is a quasi-subgradient in Figure 13.

To see that level set supports, conversely, do not necessarily correspond to quasi-subgradients, consider the function $f:E^2$ defined by

$$f\left(x_1,x_2\right)=\begin{cases}0 \text{ if } x_1+x_2<1, \text{ or } x_1+x_2=1 \;\&\; x_1\geq 0.5\\ x_1+x_2 \text{ otherwise } \left(x_1\geq 0 \;\&\; x_1\geq 0\right)\end{cases}$$

whose graph and level sets are illustrated in Figure 14 J&B. The set $L_1 f$ is supported at the point x=(0.5, 0.5) by the hyperplane $x_1+x_2=1$ (i.e., $H^1_{(1,1)}$) but unfortunately $L^o_1 f$ has a nonempty intersection with this hyperplane, and so u=(1,1) is not a quasi-subgradient of f at x=(0.5, 0.5).

The correspondence between level set supports and quasi-subgradients failed for the function in Figure 13 because $\text{cl}\, L^o_{f(x)} f \neq L_{f(x)} f$, while the failure for the function in Figure 14 results from the fact that $L^o_{f(x)} f$ contained boundary points. In general, if

$$\text{cl}\, L^o_{f(x)} f = I_{f(x)} f$$

then

$$u\in\partial^+ f\left(x\right)\Rightarrow H^{ux}_u$$

supports $L_{f(x)} f$ at x. Conversely, if $L^o_{f(x)} f$ is open, then

$$H^{ux}_u \text{ is a barrier (or support) for } L_{f(x)} f \text{ at x } \Rightarrow u\in\partial^+ f\left(x\right).$$

The importance of the quasi-subgradient derives mainly from the following properties:

(i) $0\in\partial^+ f\left(x\right)\Leftrightarrow x\in \operatorname{argmin} f\left(x\right)$

(ii) $\partial^+ f\left(x\right)\neq\Phi\Rightarrow f\left(x\right)=f^{++}\left(x\right)$

Thus our question "does b^0 lie in a surrogate gap?" is equivalent to the question "does F have a quasi-subgradient at b^0?". Toward answering this question, we may use the following sufficient conditions, the proofs of which are very straightforward. (Note that any support is a barrier, but not conversely.)

(i) If $L^o_{f(x)} f$ is a non-empty open set, and if H^{ux}_u is a barrier for $L^o_{f(x)} f$, then $u \in \partial^+ f(x)$.

(ii) If $L^o_{f(x)} f$ is non-empty and f is upper semi-continuous on some set containing $L^o_{f(x)} f$, and if H^{ux}_u is a barrier for $L^o_{f(x)} f$, then $u \in \partial^+ f(x)$.

(iii) If $L^o_{f(x)} f$ is non-empty and f is upper semi-continuous on some set containing $L^o_{f(x)} f$, and if H^{ux}_u supports $L_{f(x)} f$, then $u \in \partial^+ f(x)$.

(iv) If f is quasi-convex and $x \notin \text{cl}\, L^o_{f(x)} f$, then $\partial^+ f(x)$ is non-empty.

(v) If f is a quasi-convex function which is upper semi-continuous on $L_{f(x)} f$ for some x, then $\partial^+ f(x)$ is non-empty.

EXAMPLES

The following examples will help to illustrate the concepts which have been presented.

Example 1. Consider the problem

$$\text{Minimize } f(x) = x_1^2 + x_2^2$$

subject to

$$x_1 + x_2 \geq 1 = b_1^0$$

$$x_1 - x_2 \geq 1 = b_2^0$$

Our optimal response function, $F(b)$, is

$$F(b_1, b_2) = \min\left\{x_1^2 + x_2^2 : x_1 + x_2 \geq b_1, x_1 - x_2 \geq b_2\right\}$$

$$= \begin{cases} 0.5\left(b_1^2 + b_2^2\right) & \text{if } b_1 \geq 0, b_2 \geq 0 \\ 0.5 b_1^2 & \text{if } b_1 > 0, b_2 < 0 \\ 0.5 b_2^2 & \text{if } b_1 < 0, b_2 > 0 \\ 0 & \text{if } b_1 < 0, b_2 < 0 \end{cases}$$

as can be seen graphically (see Figure 15a). Its contours are depicted in Figure 15b.

The surrogate program corresponding to any $u \in \boldsymbol{U}$, where

$$\boldsymbol{U} = \{(u_1, u_2) : u_1 + u_2 = 1, u_1 \geq 0, u_2 \geq 0\}$$

is

$$S(b^0, u) = \inf\{x_1^2 + x_2^2 : u_1(x_1 + x_2) + u_2(x_1 - x_2) \geq 1\}$$

$$= \inf\{x_1^2 + x_2^2 : x_1(u_1 + u_2) + x_2(u_1 - u_2) \geq 1\}$$

$$= \inf\{x_1^2 + x_2^2 : x_1 + x_2(2u_1 - 1) \geq 1\}$$

which has the solution (see Figure 15c):

$$S(b^0, u) = 1 \big/ \left[1 + (2u_1 - 1)^2\right] \quad \text{for } 0 \leq u_1 \leq 1, u_2 = 1 - u_1$$

The surrogate dual is therefore

$$\hat{S}(b^0) = \sup_{u \in \boldsymbol{U}} S(b^0, u)$$

$$= S(b^0, u^0), \text{ where } u^0 = (0.5, 0.5)$$

$$= 1.$$

Thus $b^0 = (1,1)$ is not in a surrogate gap, since

$$\hat{S}(b^0) = F(b^0) = 1$$

and it is evident from Figure 15 that F has no surrogate gaps whatsoever.

Our next example illustrates the existence of surrogate gaps.

Example 2.

Consider the problem

$$\text{Minimize } x_1 + x_2$$

subject to

$$x_1 + 2x_2 \geq 4 = b_1^0$$

$$2x_1 + x_2 \geq 3 = b_2^0$$

x_1 and x_2 both nonnegative and integer

The graph of our optimal response function, F, is sketched in Figure 16a and its contours are shown in Figure 16b. Note that $F(b)$ is both isotonic and lower semi-continuous everywhere, but clearly is not quasi-convex.

The surrogate problem with parameter $u \in \boldsymbol{U}$, where (as before),

$$\boldsymbol{U} = \left\{ (u_1, u_2) : u_1 + u_2 = 1, u_1 \ge 0, u_2 \ge 0 \right\}$$

is

$$S\left(b^0, u\right) = \underset{x_i \in \{0,1,2,\ldots\}}{\text{minimum}} \left\{ x_1 + x_2 : u_1 \left(x_1 + 2x_2 \right) + u_2 \left(2x_1 + x_2 \right) \ge 4u_1 + 3u_2 \right\}$$

$$= \underset{x_i \in \{0,1,2,\ldots\}}{\text{minimum}} \left\{ x_1 + x_2 : x_1 (1+u_1) + x_2 (1+u_2) \ge 3 + u_1 \right\}$$

which has the solution

$$S\left(b^0, u\right) = \begin{cases} \left\lceil \dfrac{(3+u_1)}{(1+u_1)} \right\rceil = 1 + \left\lceil \dfrac{2}{(1+u_1)} \right\rceil \text{ if } u_1 \ge u_2 \text{, i.e., } 0.5 \le u_1 \le 1 \\ \left\lceil \dfrac{(3+u_1)}{(1+u_2)} \right\rceil = \left\lceil \dfrac{5}{(2-u_1)} \right\rceil - 1 \text{ if } u_1 < u_2 \text{, i.e., } 0 \le u_1 < 0.5 \end{cases}$$

where $\lceil z \rceil$ denotes the smallest integer greater than or equal to z. (That this is the solution may be seen in Figure 16c: the minimum will always be attained at a point on a coordinate axis.) This solution is graphed as a function of u in Figure 16d.

The surrogate dual is $\hat{S}\left(b^0\right) = \sup_{u \in \boldsymbol{U}} S\left(b^0, u\right)$ and its solution, obtained from Figure 16d, is $\hat{S}\left(b^0\right) = 3,$ and

$$\underset{u}{\text{argmin}}\, S\left(b^0, u\right) = \left\{ u \in \boldsymbol{U} : \tfrac{1}{3} \le u_1 \le 1 \right\}$$

We see, therefore, that b^0 = (4,3) is not in a surrogate gap, since from Figure 16b,

$$F(4,3) = 3.$$

It follows then that any optimal multiplier u is a quasi-subgradient, so

$$\partial^+ F(4,3) = \left\{ u : \tfrac{1}{3} \le u_1 \le 1, u_2 = 1 - u_1 \right\}.$$

An examination of Figure 16e confirms this; any direction between $u^1 = \left(\tfrac{1}{3}, \tfrac{2}{3} \right)$ and $u^2 = (1,0)$ is a barrier of $L_3^o F = L_2 F$. (It was demonstrated in [2] that b0=(4,3) is in a GLM (generalized Lagrangian multiplier) duality gap. This is evident from Figure 16a: the epigraph of $F(b)$ has supports only at the points indicated in Figure 16f, and all other points must be in a GLM duality gap.)

We might now ask, "does our F have any surrogate gaps?". Further inspection indicates that the areas indicated in Figure 16g, for example, are surrogate gap regions. That is, the triangular area

$$\left\{(b_1, b_2) : b_1 > 3, b_2 < 2, b_1 + b_2 \leq 6\right\}$$

is a surrogate gap region. For any point b in these regions, we cannot construct a hyperplane which acts as a non-intersecting barrier of $L^o_{f(x)}F$.

An important relationship is illustrated here, namely, that surrogate gaps form a subset of the GLM duality gaps, i.e., if b^0 is in a surrogate gap, so that no surrogate multiplier vector u≥0 can be found such that $S\left(b^0, u\right) = F\left(b^0\right)$, then it is also true that no GLM multiplier vector u≥0 may be found such that

$$\underset{x}{\text{minimum}} \left\{f(x) + u\left[b^0 - g(x)\right]\right\} = F\left(b^0\right).$$

Summary

We have seen that quasi-conjugacy and the quasi-subgradient provide a basis for interpreting surrogate duality, much as conjugacy and the subgradient provide a basis for understanding Lagrangian duality.

While the Lagrangian dual has gaps when F is not convex, i.e.,

$$F^{\vee\vee}(b) < F(b),$$

the surrogate dual has a reduced gap region, as a consequence of the property

$$F^{\vee\vee}(b) \leq F^{++}(b) \leq F(b).$$

That is, F^{++} provides a better approximation to F than does $F^{\vee\vee}$.

A much more complete discussion of the relationship between the surrogate and Lagrangian dual may be found in [2]. Other important properties of the quasi-conjugates and quasi-subgradients are reported in [3].

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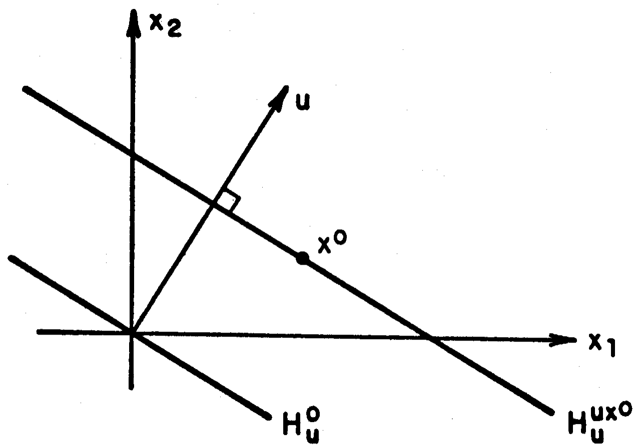


Figure 1. A hyperplane with direction vector u through x^0.

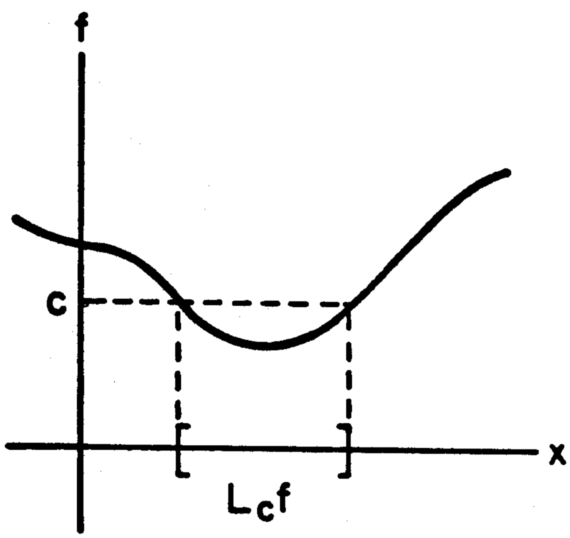


Figure 2. The c-level set of the function f.

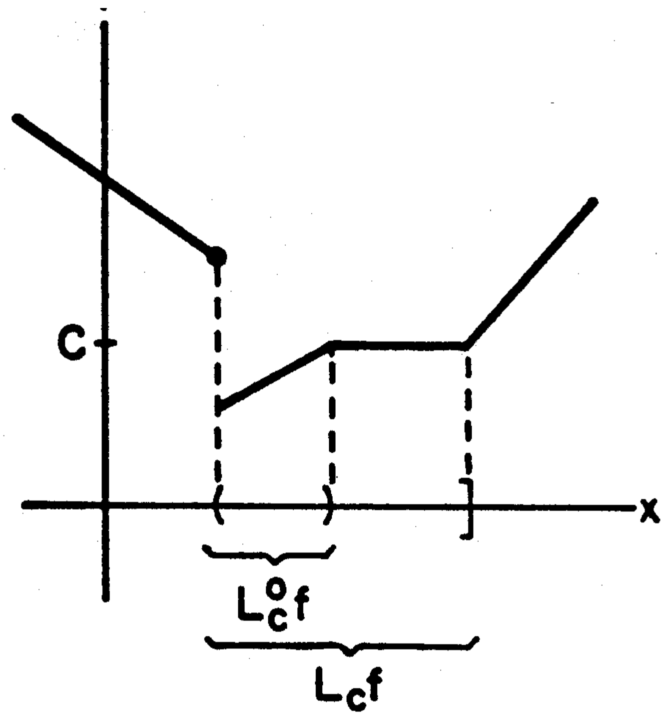


Figure 3. An example illustrating $L_c f \neq \text{cl } L^o_c f$.

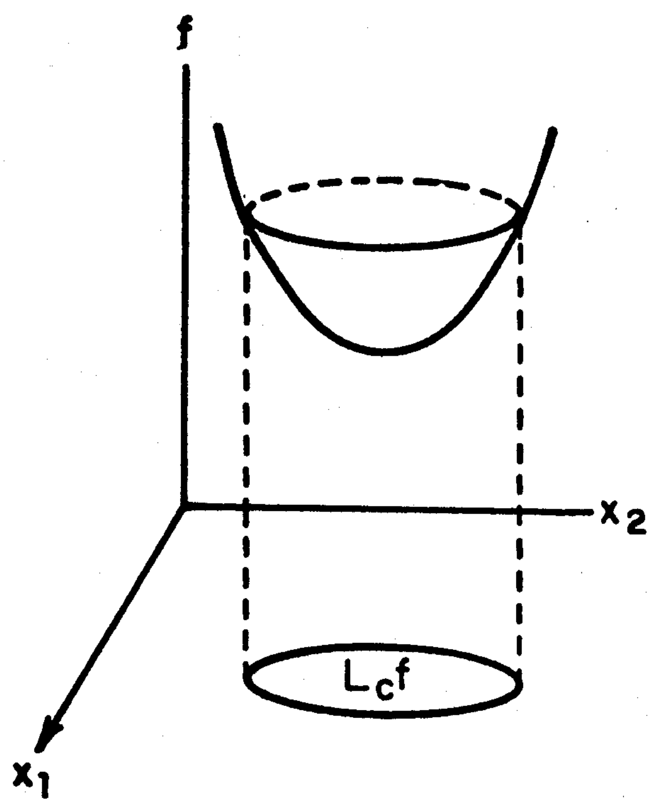


Figure 4. A c-level set of a function defined on E^2.

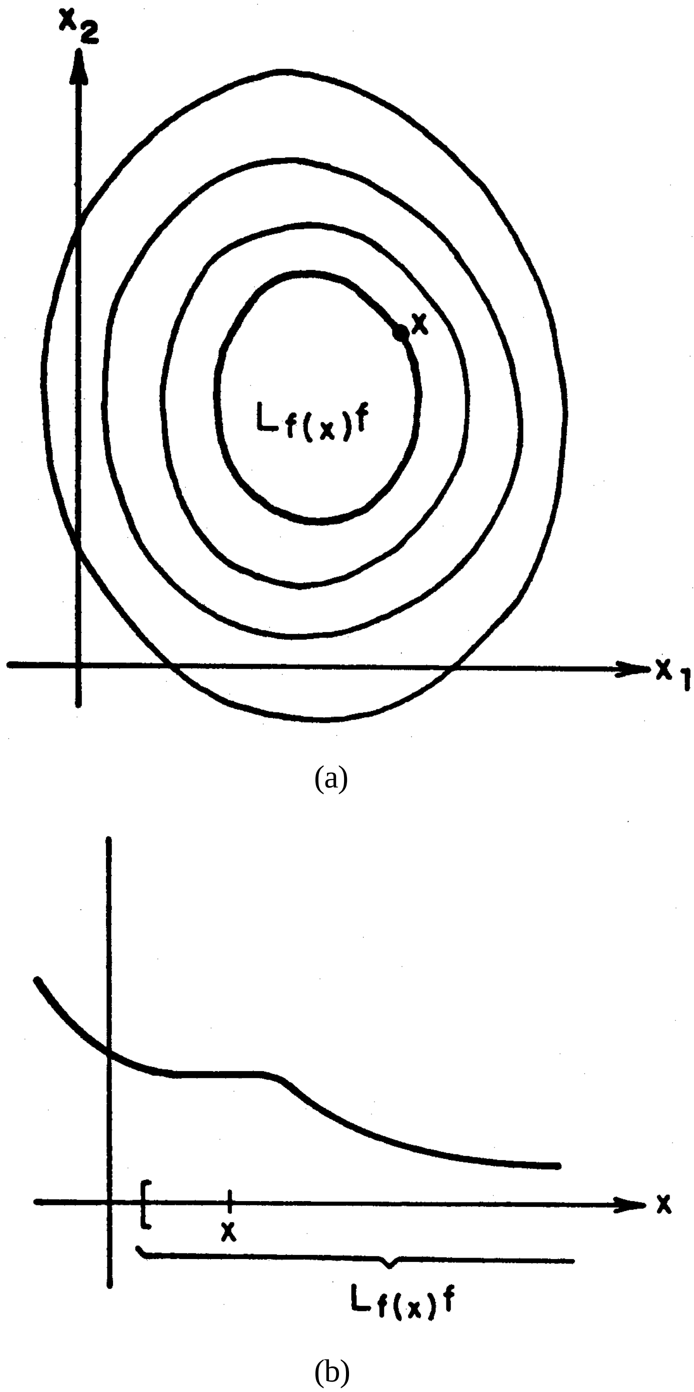


Figure 5. The level set $L_{f(x)}f$ corresponding to a point x.

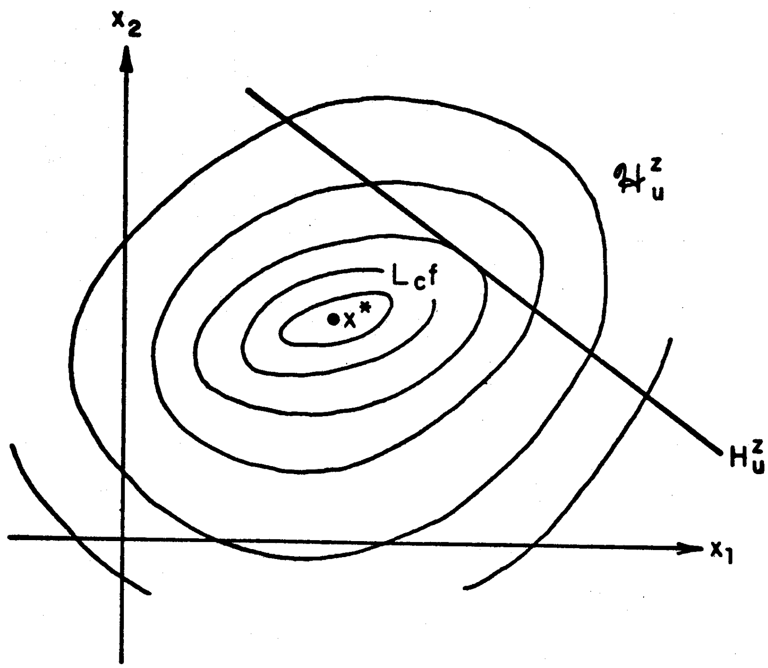


Figure 6. The hyperplane H_u^z corresponding to $f_z^+(u)$.

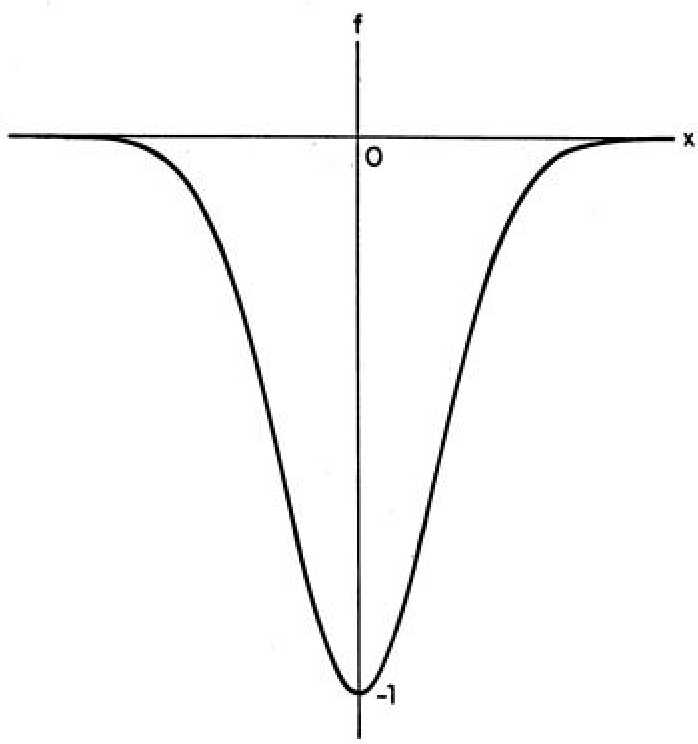


Figure 7a. The function $f(x) = -\exp(-x^2)$

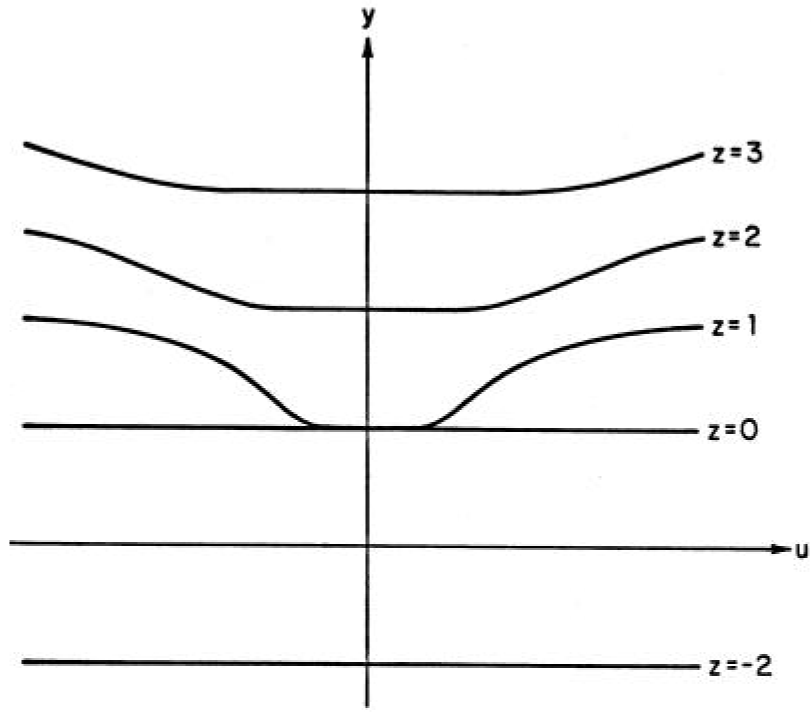


Figure 7b. Graphs of selected z-quasi-conjugates of $f(x) = -\exp(-x^2)$

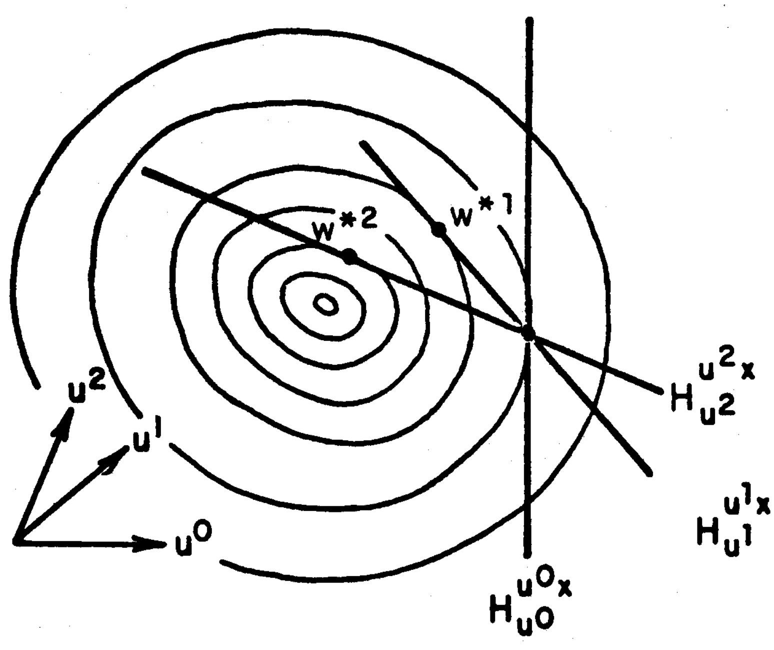


Figure 8. Geometric interpretation of f^{++} (where f is quasi-convex)

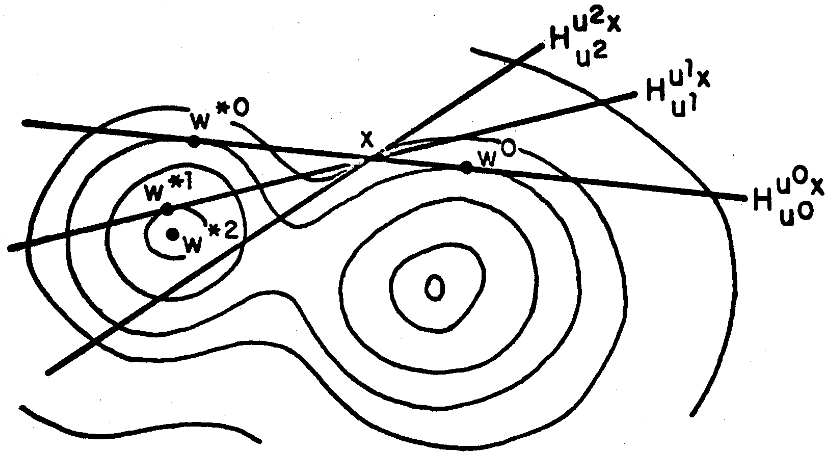


Figure 9. Geometric interpretation of f^{++} (where f is not quasi-convex).

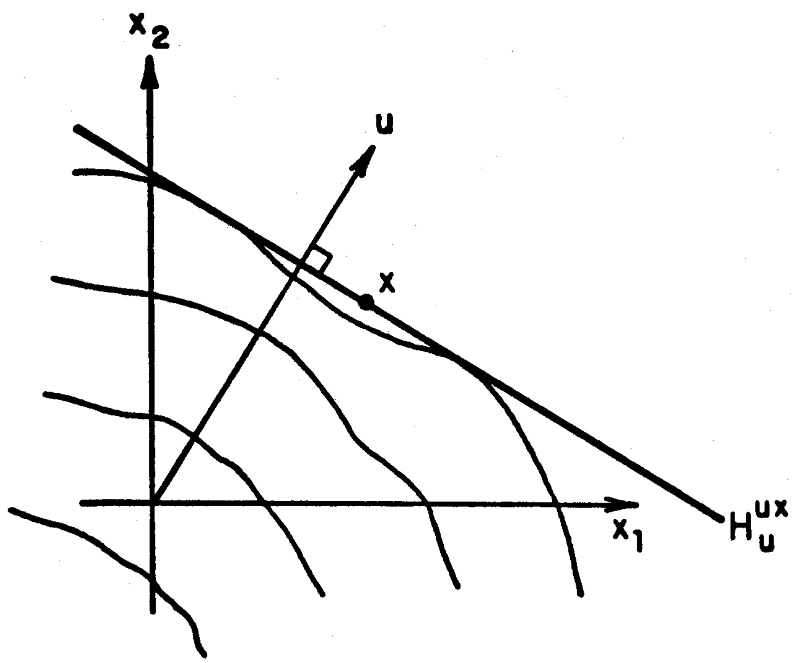


Figure 10. Level curves of an isotonic function

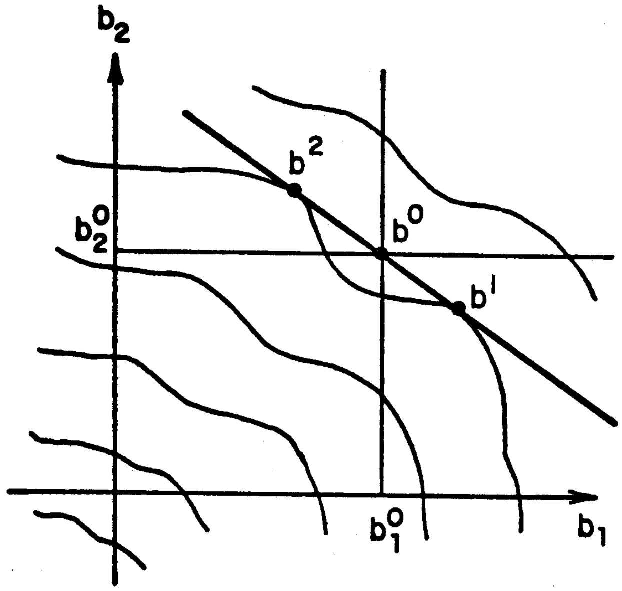


Figure 11. Illustration of a surrogate gap (at b^0).

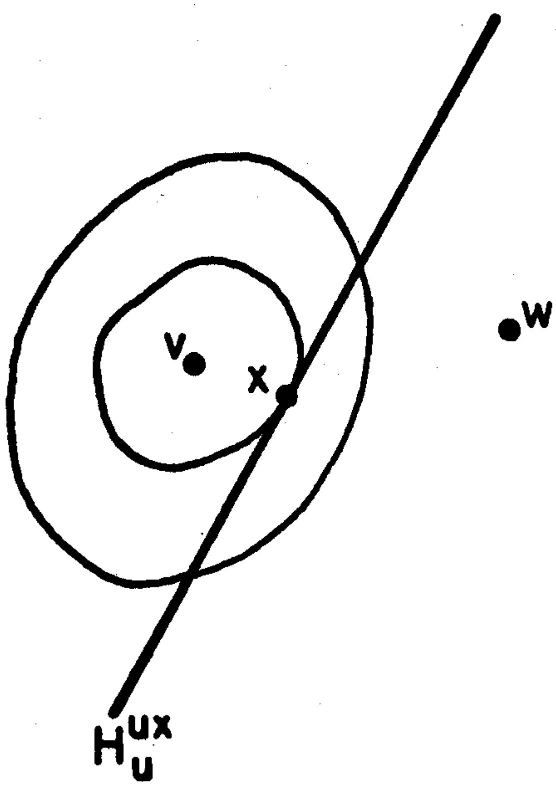


Figure 12. The hyperplane H_u^{ux} corresponding to quasi-subgradient u of the function f is a support of the level set $L_{f(x)}f$ (where f is explicitly quasi-convex).

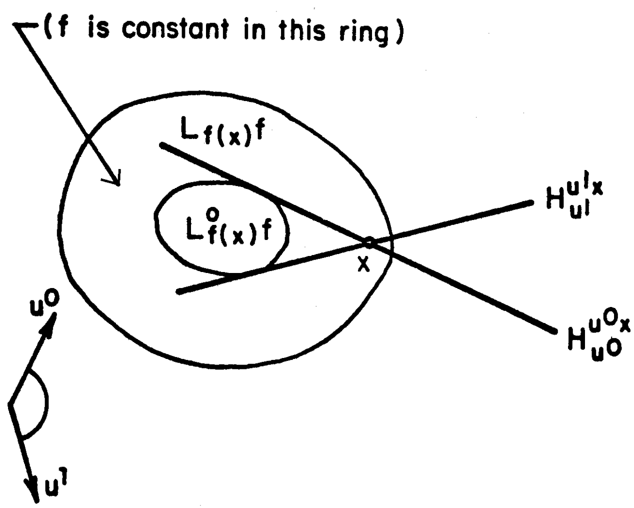
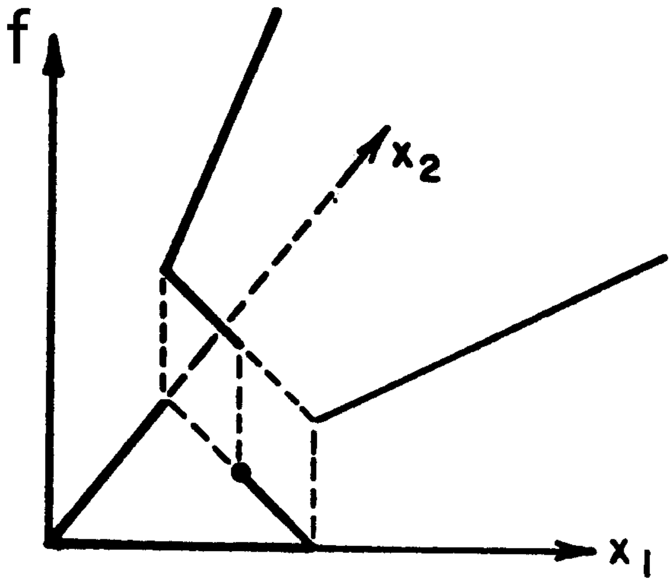
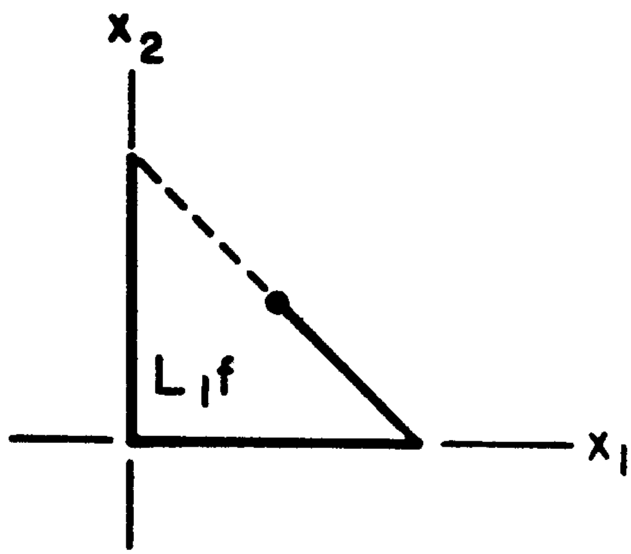


Figure 13. The quasi-subgradient set of f is the convex hull of u^0 and u^1, which do not correspond to supports of the level set $L_{f(x)}f$.



(a)



(b)

Figure 14. The graph (a) and the 1-level set (b) of an example function f

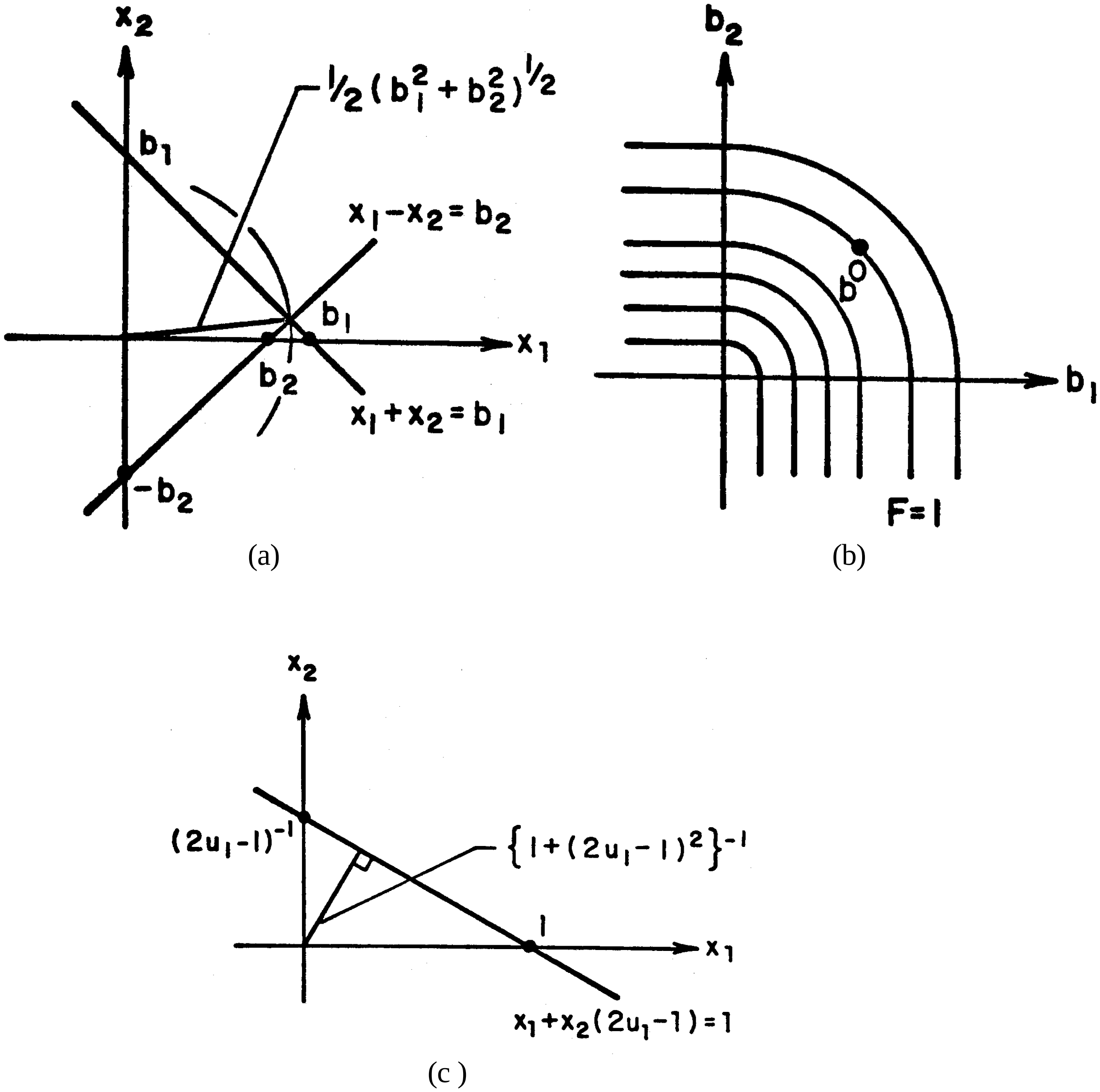


Figure 15. Example 1: (a) graphical solution; (b) contours of optimal response function F ; (c) graphical solution of surrogate problem.

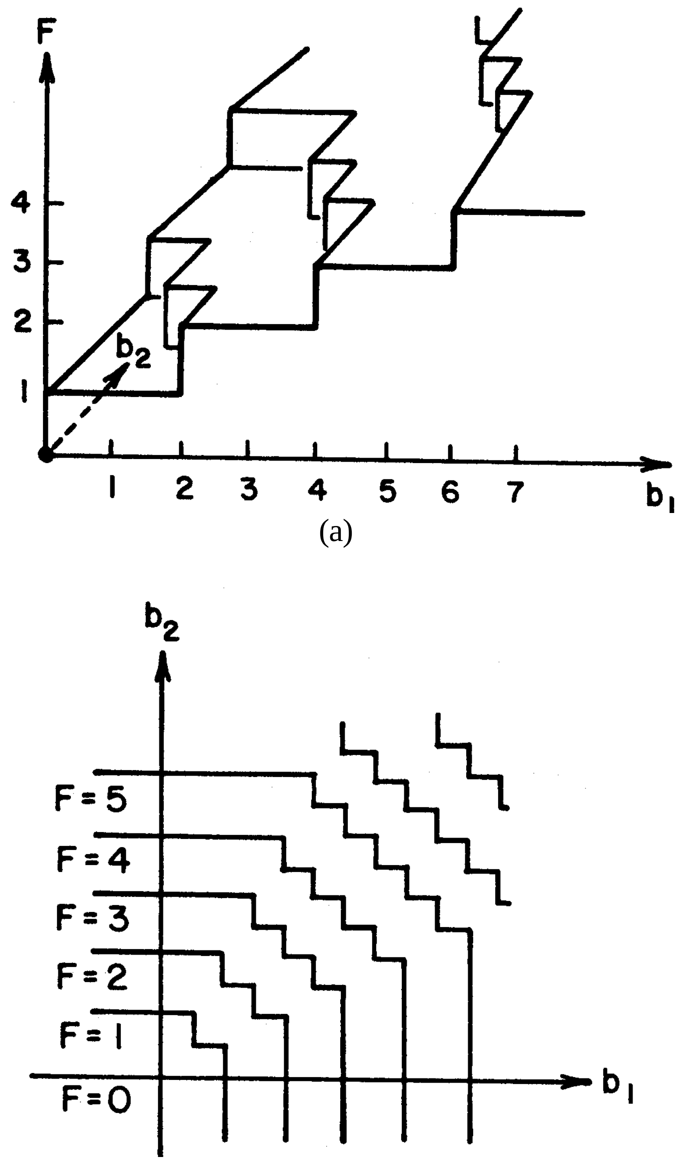
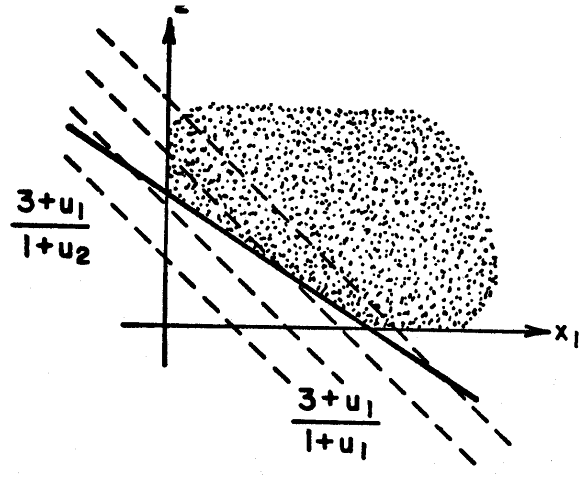
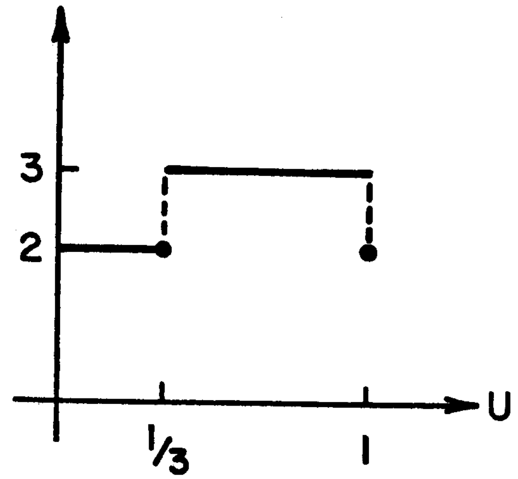


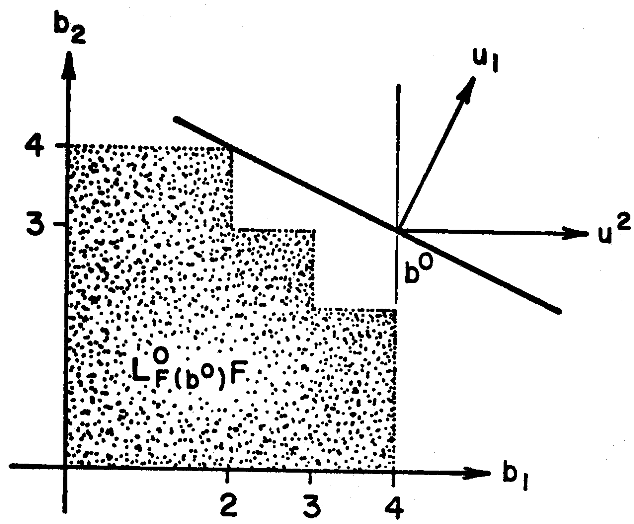
Figure 16. Example 2: (a) graph of optimal response function F ; (b) contours of optimal response function F



(c)

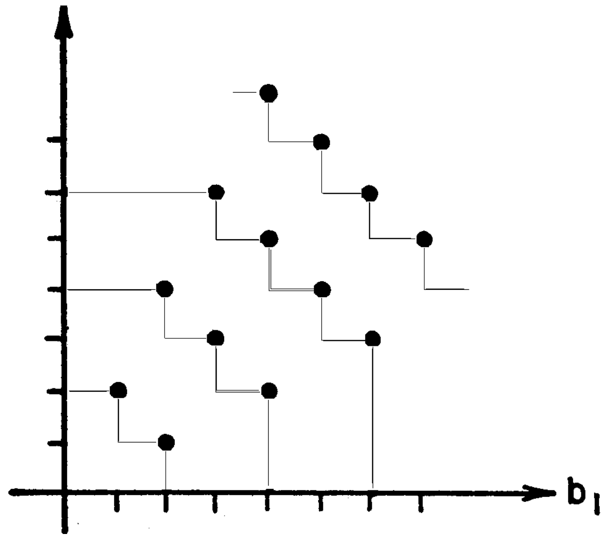


(d)

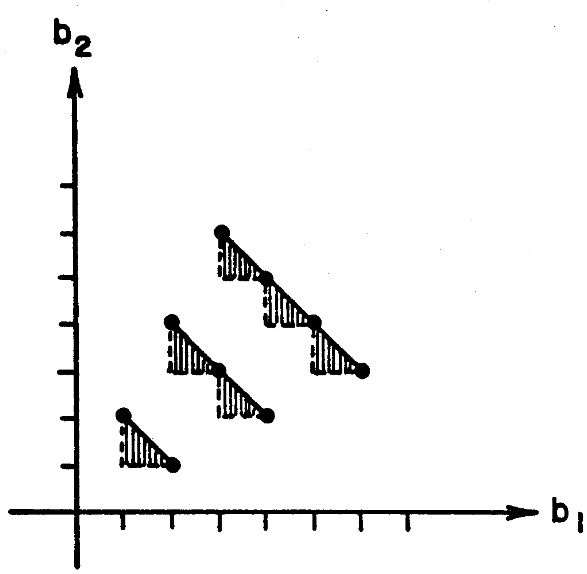


(e)

Figure 16 (continued). Example 2: (c) graphical solution of surrogate problem; (d) graphical solution of surrogate dual problem; (e) the quasi-subgradient set of F at b^0



(f)



(g)

Figure 16 (continued). Example 2: (f) Lagrangian duality gap region; (g) surrogate duality gap regions