EXAMPLE

C Capacitated
R Plant Location
DECOMPOSITION
S
S

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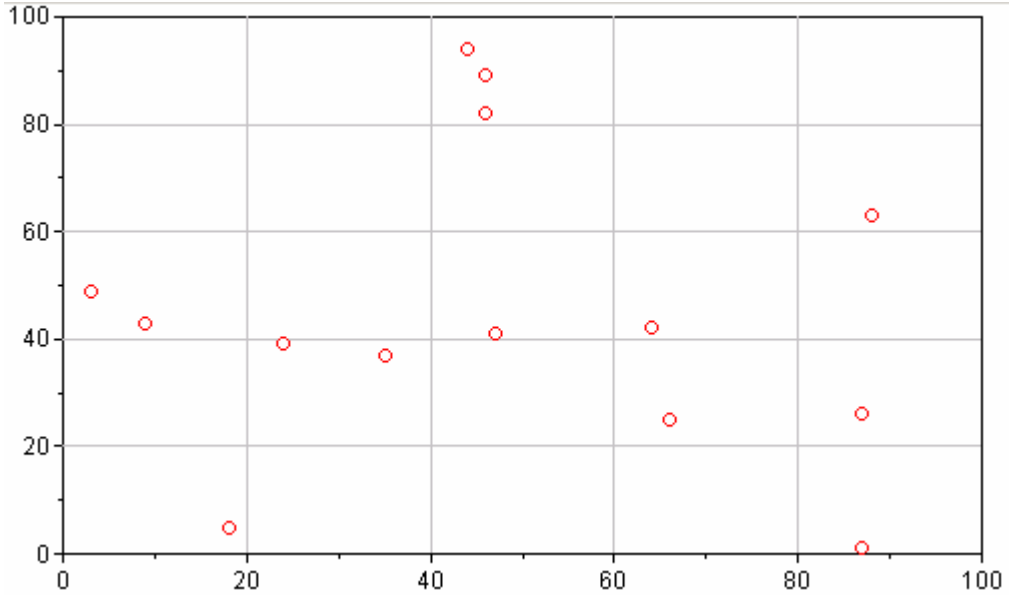
Randomly-generated problem with

- 7 potential plant sites and
- 14 demand points

Random number seed = 3432

i	X	Y	D	i	X	Y	D	i	X	Y	D	i	X	Y	D	i	X	Y	D
1	3	49	7	4	9	43	8	7	46	82	3	10	87	26	1	13	88	63	8
2	44	94	5	5	46	89	7	8	87	1	5	11	18	5	3	14	66	25	2
3	24	39	2	6	47	41	9	9	35	37	2	12	64	42	8				

Total demand: 70



Points 1 2 3 4 5 6 7 are potential plant sites,
with capacities & fixed costs

i	Ki	Fi
1	37	405
2	63	344
3	25	330
4	59	116
5	82	292
6	48	498
7	95	281

(i = plant site #, K[i] = capacity, F[i] = fixed cost)

Costs, Supplies, & Demands:

i/j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	K	F
1	0	61	23	8	59	45	54	97	34	87	46	61	86	67	37	405
2	61	0	59	62	5	53	12	102	58	80	93	56	54	72	63	344
3	23	59	0	16	55	23	48	74	11	64	35	40	68	44	25	330
4	8	62	16	0	59	38	54	89	27	80	39	55	81	60	59	116
5	59	5	55	59	0	48	7	97	53	75	89	50	49	67	82	292
6	45	53	23	38	48	0	41	57	13	43	46	17	47	25	48	498
7	54	12	48	54	7	41	0	91	46	69	82	44	46	60	95	281
Demand:	7	5	2	8	7	9	3	5	2	1	3	8	8	2	409	0

K = capacity,
F = fixed cost

We solve this problem by

"Standard" Benders' decomposition *(optimizing master problem at each iteration)*

"Standard" Cross-Decomposition

First the problem is solved by Benders' decomposition algorithm:

Benders Decomposition Algorithm

Master problem will be ***optimized*** at each iteration,
providing the Y minimizing current approximation v(Y) and
a lower bound

Iteration #1

Trial Y for primal subproblems:
open #3 6 *(initial "guess")*

Primal subproblem results:

Transport costs	2030
Fixed costs	828
Total costs	2858

****** New incumbent! ******

Solution of Master Problem
Y: open < **2 7** >
Estimated V(X): ¯4579

Iteration #2

Trial Y for primal subproblems:
open #2 7

Primal subproblem results:

Transport costs	3012
Fixed costs	625
Total costs	3637

Solution of Master Problem
Y: open <**4 6 7** >
Estimated V(X): ¯1247

Iteration #3

Trial Y for primal subproblems:
open #4 6 7

Primal subproblem results:

Transport costs	1222
Fixed costs	895
Total costs	2117

****** New incumbent! ******

Solution of Master Problem
Y: open <1 2 3 5 >
Estimated V(X): 775

Iteration #4

Trial Y for primal subproblems:
open #1 2 3 5

Primal subproblem results:

Transport costs	1723
Fixed costs	1371
Total costs	3094

Solution of Master Problem
Y: open <2 3 6 >
Estimated V(X): 1311

Iteration #5

Trial Y for primal subproblems:
open #2 3 6

Primal subproblem results:

Transport costs	1377
Fixed costs	1172
Total costs	2549

Solution of Master Problem
Y: open <1 2 5 6 7 >
Estimated V(X): 1544

Iteration #6

Trial Y for primal subproblems:
open #1 2 5 6 7

Primal subproblem results:

Transport costs	1156
Fixed costs	1820
Total costs	2976

Solution of Master Problem
Y: open <4 5 6 >
Estimated V(X): 1590

Iteration #7

Trial Y for primal subproblems:
open #4 5 6

Primal subproblem results:

Transport costs	1167
Fixed costs	906
Total costs	2073

****** New incumbent! ******

Solution of Master Problem
Y: open <4 7 >
Estimated V(X): 1619

Iteration #8

Trial Y for primal subproblems:
open #4 7

Primal subproblem results:

Transport costs	2064
Fixed costs	397
Total costs	2461

Solution of Master Problem
Y: open <4 6 >
Estimated V(X): 1836

Iteration #9

Trial Y for primal subproblems:
open #4 6

Primal subproblem results:

Transport costs	1863
Fixed costs	614
Total costs	2477

Solution of Master Problem
Y: open <4 5 >
Estimated V(X): 1909

Iteration #10

Trial Y for primal subproblems:
open #4 5

Primal subproblem results:

Transport costs	2079
Fixed costs	408
Total costs	2487

Solution of Master Problem

Y: open <2 3 4 6 >
Estimated V(X): 1740

Iteration #11

Trial Y for primal subproblems:
open #2 3 4 6

Primal subproblem results:

Transport costs	1144
Fixed costs	1288
Total costs	2432

Solution of Master Problem
Y: open <3 4 5 6 7 >
Estimated V(X): 1765

Iteration #12

Trial Y for primal subproblems:
open #3 4 5 6 7

Primal subproblem results:

Transport costs	1090
Fixed costs	1517
Total costs	2607

Solution of Master Problem

Y: open <5 6 >
Estimated V(X): 1957

Iteration #13

Trial Y for primal subproblems:
open #5 6

Primal subproblem results:

Transport costs	1779
Fixed costs	790
Total costs	2569

Solution of Master Problem
Y: open <**2 3 4 7** >
Estimated V(X): 1979

Iteration #14

Trial Y for primal subproblems:
open #2 3 4 7

Primal subproblem results:

Transport costs	1663
Fixed costs	1071
Total costs	2734

Solution of Master Problem
Y: open <6 7 >
Estimated V(X): 2001

Iteration #15

Trial Y for primal subproblems:
open #6 7

Primal subproblem results:

Transport costs	1820
Fixed costs	779
Total costs	2599

Solution of Master Problem
Y: open <2 4 6 7 >
Estimated V(X): 2014

Iteration #16

Trial Y for primal subproblems:
open #2 4 6 7

Primal subproblem results:

Transport costs	1148
Fixed costs	1239
Total costs	2387

Solution of Master Problem
Converged at iteration #16!
no trial solution

Incumbent Solution

Random Problem (Seed = 3432)

(Found at iteration #7!)

Summary

Transport cost=	1167
Fixed costs=	906
Total costs=	2073
Low bound	2073
Gap (%)	0

Plant	Fixed Cost	Supply	Surplus
4	116	59	39
5	292	82	67
6	498	48	13

Total fixed costs= 906 = 43.70% of total cost

Optimal Shipments

	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5
4	7	0	2	8	0	0	0	0	0	0	3	0	0	0	39
5	0	5	0	0	7	0	3	0	0	0	0	0	0	0	67
6	0	0	0	0	0	9	0	5	2	1	0	8	8	2	13

(*Demand pt #15 is dummy demand for excess capacity.)*

Dual Solution of Transportation Problem

Supply constraints

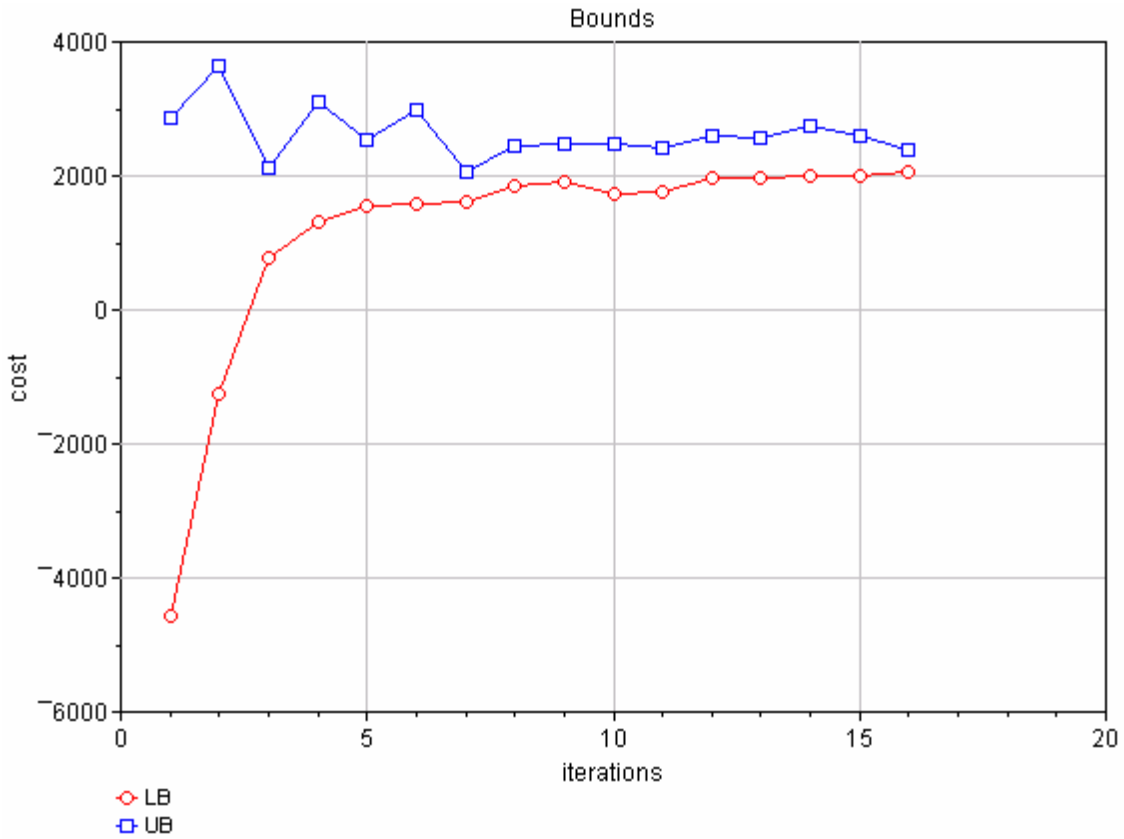
i	Ui	i	Ui
1	8	4	16
2	11	5	16
3	0	7	9

Demand constraints

j	Vj	j	Vj	j	Vj	j	Vj	j	Vj	j	Vj	j	Vj
1	8	3	0	5	16	7	9	9	3	11	23	13	31
2	11	4	16	6	16	8	41	10	27	12	1	14	9

Reduced costs: COST - U°.+V

I \ J=	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	64	15	16	67	53	55	48	29	52	15	52	47	50
2	58	0	48	67	10	58	10	50	50	42	59	44	12	52
3	31	70	0	32	71	39	57	33	14	37	12	39	37	35
4	0	57	0	0	59	38	47	32	14	37	0	38	34	35
5	51	0	39	59	0	48	0	40	40	32	50	33	2	42
6	37	48	7	38	48	0	34	0	0	0	7	0	0	0
7	53	14	39	61	14	48	0	41	40	33	50	34	6	42



Lower bound is monotonically increasing!

Cross-Decomposition Algorithm

Current parameters for cross-decomposition

```
Method for generating Y for primal subproblems
     Most recent
----------------------------------------
Method for updating Lagrangian multipliers for use in dual
subproblems
     Most recent
----------------------------------------
```

Iteration #1

Dual subproblem results:
Using multipliers:

i	1	2	3	4	5	6	7
Mu[i]	2.621	1.619	2.96	1.508	1.182	1.187	0.957

Subproblem in X:
Optimal cost= 1009, X=

|	1	2	3	4	5	6	7	8	9	0	1	2	3	4	sum
1	7	0	0	0	0	0	0	0	0	0	0	0	0	0	7
2	0	5	0	0	0	0	0	0	0	0	0	0	0	0	5
3	0	0	2	0	0	0	0	0	2	0	3	0	0	0	7
4	0	0	0	8	0	0	0	0	0	0	0	0	0	0	8
5	0	0	0	0	7	0	0	0	0	0	0	0	0	0	7
6	0	0	0	0	0	9	0	5	0	1	0	8	0	2	25
7	0	0	0	0	0	0	3	0	0	0	0	0	8	0	11

Subproblem in Y: Objective coefficients:

i	1	2	3	4	5	6	7
cost	308	242	256	27	195	441	190

Optimal cost= 697, by opening plants **3 6**

Total cost (Lower bound): 1706

Primal Subproblem

Trial Y for primal subproblem is:
open plants #3 6
with fixed costs 828
Primal subproblem solution:
Transportation cost = 2030
Total cost = 2858
Dual variables: **30 0 53 37 5 53 12**
***** new incumbent! *****

Iteration #2

Dual subproblem results:
Using multipliers:

i	1	2	3	4	5	6	7
Mu[i]	30	0	53	37	5	53	12

Subproblem in X:
Optimal cost= 2222, X=

|	1	2	3	4	5	6	7	8	9	0	1	2	3	4	sum
1	7	0	2	0	0	0	0	0	0	0	3	0	0	0	12
2	0	5	0	0	7	0	0	5	0	1	0	0	8	0	26
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	8	0	0	0	0	0	0	0	0	0	0	8
5	0	0	0	0	0	0	0	0	0	0	0	8	0	0	8
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	9	3	0	2	0	0	0	0	2	16

Subproblem in Y: Objective coefficients:

i	1	2	3	4	5	6	7
cost	¯705	344	¯995	¯2067	¯118	¯2046	¯859

Optimal cost= ¯6790,
by opening plants **1 3 4 5 6 7**

Total cost (Lower bound): ¯4568

Primal Subproblem

Trial Y for primal subproblem is:
open plants #1 3 4 5 6 7
with fixed costs 1922
Primal subproblem solution:
Transportation cost = 1034
Total cost = 2956
Dual variables: **5 0 5 5 5 5 5**

Iteration #3

Dual subproblem results:
Using multipliers:

i	1	2	3	4	5	6	7
Mu[i]	5	0	5	5	5	5	5

Subproblem in X:
Optimal cost= 1044, X=

	1	2	3	4	5	6	7	8	9	0	1	2	3	4	sum
1	7	0	0	0	0	0	0	0	0	0	0	0	0	0	7
2	0	5	0	0	7	0	0	0	0	0	0	0	0	0	12
3	0	0	2	0	0	0	0	0	2	0	3	0	0	0	7
4	0	0	0	8	0	0	0	0	0	0	0	0	0	0	8
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	9	0	5	0	1	0	8	0	2	25
7	0	0	0	0	0	0	3	0	0	0	0	0	8	0	11

Subproblem in Y: Objective coefficients:

i	1	2	3	4	5	6	7
cost	220	344	205	¯179	¯118	258	¯194

Optimal cost= ¯491, by opening plants **4 5 7**

Total cost (Lower bound): 553

Primal Subproblem

Trial Y for primal subproblem is: open plants #4 5 7
with fixed costs 689
Primal subproblem solution:
Transportation cost = 1980
Total cost = 2669
Dual variables: **30 33 22 38 38 0 38**
***** new incumbent! *****

Iteration #4

Dual subproblem results:
Using multipliers:

i	1	2	3	4	5	6	7
Mu[i]	30	33	22	38	38	0	38

Subproblem in X:
Optimal cost= 1153, X=

	1	2	3	4	5	6	7	8	9	0	1	2	3	4	sum
1	7	0	0	8	0	0	0	0	0	0	0	0	0	0	15
2	0	5	0	0	7	0	0	0	0	0	0	0	0	0	12
3	0	0	2	0	0	0	0	0	0	0	0	0	0	0	2
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	9	0	5	2	1	3	8	8	2	38
7	0	0	0	0	0	0	3	0	0	0	0	0	0	0	3

Subproblem in Y: Objective coefficients:

i	1	2	3	4	5	6	7
cost	¯705	¯1735	¯220	¯2126	¯2824	498	¯3329

Optimal cost= ¯10939,
by opening plants **1 2 3 4 5 7**

Total cost (Lower bound): ¯9786

Primal Subproblem

Trial Y for primal subproblem is:
open plants #1 2 3 4 5 7
with fixed costs 1768
Primal subproblem solution:
Transportation cost = 1572
Total cost = 3340
Dual variables: **27 27 23 27 27 0 27**

```
Convergence occurs in iteration #29:
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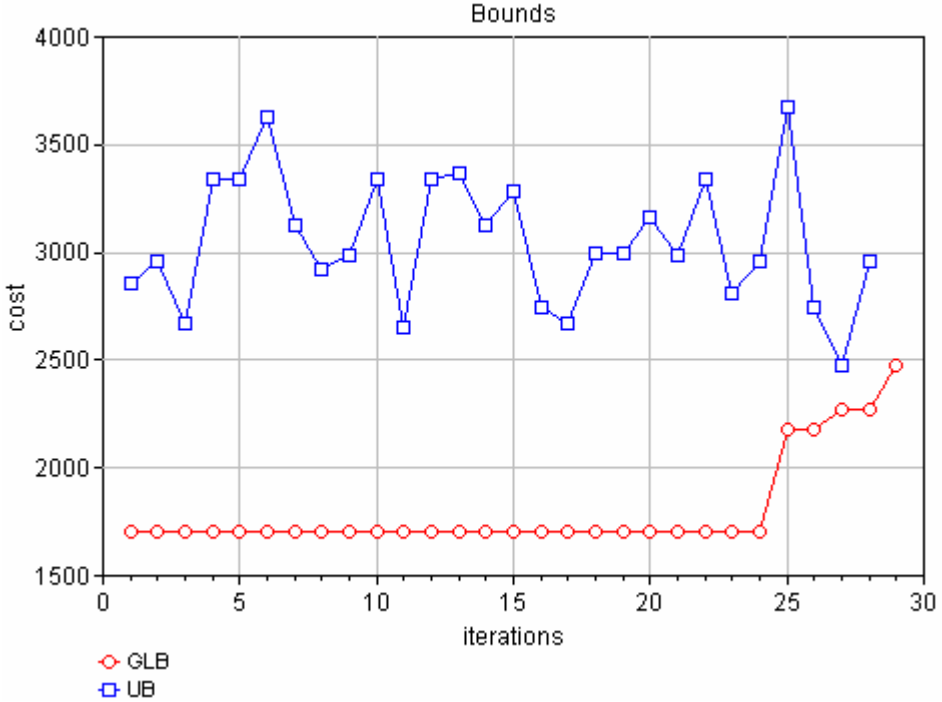

Upper Bound

vs

Greatest Lower Bound