

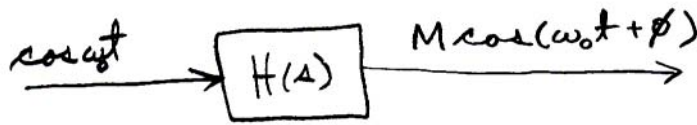
Bode Plots

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1 Overview of frequency response

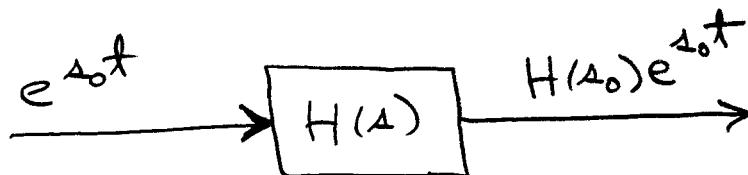
Bode plots are a graphical representation of the response of a system to a *steady-state* sinusoidal input. Specifically, for a system with a transfer function $H(s)$, the response to a sinusoid may be shown as:



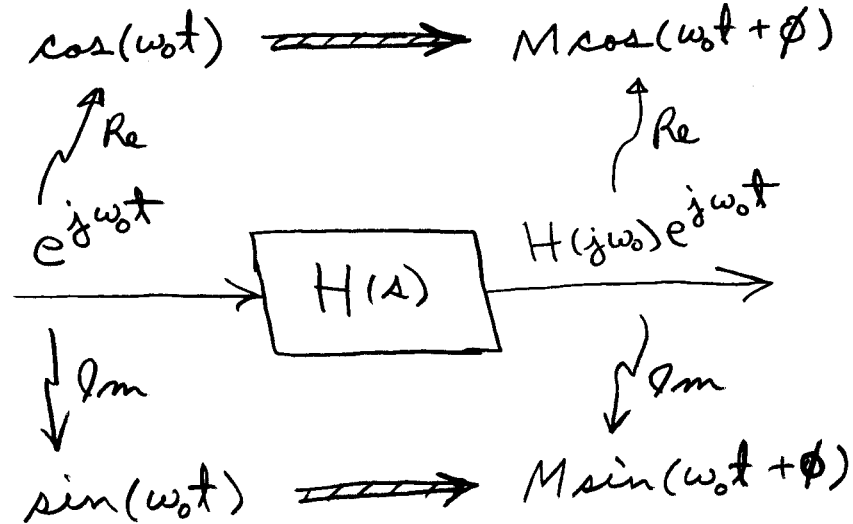
where $M = |H(j\omega_0)|$ and $\phi = \angle H(j\omega_0)$. Of course, the same picture holds if the input is a sinusoidal function of time instead of a cosine input.

Here we use the notation ω_0 to indicate that we are considering a sinusoid with a specific input frequency. The key result is that an input sinusoid leads, in steady-state, to an output sinusoid of the same frequency. The amplitude at the output is scaled by a factor of M relative to the input amplitude, and the phase at the output is shifted by an amount ϕ relative to the input. The terms M and ϕ are calculated by evaluating the magnitude and phase of the transfer function with the pure imaginary argument $j\omega_0$.

The underlying mathematics behind the result stated above come from recognizing the particularly simple form of the particular solution when the input driving function is a complex exponential. That is, if a system input has the form $e^{s_0 t}$, a particular solution is given by $H(s_0)e^{s_0 t}$. This is shown graphically below:



In the case that we are interested in the *sinusoidal steady-state*, we set $s_0 = j\omega_0$, since the real and imaginary parts of $e^{j\omega_0 t}$ are $\cos \omega_0 t$ and $\sin \omega_0 t$, respectively. It is computationally far simpler to work with the complex exponential input and its associated particular solution, and then take the real or imaginary parts after the fact to find the solution to a sinusoidal input. The operation of taking the real or imaginary part is valid due to the linearity of the the system $H(s)$. Because the system is linear, it simultaneously and independently processes the real and imaginary parts of the signal, and these two components can be separated at the output. This situation is shown below



where, as before, $M = |H(j\omega_0)|$ and $\phi = \angle H(j\omega_0)$.

This last result can be demonstrated as follows. If we define M and ϕ as above, then it holds that

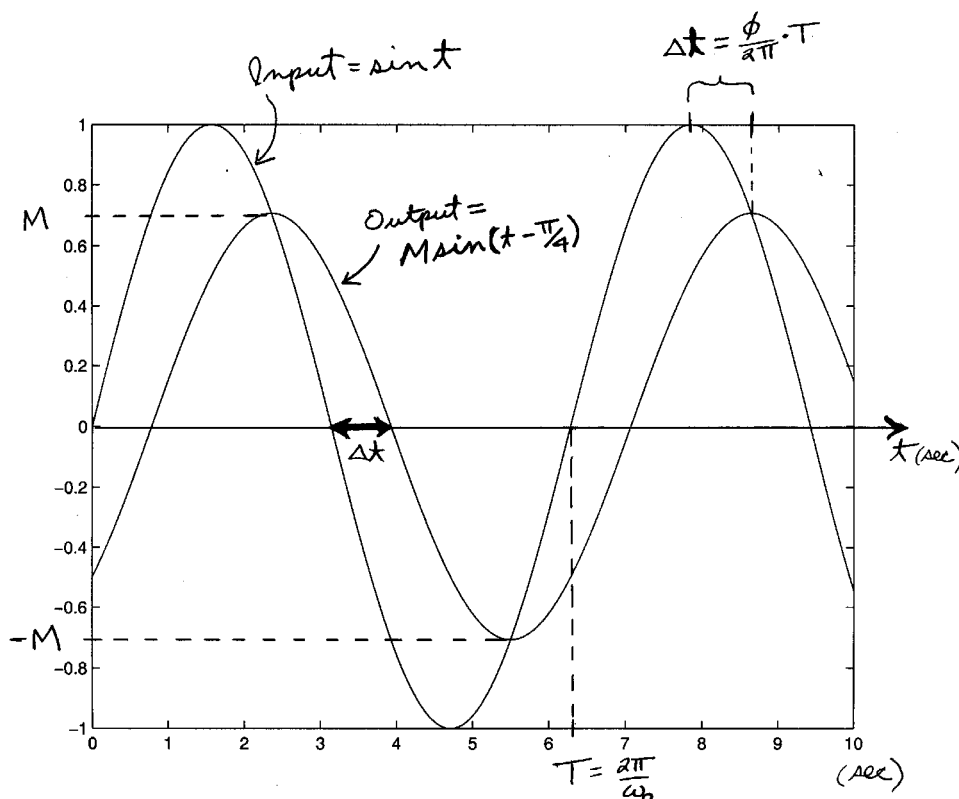
$$H(j\omega_0) = Me^{j\phi}, \quad (1)$$

and thus that

$$H(j\omega_0)e^{j\omega_0 t} = Me^{j\phi}e^{j\omega_0 t} = Me^{j(\omega_0 t + \phi)}. \quad (2)$$

The real part of the last term above is $M \cos(\omega_0 t + \phi)$, and the imaginary part is $M \sin(\omega_0 t + \phi)$ as shown in the figure. Thus we have shown that a sinusoidal input is scaled in amplitude by the factor M , and shifted in phase by an amount ϕ .

A time domain view of this result is shown below:



Here, the input is chosen as $u(t) = \sin t$, and the output is $M \sin(t+\phi)$, where we have set $M = 0.707$ and $\phi = -\pi/4$. For the purposes of this plot, we set $\omega_0 = 1$, but if the time axes were redefined as $t' = \omega_0 t$, then this plot is valid for all ω_0 .

There are a couple points to make about the plot. First, the period T of the sinusoids is $T = 2\pi/\omega_0$, and thus has the value $T = 2\pi$ sec for $\omega_0 = 1$. The input oscillates between peak values of ± 1 , whereas the output oscillates between peak values of $\pm M$. Finally, the time shift between equivalent phase points on the output relative to the input is related to the phase shift ϕ as

$$\frac{\Delta t}{T} = \frac{\phi}{2\pi} \quad (3)$$

and thus

$$\Delta t = \frac{\phi}{2\pi} T. \quad (4)$$

We count a time delay as *negative* phase and thus a negative time shift. In the plot as shown, $\phi = -\pi/4$, and thus, for $\omega_0 = 1$, $\Delta t = -\pi/4$ sec.

As a final comment, note that we have selected the input amplitude as unity, and the input phase shift as zero. However, if we were to scale the input by some factor, this would simply multiply both waveforms, and so would just be a redefinition of the scale-factor of the axes. The *ratio* of output to input would remain equal to M . Further, if we were to give the input a nonzero phase

shift, this would just shift both waveforms in time, with the *relative* phase shift remaining equal to ϕ . This would just be a redefinition of the point $t = 0$. The conclusion then is that the plot above shows the essential features for understanding frequency response for any choice of input amplitude or phase shift. The linear system H acts to scale the amplitude by M , and shift the phase by ϕ .

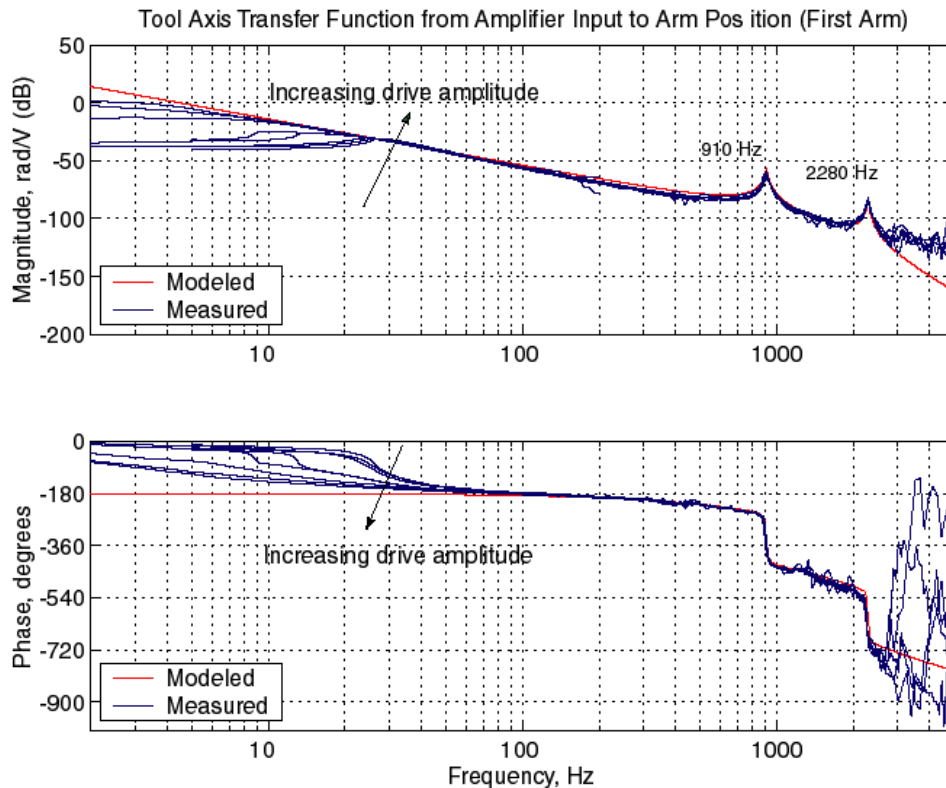
In the discussion above, we have assumed a given input frequency ω_0 , but if we now let the frequency vary, then it is clear that we can characterize the frequency response for *any* input frequency ω , by plotting M and ϕ as a function of ω .

In a Bode plot, we plot $M(\omega)$ on a log-log scale, and $\phi(\omega)$ on a semi-log scale. The reason for doing this is that we are frequently interested in a wide range of frequency and amplitudes, and a log-log plot allows showing such wide dynamic range with equal resolution in each decade of response. However, since phase typically varies in a limited range, it is plotted on linear axes versus frequency on log axes.

Another strong advantage of taking a frequency response viewpoint is that the Bode plot for real systems can be readily measured using an instrument referred to as a *dynamic analyzer*. It is usually far more accurate to develop a model for a linear system by taking frequency response measurements and then fitting these measurements to some model, as opposed to making a time-response measurement for an input such as a step.

2 Example system Bode plot

An example experimentally-determined Bode plot appears as follows. This plot is taken from the Doctoral thesis work of Steve Ludwick¹, and shows the response for an electromagnetically-driven tool axis which carries and positions a diamond cutting tool in a novel fast tool servo which Ludwick developed. The Bode plot represents the response from motor torque command in volts to angular output in radians as measured by a rotary encoder.



In this plot, the frequency response is presented with magnitude versus frequency in the top panel, and phase versus frequency in the bottom panel. This is the typical Bode plot format. The frequency axis is in units of Hertz (cycles/sec) rather than the radians/sec units that the theory presented earlier assumes. Both are acceptable units for frequency, however it is required that the plot clearly label the associated units. The relationship between these is $\omega = 2\pi f$, where f is in Hz and ω is in units of rad/sec. Note also the logarithmically-scaled axes for frequency and magnitude, and the linear axes for phase shift.

In this particular plot, the logarithmic scaling of the magnitude is accomplished by plotting magnitude in units of *decibels*, which are abbreviated as dB. The definition of dB is

$$\text{dB} = 20 \log_{10}(M) \quad (5)$$

¹Ludwick, Stephen, "High-Speed Lens Cutting Machine," Ph.D. Thesis, MIT Dept. of Mechanical Engineering, June, 1999

where M is the transfer function magnitude as defined earlier. Thus a magnitude of $M = 100$ is equivalent to 40 dB, and so on. Bode plots can be labelled either with the magnitude M on a logarithmically-scaled axis, or with the magnitude in dB on a linearly scaled axis. It is my preference to label with M directly, since this is the quantity of most direct interest, and avoids computations to convert back from dB. However, common plotting software, such as the `bode.m` routine in Matlab, plot in terms of dB.

While it is not strictly correct to take the log of a quantity with dimensions, the standard practice is not to worry about the units when computing dB. Thus, while the indicated transfer function magnitude is in units of rad/V, when computing the magnitude in dB, we simply take $20 \log_{10}(M)$ without consideration for the units of M .

The magnitude plot passes through -60 dB at a frequency of about 100 Hz. Given the definition of dB above, this means that the transfer function magnitude has a value of 10^{-3} at a frequency of 100 Hz. Further, the phase shift is equal to -180° at this frequency.

Putting this information together, if the input voltage has an amplitude of 1 V at a frequency of 100 Hz, then we can write the input as $v_{in}(t) = 1 \sin(2\pi 100t)$ V. Given the Bode plot magnitude and phase at this frequency, then we can compute the output angle as $\theta_{out}(t) = 10^{-3} \sin(2\pi 100t - \pi)$ rad. That is, the output amplitude is 1 milliradian, with a phase shift of $-\pi$ relative to the input voltage. Note here that we've converted 100 Hz to $2\pi 100$ rad/sec, and the phase shift of -180° to $-\pi$, since the argument of sine must be in radians.

In this design, it was key to have the tool axis transfer function with its resonant frequencies as high as possible. The peaks in the magnitude plot show that the first resonant frequency is at 910 Hz, and the second resonant frequency is at 2280 Hz. These are remarkably high for such an electromechanical axis. By keeping the resonant frequencies high, it was possible to implement a relatively high-bandwidth closed-loop control bandwidth, and thereby achieve accurate and rapid control of the axis.

The experimental data in this plot was taken for a number of varying input amplitudes, which results in the variations in the plots shown at low frequency. This effect is due to the rolling characteristics of the axis bearings as a function of amplitude. As seen in the plot, however, the measurements are largely amplitude-independent for frequencies above about 30 Hz. Also note that the experimental data becomes somewhat noisy above about 2200 Hz, and thus is not likely reliable above this frequency.

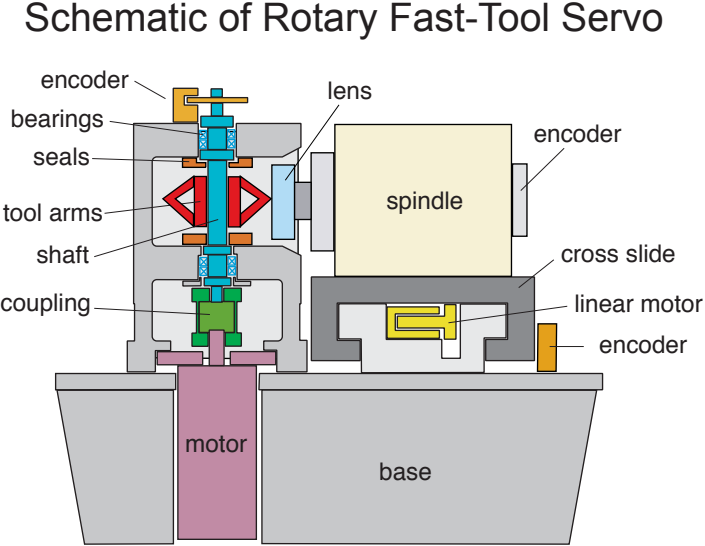
A transfer function model was fitted to this data, and the Bode plot for the model is overlaid on the plot. The fitted model is

$$= \frac{800}{s^2} \cdot \frac{\omega_{n1}^2}{s^2 + 2\zeta_1\omega_{n1}s + \omega_{n1}^2} \cdot \frac{\omega_{n2}^2}{s^2 + 2\zeta_2\omega_{n2}s + \omega_{n2}^2} \cdot e^{-2 \times 10^{-4}s}. \quad (6)$$

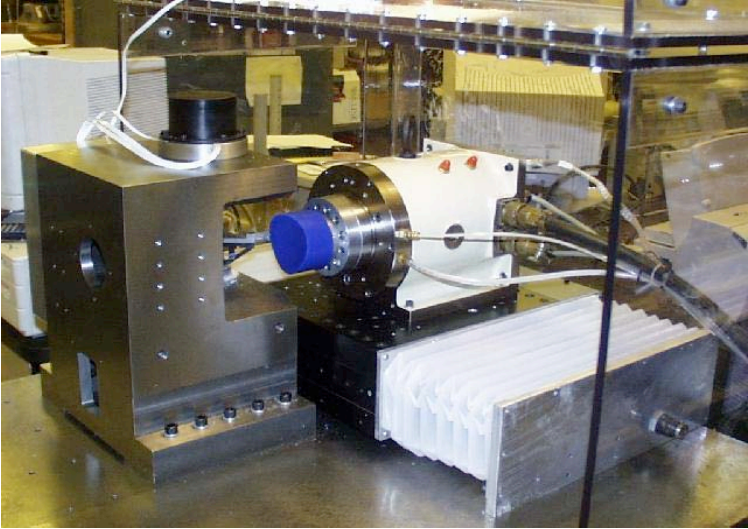
where $\omega_{n1} = 2\pi 910$ rad/sec, $\omega_{n2} = 2\pi 2285$ rad/sec, $\zeta_1 = 0.01$, and $\zeta_2 = 0.005$. The model is of sixth order, plus a time delay. The time delay has a value of 200 μ sec, and is due to the finite computational time of the computer-based controller associated with this axis, computations in the encoder interface electronics, as well as the characteristics of the sampling operation associated with such digital controllers. The term $e^{-2 \times 10^{-4}s}$ represents this time delay (we later show that

the transfer function of a time delay of T seconds is e^{-Ts}). The time delay affects only the phase plot, and is responsible for the sagging of the phase to more negative values at high frequencies.

The machine itself which houses this axis is shown schematically in cross-section with the rotational axis oriented vertically at the left of the figure:



A picture of the hardware setup is shown below. The tool axis is housed in the cast-iron block at the left in the picture.

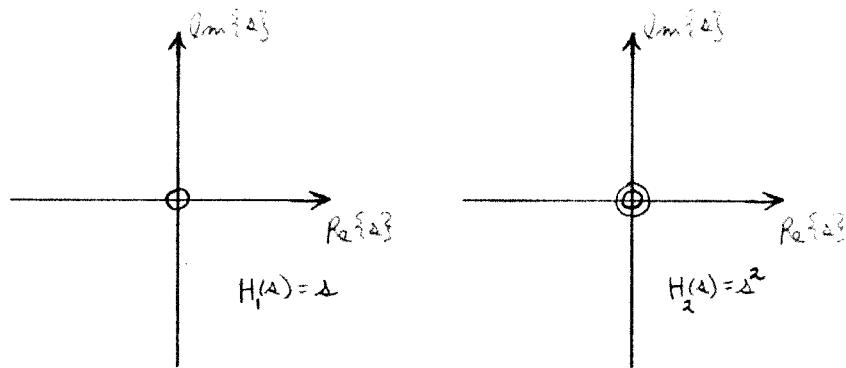


3 Bode plots for some basic building blocks

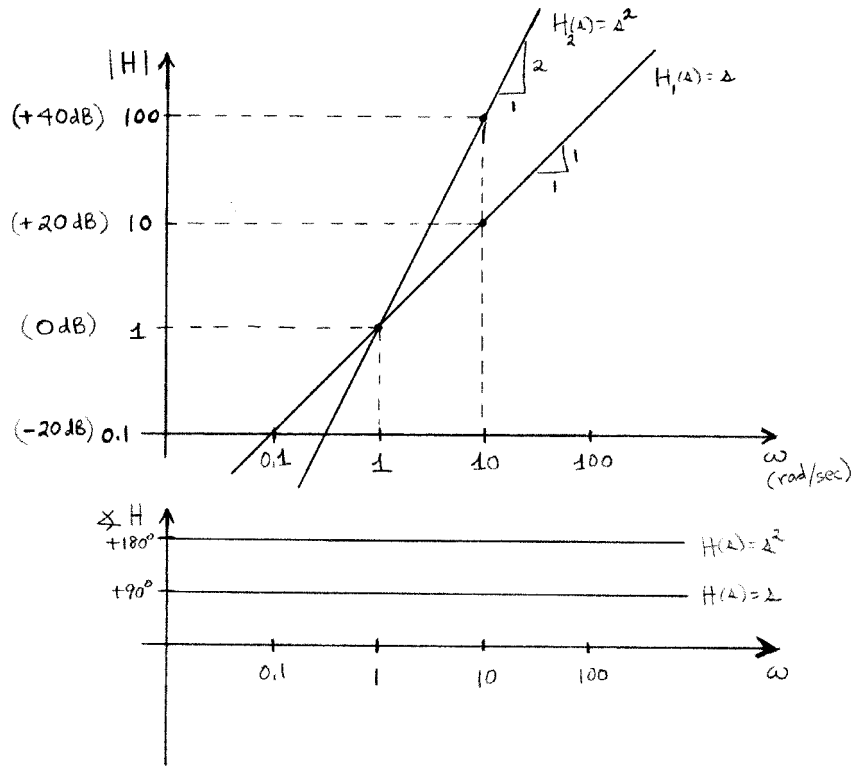
The goal of this section is to show how to sketch Bode plots for some basic terms which appear in transfer functions. Once the plots for these building blocks are well-understood, it is a fairly simple matter to draw the Bode plot for more complex transfer functions. We start with single real-axis zeros and poles and then move on to discussing complex conjugate zeros and poles.

3.1 Zeros at the origin

This section develops the Bode plots for $H_1(s) = s$ and $H_2(s) = s^2$. These have one and two zeros at the origin, respectively:

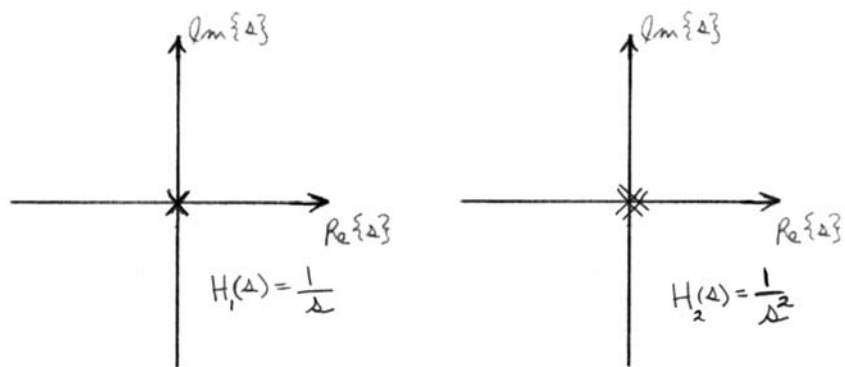


The corresponding Bode plots are

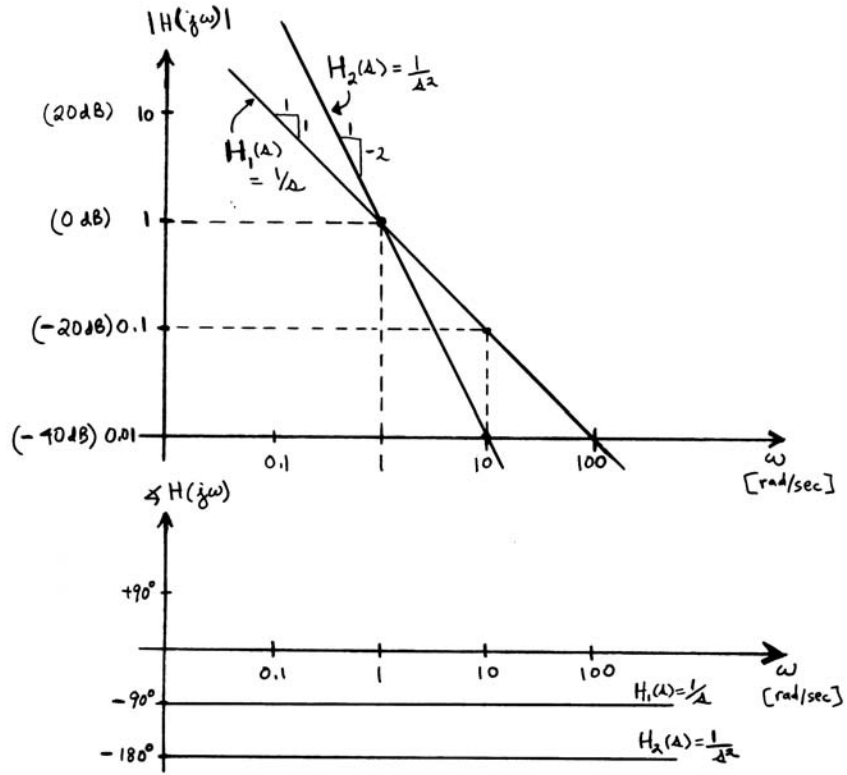


3.2 Poles at the origin

This section develops the Bode plots for $H_1(s) = 1/s$ and $H_2(s) = 1/s^2$. These have one and two poles at the origin, respectively:



The corresponding Bode plots are



4 Bode plot sketching rules