# 58:080

# EXPERIMENTAL ENGINEERING

# LECTURE NOTES

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# LECTURE NOTES 0-- INTRODUCTION

## 1. Introduction

Measurements are important for quality assurance and process control, and to obtain process information. Three aspects will be covered in the Experimental Engineering class:

- Sensors-- fundamentals of sensors for mechanical and thermal quantities.
- Systems-- response and configuration.
- Experimental methods-- planning, acquisition, and analysis.

Quantities of interest include displacement, strain, temperature, pressure, force, torque, moment, velocity, acceleration, volumetric flow rate, mass flow rate, frequency, time, heat flux, etc.

### 1.1 Definitions Commonly used in Sensors and Instrument

- Readability-- scales in analog instrument.
- Least Count-- smallest difference between two indications.
- Static Sensitivity-- displacement versus input, e.g., scale in oscilloscope (cm/mV), etc.
- Hysteresis-- measured quantity which depends on the history to reach that particular condition; generally it is a result of friction, elastic deformation, magnetic, or thermal effects.
- Accuracy-- deviation of a reading from a known input.
- Precision-- related to reproducibility of measurement.
- Error-- deviation from a known input, a measure of accuracy.
- Uncertainty-- data scatter, a measure of precision.

### 1.2. Calibration

Calibration involves a comparison of a particular instrument with respect to a known quantity provided from (1) a primary standard, (2) a secondary standard with a higher accuracy than the instrument to be calibrated, or (3) a known input source.

### 1.3. Standards

The National Institute of Standards and Technology (NIST) has the primary responsibility to maintain standards for such quantities as length, time, temperature, and electrical quantities for the US.

<u>Mass.</u> International Bureau of Weights and Measurements (Sevres, France) maintains several primary standards, e.g., the kilogram is defined by the mass of a particular platinum-iridium bar maintained at very specific conditions at the Bureau.

<u>Time.</u> One second has been defined as the time elapsed during 9,192,631,770 periods of the radiation emitted between two excitation levels of the fundamental state of cesium-133. The Bureau Internationale de l'Hueure (BIH) in Paris, France maintains the primary standard for clock time. The standard for cyclical frequency is based on the time standard, 1 Hz = 1 cycle/second, or 1 Hz = $2\pi$ radian/second.

<u>Length.</u> One meter is defined as the length traveled by light in $3.335641 \times 10^{-9}$ second (based on the speed of light in a vacuum).

<u>Temperature.</u> The absolute practical scale is defined by the basic SI unit of a Kelvin, K. The absolute temperature scale, Kelvin, is based on the polynomial interpolation between the equilibrium phase change points of a number of pure substances from the triple point of the equilibrium hydrogen (13.81 K) to the freezing point of gold (1337.58 K). Above 1337.58 the

scale is based on Planck's law of radiant emissions. The details of the temperature standard are governed by the International Temperature Scale-1990.

Electric Dimensions; volt (V), ampere (A), and ohm (Ω). One ampere absolute is defined by 1.00165 times the current in a water-based solution of $AuN_2$ that deposits Au at an electrode at a rate of 1.118 x $10^{-5}$ kg/s. One ohm absolute is defined by 0.9995 times the resistance to current flow of a column of mercury that is 1.063 m in length and has a mass of 0.0144521 kg at 273.15 K. The practical potential standard makes use of a standard cell consisting of a saturated solution of cadmium sulfate. The potential difference of two conductors connected across such a solution is set at 1.0183 V at 293 K.

Laboratory calibration is made with the aid of secondary standards, e.g. standard cells for voltage sources and standard resistors, etc.

1.4. Dimensions and Units

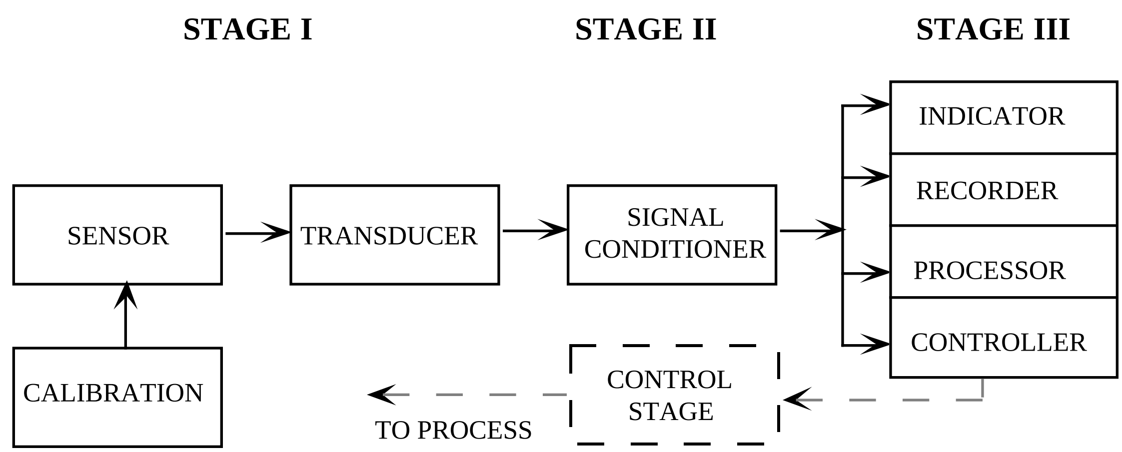
Fundamental dimensions are: length, mass, time, temperature, and force. Basic SI units are: m, kg, s, A, K, cd (candela, luminous intensity), and supplemental units are rad (radian, plane angle) and sr (steradian, solid angle). There are many derived SI units, for example, N, J, W, C (Coulomb = A • s), V (W/A), Ω(V/A), Hz, W/m2, N/m2 (Pa), Hz (1/s), etc. Conversion factors between the SI and US engineering units are fixed, e.g. 1 in. = 0.02540005 m, 1 $lb_m$ =0.45359237 kg., ($^oC$) = (K) - 273.15, ($^oF$) = (K) -459.67, etc.

## 2. General Measurement System

Most measurement systems can be divided into three parts:

Stage I -- A detector-transducer or sensor stage,
Stage II -- An intermediate stage (signal conditioning), and
Stage III-- A terminating or read-out stage ( sometimes with feedback signal for control).

The dynamic response of a generalized measurement system can be analyzed by a mechanical system. A schematic of the generalized measurement system is shown below.




## 3. Types of Input Quantities

- Time relationship
  Static-- not a function of time.
  Dynamic-- steady-state, periodic, aperiodic, or transient (single pulse, continuing, or

random).

- Analog or digital
  Analog-- temperature, pressure, stress, strain, and fluid flow quantities usually are analog (continuous in time).
  Digital-- quantities change in a stepwise manner between two distinct magnitudes, e.g., TTL signals.

The time relationship is important in selecting an instrument adequate for the required time response, and proper but different signal conditioners are usually needed depending on the input signal is digital or analog.

## 4. Error Classification

Three types of error can be identified: systematic, random and illegitimate errors. Systematic errors are not susceptible to statistical analysis, and generally result from calibration errors, certain type of consistently recurring human error, errors of technique, uncorrected loading errors, and limits of system resolution. Random or accidental errors are distinguished by lack of consistency. They involve errors stemming from environmental variations, certain type of human errors, errors resulting from variations in definition, and errors derived from insufficient definition of the measuring system. Illegitimate errors are those should not exist-- blunders or mistakes, computational errors, and chaotic errors. Error analysis is necessary for measurements.

## 5. Calibration (Output versus Known Input)

Static Calibrations

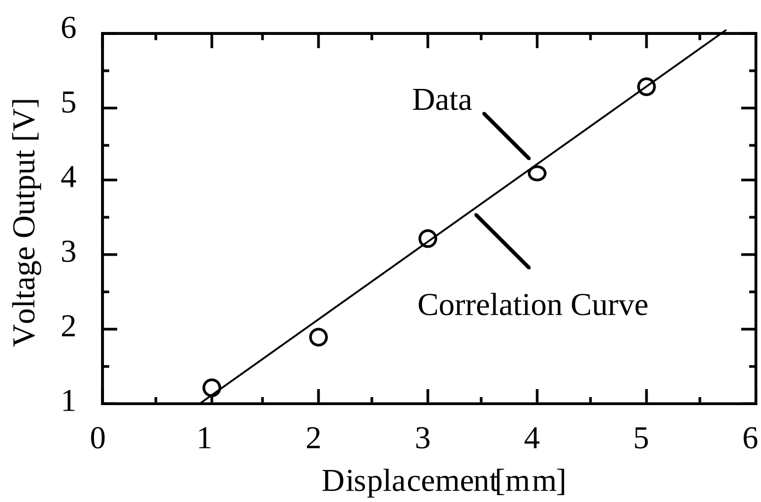
Static ⇔ independent of time
Only the magnitude of the known input is important in static calibrations.

Dynamic Calibrations

Time dependent variables are measured in dynamic calibrations.

Calibration Curve

Usually plotted in terms of output versus input of known values or standards.



5.1 Static Sensitivity

$$K = K(x_1) = \left(\frac{dy}{dx}\right)_{x\ =\ x_1}$$

5.2 Range

Input range : $\nu_i = x(\max) - x(\min)$
Output range: $\nu_o = y(\max) - y(\min)$

5.3 Accuracy

Absolute error, $\varepsilon$ = true value - indicated error

Relative accuracy, $A = \left| 1 - \frac{\varepsilon}{\text{true value}} \right|$

5.4 Sequence Calibration

Hysteresis error, $\varepsilon_h$; $\varepsilon_h = \left| (y)_{\text{upscale}} - (y)_{\text{downscale}} \right|$

5.5 Random Calibration

Linearity Error
Sensitivity and Zero Errors
Instrument Repeatability

## 6. Experimental Test Plan

A well thought-out experimental test plan includes
(1) An identification of pertinent process variables and parameters.
(2) A measurement pattern.
(3) A selection of a measurement technique and required equipment.
(4) A data analysis plan.

- Random tests-- a random order set to the applied independent variables.
- Replication-- an independent duplication of a set of measurements under similar controlled conditions.
- Concomitant Methods-- two or more estimates for the result, each based on a different method.

## 7. Measurement Overview

The overall planning of experiments should include
(1) Objective
(2) Plan -- to achieve the objectives
(3) Methodology
(4) Uncertainty Analysis
(5) Costs
(6) Calibration
(7) Data Acquisition
(8) Data Analysis

# LECTURE NOTES I-- UNCERTAINTY ANALYSIS

# I. Statistical Analysis

## I.1 Introduction

Variations are usually observed in engineering measurements repeatedly taken under seemingly identical conditions. Source of the variation can be identified as follows:

<u>Measurement System</u>
Resolution and Repeatability

<u>Measurement Procedure and Technique</u>
Repeatability

<u>Measured Variable</u>
Temporal variation and spatial variation

Statistical analysis provides estimates of
(1) single representative value that best characterizes the data set,
(2) some representative value that provides the variation of the data, and
(3) an interval about the representative value in which the true value is expected to be.

Repeated measurements of x will yield a most probable value "$\bar{x}$ ", and the true value x' will lie in the interval $\bar{x} - u_x$ and $\bar{x} + u_x$, or

$$\bar{x} \pm u_x \text{ (P\%)}$$

with some probability level (or confidence level), i.e., at P(%)

## I.2 Statistical Properties of a Single Point Measurement

Statistical properties of a single-point measurement can be illustrated with the example of calibration of an instrument. Let's consider the calibration of a pressure gauge using a dead-weight tester and 20 readings are obtained. The test set-up is shown:

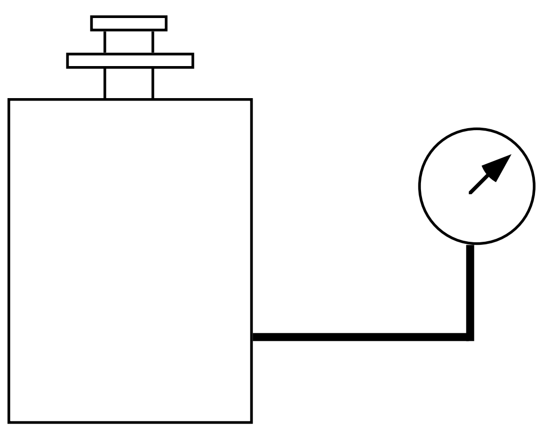


Figure 1. Schematic of a Pressure Calibration Set-Up.

The data can be grouped into 5 groups, e.g.

| Range of Group (kPa) | No. in the Group |
|---|---|
| 9.8-9.9 | 3 |
| 9.9-10.0 | 4 |
| 10.0-10.1 | 8 |
| 10.1-10.2 | 3 |
| 10.2-10.3 | 2 |

The histogram is plotted in the following:

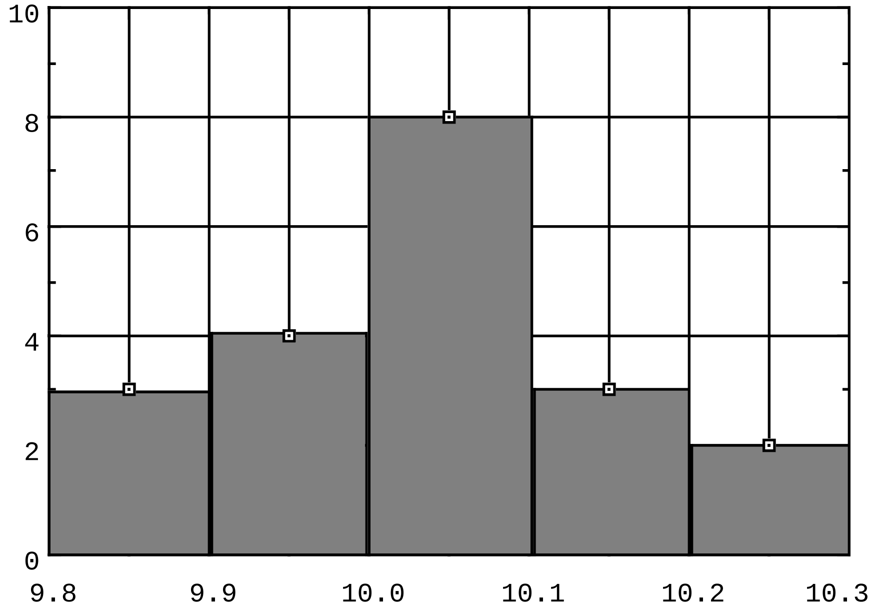



Figure 2. Histogram.

When a larger sample is available, the probability density function (*pdf*) of the distribution can be defined in the following manner:

$$f\,\Delta x = \frac{\text{No. of Readings in } \Delta x}{\text{Total No. of Readings}}$$

As the total number of readings increases and $\Delta x$ decreases, f then approaches a continuous function:

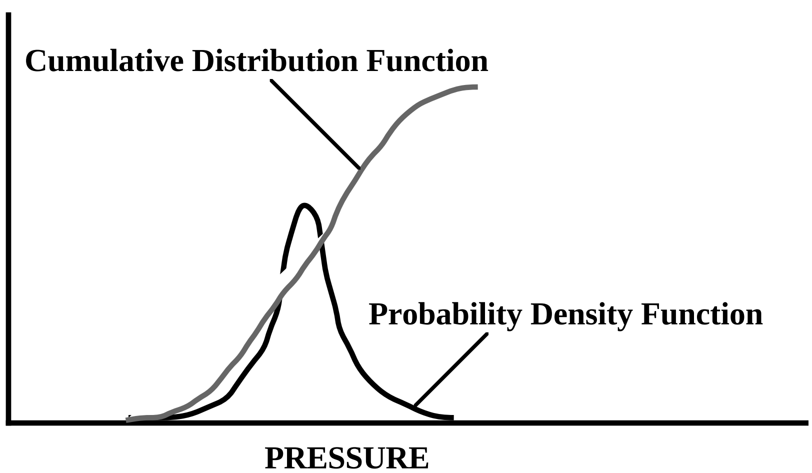



Figure 3. Schematic Diagrams of Probability Density Function and Cumulative Distribution Function.

The mathematical expression for the cumulative distribution function (F) is as follows:

$$P\ (a<x<b) = \sum f_i\ \Delta x_i$$

$$F\ (a) = P\ (p<a) = \int_{-\infty}^{a} f\ dp$$

A probability distribution function can be characterized by its moments. Suppose n readings are given, e.g. $x_1$, $x_2$, .... and $x_n$, then

<u>Mean value</u>

$$x' = \lim_{n\to\infty} \frac{\sum x_i}{n} \text{ (true mean)}$$

$$\bar{x} = \frac{\sum x_i}{n} \text{ (finite sample mean)}$$

<u>2nd moment</u>

$$m_2 = \lim_{n\to\infty} \frac{\sum (x_i - x')^2}{n}$$

Variance

$$\sigma^2 = \lim_{n\to\infty} \frac{\sum (x_i - x')^2}{n}$$

Sample Variance

$$S_x^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1}$$

Standard Deviation $\sigma$

Sample Standard Deviation $S_x$

<u>3rd moment</u> (measure of symmetry, skewness)

$$m_3 = \lim_{n\to\infty} \frac{\sum (x_i - x')^3}{n}$$

<u>4th moment</u> (measure of peakness, kurtosis)

$$m_4 = \lim_{n\to\infty} \frac{\sum (x_i - x')^4}{n}$$

Several mathematical functions are often used as a *pdf*, for example, the two-parameter functions of the normal (or Gaussian) distribution function and the log normal distribution function, e.g., see Table 4-2 Textbook (p. 112) for examples of distribution functions.

In engineering applications, the Gaussian distribution function can be used to describe the distribution function:

$$p\ (x) = \frac{1}{\sigma\ (2\pi)^{1/2}} \exp\left(\frac{-\ (x - x')^2}{2\ \sigma^2}\right)$$

where x' and $\sigma$ can be estimated from $\bar{x}$ and $S_x$ of a finite sample size of n.

The calibration of the pressure gauge example has

$\bar{x}$ = 9.985 (or 10.0) and

$S_x$ = 0.1182 (or 0.12).

If the distribution function is exactly the Gaussian distribution, then 68 % of the readings fall in $\bar{x} \pm S_x$, 95 % in $\bar{x} \pm 2S_x$, and 99.7 % in $\bar{x} \pm 3S_x$. Examining the histogram, one observes that 75 % of the readings are in the range of 9.98 ± 0.12 kPa, and 90 % of the readings are in the range of 9.98 ± 0.24 kPa. Questions then arise as to how close the distribution follows a Gaussian or Error distribution. To test the "normality" of a distribution function, the $\chi^2$ test may be performed.

## I.3 Test of Data Outliers

Chauvenet's Criterion

First let's examine a way of rejecting "bad " data by using Chauvenet's Criterion. It states that a reading may be rejected if the probability of obtaining the particular deviation from the mean is less than 1/(2n). The maximum allowable deviation, $d_{max}$, can be obtained from the normal (Gaussian) distribution function and reject $x_i$'s which lie outside the $d_{max}$ range.

$$|x_i - \bar{x}| > d_{max}$$

As an example, let's consider n = 3. The CDF value for n = 3 is obtained from 1- (1/2n)/2, i.e., it assumes a value of 0.9167; the corresponding departure from the mean based on the standard deviation ($d_{max}/\sigma$) is 1.38. Similarly one can obtain the "multiplier" values (to standard deviation) to be 1.73 for n = 6, and 1.96 for n = 10, etc. (as shown below).

| n | 3 | 6 | 10 | 25 | 50 | 100 | 500 | 1000 |
|---|---|---|---|---|---|---|---|---|
| $d_{max}/\sigma$ | 1.38 | 1.73 | 1.96 | 2.33 | 2.57 | 2.81 | 3.29 | 3.48 |

The multiplier value can also be obtained based on different assumptions.

Student's t Distribution

The data point outside the 99.8% of the population based on Student's t Distribution is considered an outlier. Therefore, $t_{v,99.8}$ is to be used as the multiplier value as discussed in Textbook.

Modified Thompson $\tau$ Technique
(Measurement Uncertainty, ANSI/ASME, 1986; Wheeler and Ganji, 1995)

| n | 3 | 5 | 7 | 10 | 15 | 20 | 25 | 30 | 35 |
|---|---|---|---|---|---|---|---|---|---|
| $d_{max}/\sigma$ | 1.150 | 1.572 | 1.711 | 1.798 | 1.858 | 1.885 | 1.902 | 1.911 | 1.919 |

Tchebychev Inequality

It should be mentioned that when all else fails for the test, one can apply the Tchebychev's inequality. The result of this analysis is "distribution free." It states that for any distribution having a finite mean and variance:

$P(|x - \bar{x}| \le k\,S_x) > 1 - k^{-2}$, or
$P(|x - \bar{x}| \le S_x) > 0$
$P(|x - \bar{x}| \le 2\,S_x) > 0.75$
$P(|x - \bar{x}| \le 3\,S_x) > 0.89$, etc.

To yield a 5% uncertainty, it requires that

$$P(|x - \bar{x}| \le 4.5\,S_x) > 0.95$$

Thus, the multiplier is 4.5 based on Tchebychev's inequality to reject data outside the 95% of the sample, and 22.36 outside the 99.8%. Since the criterion is excessively broad, there are attractive gains to be made by treating whether it is reasonable to assume a particular distribution function is satisfactory.

## I.4 Goodness-of-Fit: Chi-squared Test

It should also be noted that the $\chi^2$ test is only one of the many approaches for testing the normality of a distribution function. The procedure of the test follows.

Given: m readings of $x_1, x_2, \ldots\ldots\ldots\ldots, x_m$

(1) Group the m readings in ranges to yield n groups and $n \ge 4$, and preferably more than 5 readings in each group. Let's identify $n_{oi}$ as the number of readings in group i.

(2) Compute $n_{ei}$, the expected number of readings in group i should the distribution follow a Gaussian distribution. The mathematical expression is

$$n_{ei} = m\,P_i = m \int_{x_{i-1}}^{x_i} p(x)\,dx \quad \text{(or the integration over the range of group i) .}$$

(3) Compute the $\chi^2$ values for the distribution:

$$\chi^2 = \frac{\sum (n_{oi} - n_{ei})^2}{n_{ei}} \quad \text{(summation over the n groups)}$$

(4) Determine the number of degrees of freedom:

$\nu = n - 3$

where 3 is used in the calculation of $\nu$ to account for $\bar{x}$ and $S_x$, and "grouping" in estimate of $u_{ei}$, i.e., the constraint is 3 not 2.

(cf., Doebelin, E.O., 1983, *Measurement Systems Application and Design*, 3rd Ed., p. 53, McGraw-Hill, or Beckwith, T.G., Buck, N.L., and Maraugoui, R.D., 1982, *Mechanical Measurements*, 3rd Ed., Addison-Wesley, p. 284.)

(5) Determine if the $\chi^2$ value falls in the limit given by 5 % and 95 %, or find $P(\chi^2)$ for the distribution function *p(x)* to see if it is within the limits. The table in the handout illustrates the $\chi^2$ values as a function of f for different probability levels.

Example 1. Perform the $\chi^2$ test for the temperature measurement using the data set given.

| T(K) | 1850 | 1900 | 1950 | 2000 | 2050 | 2100 |
|---|---|---|---|---|---|---|
| Occurrence | 1 | 9 | 6 | 18 | 10 | 2 |

Solution:

- First, determine mean and standard deviation of the data, $\bar{x}$ = 1985.9 K, $S_x$ = 60.2 K
- Second, calculate the $\chi^2$ values:

| Group | T-Range | $n_o$ | z-Range | $n_{ei}$ | $(n_o-n_e)_i^2/n_{ei}$ |
|---|---|---|---|---|---|
| 1 | -∞ to 1925 | 10 | -∞ to -1.01 | 7.185 | 1.103 |
| 2 | 1925 to 1975 | 6 | -1.01 to -0.18 | 12.530 | 3.403 |
| 3 | 1975 to 2025 | 18 | -0.18 to 0.65 | 14.426 | 0.885 |
| 4 | 2025 to ∞ | 12 | 0.65 to ∞ | 11.859 | 0.002 |

Thus, $\chi^2$ = 5.393. Note that a transformation was used in the above table; namely,

$$z = \frac{x - x'}{\sigma}$$

where $\bar{x}$ and $S_x$ are the best estimates of x' and $\sigma$.

$$n_{ei} = m\,P_i = m \int p(x)\,dx \;=\; m \int p(z)\,dz \qquad n_{ei} = m\left( \int_{-\infty}^{z_i+\Delta z_i} p(z)\,dz - \int_{-\infty}^{z_i} p(z)\,dz \right)$$

$n_{ei} = m\,[F(z_i+\Delta z_i) - F(z_i)]$, Also note that $F(-|z|) = 1 - F(z)$

- Third, compute the number of degrees of freedom, $\nu = n - 3 = 1$
- Fourth, consult the $\chi^2$ table for normality.

For this particular example, the interpolation shows that P = 0.022, suggesting that the distribution is not likely a Gaussian distribution, following the 5-95 criterion. The 10-90 criterion can be used.

## I.5 Number of Measurements Required

Precision Interval, CI

$$CI = x' - \bar{x} = \pm\, t_{\nu,\,95} \frac{S_x}{N^{1/2}} \quad (95\%)$$

where $S_x$ is a conservative estimate based on prior experience, manufacturer's information. The deviation from the mean usually is symmetric, thus one may define d = CI/2 or

$$N \approx \left( \frac{t_{\nu,95}\, S_x}{d} \right)^2 \quad (95\%)$$

It is suggested that when the precision interval is considerably smaller than the variance, the value of N required will be large, say N > 60, and the t estimator will be approximately equal to 2. A trial and error method is required because *t* estimator is a function of the degrees of freedom in the sample variance.

One approach to estimate the sample size is to conduct a preliminary small number of measurements, $N_1$, to obtain an estimate of the sample variance, $S_1$. Then $S_1$ is used to estimate the total number of measurements, $N_T$,

$$N_T \approx \left( \frac{t_{N-1,\,95} S_1}{d} \right)^2 \quad (95\%)$$

($t_{N-1,95}$ should be used)

## I.6 Student's t distribution

Small sample size introduces the bias error. This bias error can be quantified by

$$[\bar{x} - t S_{\bar{x}}] < x' < [\bar{x} + t S_{\bar{x}}]$$

where t is the student's t distribution. The student's t distribution is defined by a relative-frequency equation f(t) :

$$f(t) = F_0 \left( 1 + \frac{t^2}{\nu} \right)^{(\nu+1)/2}$$

where $F_0$ is the relative frequency at t = 0 to make the total area under f(t) curve equal to unity, and $\nu$ is the number of degrees of freedom. When $t \to \infty$, student's t distribution approaches the normal distribution. Table 4.4 (cf., p. 119 of Textbook) summarizes the t - estimator.

Student t distribution can also be interpreted as an estimate of the variation from the mean. For a normal distribution of x about some sample mean, one can state that

$$x_i = \bar{x} + t_{V,P} S_x \quad (P\%)$$

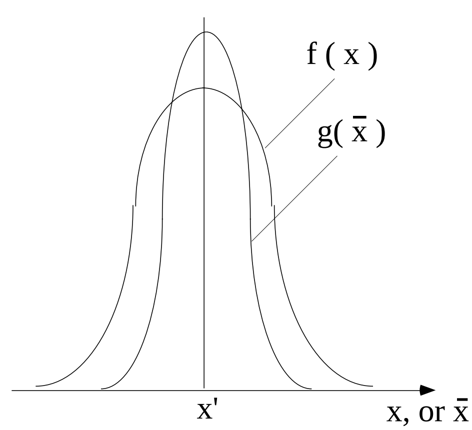
where $t_{V,P}$ is obtained from a new weighting function for finite sample set which replaces the "z-variable" in our earlier discussion of the Gaussian distribution. The standard deviation of the means represents a measure of the precision in a sample mean. For example, the range over which the true mean value may lie is at probability level P is given by

$$\bar{x} \pm t_{V,P} S_{\bar{x}} \quad (P\%)$$

where the standard error is defined

$$S_{\bar{x}} = \frac{S_x}{\sqrt{n}}$$

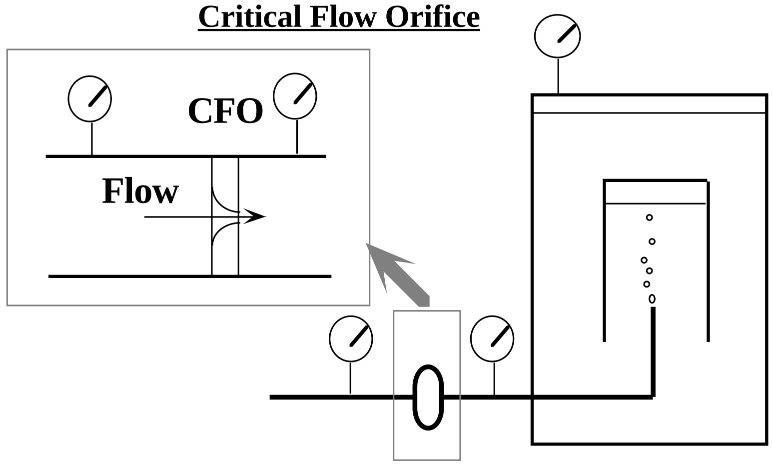
Graphically, it can be shown as



I.7 Least Squares-- a Multiple Point Correlation

Quite often a range of operating condition is desired, for example, the calibration of an instrument over the range of operation. The previous section only deals with a single point statistical analysis. The least squares analysis is introduced in this section. Calibration of a pressure gauge or mass flow meter (critical orifice meter) is used to illustrate the least squares analysis.

**Critical Flow Orifice**




To present the data, often a single curve is desired and estimate of the uncertainty of a measurement when this instrument is used needs to be specified. To accomplish this, regression techniques may be used. We will discuss the linear regression here.

Suppose that the data can be represented by a linear function,

$$y = a + b\,x$$

Let's define a linear calibration line of

$$y_c = a + b\,x$$

To find a best estimate of a and b, and to determine the uncertainty, one may define a function of the sum of the squares of the deviation between the data and the calculated value:

$$S_{yx} = \Sigma\,(y_i - y_{ci})^2 = \Sigma\,(y_i - (a + b\,x_i))^2$$ , where $\Sigma$ is over n data points.

To minimize the uncertainty, one may take a derivative of S

$$dS = \frac{\partial S}{\partial a}\,da + \frac{\partial S}{\partial b}\,db$$

Let $\frac{\partial S}{\partial a} = \frac{\partial S}{\partial b} = 0$, or

$$\frac{\partial S}{\partial a} = -2\,\Sigma\,[y_i - (a + b\,x_i)] = 0; \quad \frac{\partial S}{\partial b} = -2\,\Sigma\,x_i\,[y_i - (a + b\,x_i)] = 0.$$

$$-2\sum_i [y_i - (a + bx_i)] = 0; \quad 2\sum_i \{x_i [y_i - (a + bx_i)]\} = 0$$

$$\sum_i y_i - (n a + b \sum_i x_i) = 0 ; \quad \sum_i x_i y_i - a \sum_i x_i - b \sum_i x_i^2 = 0$$

$$a = \frac{1}{n}( \sum_i y_i - b \sum_i x_i ) ; \quad \sum (x_i y_i) - a (\sum x_i) - b \sum x_i^2 = 0$$

Therefore, one obtains

$$\boxed{a = \bar{y} - b\bar{x}}$$

where $\bar{y} = \frac{\Sigma y_i}{n}$ and $\bar{x} = \frac{\Sigma x_i}{n}$ . Now consider

$$\sum x_i (\sum y_i - n a - b \sum x_i) - n \sum (x_i y_i) + n a \sum x_i + n b \sum x_i^2 = 0, \text{ or}$$

$$\sum x_i \sum y_i - b (\sum x_i )^2 - n \sum (x_i y_i) + n b (\sum x_i^2 ) = 0$$

$$\boxed{b = \frac{n\sum (x_i y_i) - (\sum x_i)(\sum y_i)}{n\sum x_i^2 - (\sum x_i)^2}}$$

The variance of the estimate (or standard error of the fit) is

$$S_{yx}{}^2 = \frac{\Sigma (y_i - y_{ci} )^2}{n - 2}$$

Note that n-2 appears in the equation is due to the degrees of freedom is reduced by 2 (a and b are determined). It can be shown that

$$S_{yx}{}^2 = \frac{n-1}{n-2}(S_y{}^2 - b^2 S_x{}^2)$$

where $S_x{}^2 = \Sigma (x_i - \bar{x} )^2 /(n-1)$ and $S_y{}^2 = \Sigma (y_i - \bar{y} )^2 /(n-1)$.

<u>Correlation Coefficient</u>

$$r^2 = 1 - \frac{S_{yx}^2}{S_y^2}$$

where $S_y^2 = \frac{1}{n-1}\sum_{i=1}^{n} (y_i - \bar{y})^2$

The correlation coefficient can also be written as $r = \frac{\Sigma (x_i - \bar{x})(y_i - \bar{y})}{(n-1) S_x S_y} = b \frac{S_x}{S_y}$

The correlation coefficient is bounded by ± 1, with perfect correlation having r = 1 or -1 (y decreases with x). For 0.9 < r < 1.0 or -1.0 < r < -0.9, a linear regression can be considered as a reliable relation between y and x. When r = 0 the regression explains nothing about the variation of y versus x (a horizontal line).

Standard Error of b is

$$S_b = \frac{S_{yx}}{S_x (n-1)^{1/2}}$$

Slope of Fit

$$S_b = S_{yx} \sqrt{\frac{n}{n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}}$$

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Example 2. The calibration of a pressure gauge using a dead weight tester yields

| i | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| x ($p_{ind}$,kPa) | 20 | 30 | 50 | 70 | 80 | 100 |
| y ($p_{act}$,kPa) | 30 | 50 | 60 | 80 | 100 | 110 |

Find a least-squares -fit for the above data set.

Solutions:
$\Sigma x_i^2 = 25100$, $\Sigma y_i^2 = 35500$, $\Sigma x_i y_i = 29700$

b=0.9858, $\bar{y}$=71.67, $\bar{x}$=58.33, a= $\bar{y}$-b $\bar{x}$=14.17 (or 14.16 from direct calculation), r=0.9857

$p_c = 14.17 + 0.9858\ p_g$

t-estimator can be used to establish a precision or confidence interval about the linear regression. This interval can be expressed as $\pm t_{4,95} S_{yx}$ when 95% interval is used. For the example considered, $t_{4,95} = 2.770$ and $S_{yx} = 5.75$. Thus, the least-squares-fit can be expressed as

$p_c = 14.17 + 0.9858\ p_g \pm 15.93$ (kPa)

---

## II. Uncertainty Analysis

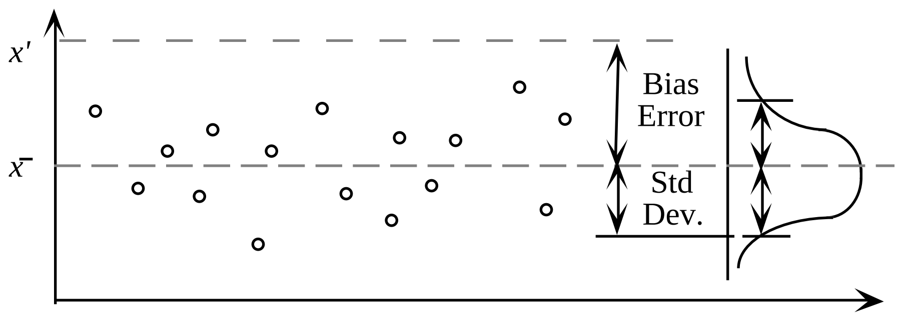
### II.1 Introduction

Systematically quantifying error estimates - uncertainty analysis.
Uncertainty in Experiments ⇔ Tolerance in Design.

II.2 Measurement Errors

Bias Errors

Precision Error




Statistical representation of the measurements can be given by

$$x' = \bar{x} \pm u_x \ (P\%)$$

where $u_x$ is the uncertainty of the data set.

II.3 Error Sources

- Calibration errors
- Data acquisition errors
- Data reduction errors

3.1 Calibration Errors

Elemental errors can enter the measurement system during the process of calibration. There are two principal sources : (1) the bias and precision errors in the standard used in the calibration, and (2) the manner in which the standards is applied. Table 5.1 of Textbook summarizes the calibration sources.

3.2 Data Acquisition Errors

Errors due to the actual act of measurement are referred to as data acquisition errors. Power settings, environmental conditions, sensor locations are some examples of data acquisition errors.

3.3 Data Reduction Errors

The errors due to curve fits and correlations with their associated unknowns are known as the data reduction errors. Table 5.3 summarizes the error source group.

II.4 Bias and Precision Errors

4.1 Bias Error

Statistical analysis can not discover the bias error; estimates of bias errors can be made by
(1) Calibration
(2) Concomitant methodology
(3) Inter-laboratory comparisons
(4) Experience

4.2 Precision Error

Precision error is affected by
a. Measurement System-- repeatability and resolution
b. Measurand-- temporal and spatial variations
c. Process-- variations in operating and environmental conditions
d. Measurement Procedure and Technique-- repeatability

II.5 Uncertainty Analysis: Error Propagation

Consider a measurement of measured variables, x, which is subject to k elements of error, $e_j$, j = 1, 2, . . . , k. The root-sum-squares method (RSS) estimates the uncertainty in the measurement, $u_x$, to be

$$u_x = \pm\sqrt{e_1^2 + e_2^2 + \ . \ . \ . + e_k^2} \qquad e_i : \text{elemental error}$$

$$u_x = \pm\sqrt{\sum_{k=1}^{K} e_k^2} \qquad (P\%)$$

A general rule is to use the 95% confidence level throughout the uncertainty calculations.

5.1 Propagation of Uncertainty to a Result

Examples :
- Normal stress derived from force and cross-sectional area measurements
  $\sigma = f(F, A)$
- Surface area derived from measured diameter

---

<u>Example</u> Measured diameter of a quarter yields a mean of 24 mm and a standard deviation of 1.2 mm; i.e., $\bar{x}$ = 24 mm, $S_x$ = 1.2 mm. Find the expected mean area of the quarter.

Solution:

Statistically, the true mean is

$$\bar{A} = \int_{-\infty}^{\infty} \frac{\pi D^2}{4} \ P(D)\, dD = \frac{\pi}{4}\int_{-\infty}^{\infty} D^2 \ P(D)\, dD; \text{ where P(D) is the pdf}$$

whereas

$$x' = \int_{-\infty}^{\infty} x \; P(x) \; dx$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x - x')^2 \; P(x) \; dx$$

---

Consider $y = f(x)$ , and confidence interval $\pm t S_{\bar{x}}$

$$x' = \bar{x} \pm t S_{\bar{x}} \quad ; \bar{x} - t S_{\bar{x}} < x' < \bar{x} + t S_{\bar{x}}$$

$$\bar{y} \pm \delta y = f(\bar{x} \pm t S_{\bar{x}}) = f(\bar{x}) \pm \left[ \left(\frac{dy}{dx}\right)_{\bar{x}} t S_{\bar{x}} + \frac{1}{2}\left(\frac{d^2y}{dx^2}\right)_{\bar{x}} (t S_{\bar{x}})^2 + \dots \right]$$

Define

$$\delta y = \left(\frac{dy}{dx}\right)_{\bar{x}} t S_{\bar{x}} + \frac{1}{2}\left(\frac{d^2y}{dx^2}\right)_{\bar{x}} (t S_{\bar{x}})^2 + \dots$$

First order approximation, $\delta y \approx \left(\frac{dy}{dx}\right)_{\bar{x}} t S_{\bar{x}}$

Thus, the precision interval is $\left(\frac{dy}{dx}\right)_{\bar{x}} t S_{\bar{x}}$

In general, errors contribute to the uncertainty in x, $u_x$, is related to the uncertainty in the estimate of resultant y,

$$\boxed{u_y = \left(\frac{dy}{dx}\right)_{\bar{x}} u_x}$$

The above analysis applies to finite sample size when the distribution function is not known and student's t- distribution is used to approximate the distribution.

Consider a result, R, is a function of L independent variables:

$$R = f_1 (x_1, x_2, \dots, x_L)$$

The best estimate of true mean value, R',

$$R' = \bar{R} \pm u_R \quad (P\ \%)$$

where

$$\bar{R} = \bar{R}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_L) = f_1(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_L)$$

and the uncertainty in $\bar{R}$ is

$$u_R = f_2 (u_{x1}, u_{x2}, u_{x3}, \dots u_{xL})$$

$$u_R = \pm \sqrt{\sum_{i=1}^{L} (\theta_i u_{xi})^2} \qquad (P\%) \qquad \text{Kline-McClintock Second Power Law}$$

where $\theta_i = \left(\frac{\partial R}{\partial x_i}\right)_{x_i = \bar{x}_i}$

$\theta_i$ is known as the sensitivity index

## II.6 Design-Stage Uncertainty Analysis

$$u_d = \sqrt{u_0^2 + u_c^2}$$

$u_0$: zero-order uncertainty of the instrument

$$u_0 = \pm \frac{1}{2} \text{ resolution} \quad (95\%)$$

An arbitrary rule, shown above, assigns a numerical value to $u_0$ to one-half of the instrument resolution with a probability of 95%.

$u_c$: manufacturer's statement concerning the instrument error $u_c$ may have more than one elemental error, for example, due to linearity and repeatability of the instrument. (Example 5.2 of Textbook)

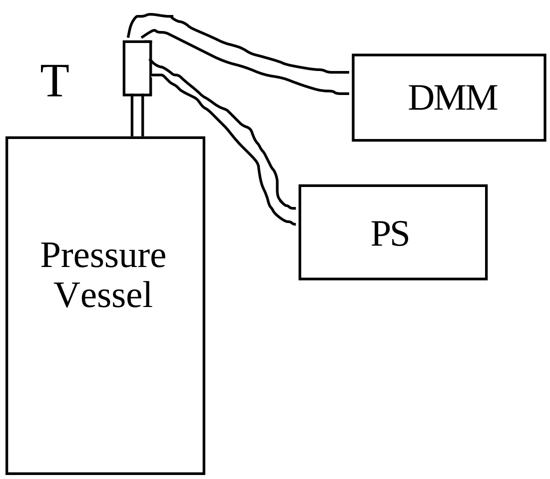
$$u_c = \sqrt{\sum_{k=1}^{K} e_k^2}$$

Multiple Instruments - Example 5.3 (transducer and DMM)

---

Example

Design Stage Uncertainty of Pressure Measurements




Expected Pressure: 3 psi

Transducer

Range: $\pm$ 5 psi
Sensitivity: 1 V/psi
Input Power: 10VDC $\pm$ 1 %
Output: 5V
Linearity: within 2.5 mV/psi over range
Repeatability: within 2 mV/psi over range
Resolution: negligible

DMM

Resolution: 10 $\mu$V
Accuracy: within 0.001% of reading

Analysis:

Assumptions - 95% probability for the values specified
RSS applicable

Voltmeter

$$(u_d)_E = \pm\sqrt{(u_0)_E^2 + (u_c)_E^2}$$

$$(u_0)_E = \pm 5\ \mu V \qquad \text{(resolution)}$$

$$(u_c)_E = \pm (3\text{ psi})\ (1V/\text{psi}) \times 10^{-5} = \pm 30\ \mu V$$

$$(u_d)_E = \pm 30.4\ \mu V = \pm 30\ \mu V$$

Transducer

$(u_c)_p = \sqrt{e_1^2 + e_2^2}$
$e_1 = 2.5$ mV/psi x 3 psi (linearity)
$e_2 = (2$ mV/psi$)$ $(3$ psi$)$ (repeatability)

$(u_c)_p = \pm 9.60$ mV

$(u_0)_p \approx 0$ V/psi
$(u_d)_p = \pm$ 9.60 mV or $(u_d)_p = \pm 0.0096$ psi

Combined Uncertainty

$u_d = \sqrt{(u_d)_E^2 + (u_d)_p^2}$
$u_d = \pm 9.60$ mV
$u_d = \pm 0.0096$ psi (95%)
$u_d = \pm 0.010$ psi

---

II.7 Multiple - Measurement Uncertainty Analysis

Three sources of errors (elemental) are
- calibration (i = 1)
- data acquisition (i = 2)
- data reduction (i = 3)

For multiple measurements, the procedures for uncertainty analysis are
(1) identify the elemental errors,
(2) estimate the magnitude of bias and precision error in each of the elemental errors, B and P,
(3) estimate any propagation of uncertainty through to the result.

Source Precision Index, $P_i$ (i = 1, 2, 3)

$$P_i = \sqrt{P_{i_1}^2 + P_{i_2}^2 + \ . \ . \ . \ + P_{i_k}^2 +} \quad i = 1, 2, 3$$

Measurement Precision Index, P

$$P = \sqrt{P_1^2 + P_2^2 + P_3^2}$$

Source Bias Limit, $B_i$ (i = 1, 2, 3)

$$B_i = \sqrt{\sum_{j=1}^{K} B_{i_j}^2} \quad i = 1, 2, 3$$

$$B = \sqrt{B_1^2 + B_2^2 + B_3^2}$$

The measurement uncertainty in x, $u_x$, is a combination of B and P:

$$u_x = \sqrt{B^2 + (t_{\nu,95}\ P)^2} \qquad (95\%)$$

Degrees of Freedom, ν

The degrees of freedom of $P_i$ and $B_i$ are different. The Welch-Satterthwaite formula is used (cf., Textbook), stateing that

$$\nu = \frac{\left(\sum_{i=1}^{3}\sum_{j=1}^{K} P_{ij}^{2}\right)^{2}}{\sum_{i=1}^{3}\sum_{j=1}^{K}\left(P_{ij}^{4}/\nu_{ij}\right)}$$

i = 1, 2, 3 the three sources of elemental errors

j is referred to each elemental error within each group, $\nu_{ij} = N_{ij} - 1$

---

Example Estimate the precision and bias Errors (i = 2) during the data acquisition summarized below.

Solution:

Force Measurements - Load Cell
Resolution: 0.25 N
Range: 0 to 200 N
Linearity: within 0.20 N over range
Repeatability: within 0.30 N over range

| n | F(N) | n | F(N) |
|---|---|---|---|
| 1 | 123.2 | 6 | 119.8 |
| 2 | 115.6 | 7 | 117.5 |
| 3 | 117.1 | 8 | 120.6 |
| 4 | 125.7 | 9 | 118.8 |
| 5 | 121.1 | 10 | 121.9 |

Precision Error:

$\bar{F}$ = 120.1 [N] $S_F$ = 3.04 [N] $P_{ij} = \frac{S_F}{N^{1/2}}$ = 0.96 [N]

Bias Error:

Elemental errors due to instruments are considered to be data acquisition source error, i = 2. Since the information as to the statistics used to generate the numbers, linearity and repeatability must be considered as bias errors.

$e_1$ = 0.20 N $e_2$ = 0.30 N

$B_{22} = \sqrt{e_1^2 + e_2^2}$ = 0.36 N

In the absence of specific calibration data, manufacturer specifications are considered as bias errors contributing to the data acquisition source.

---

Example Estimate the data reduction errors for the data set below

Solution:

LVDT measurements - Linear Regression (Least Squares)

| x(cm) | y (V) | $y_c - y_i$ |
|---|---|---|

| | | |
|---|---|---|
| 1.0 | 1.2 | -0.14 |
| 2.0 | 1.9 | 0.20 |
| 3.0 | 3.2 | -0.06 |
| 4.0 | 4.1 | 0.08 |
| 5.0 | 5.3 | -0.08 |

Analysis:

Least Squares

$y_c = 0.02 + 1.04x$ [V]

$r = 0.9965$

Precision Index

$S_{y_x}^2 = \frac{4}{3}(S_y^2 - b^2 S_x^2) = 0.02533$; $S_{yx} = 0.159$

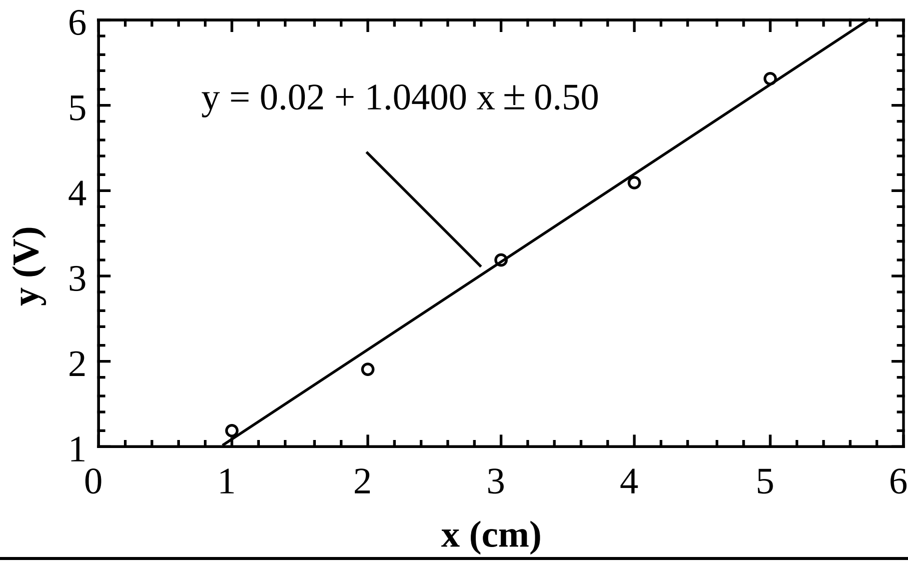
$P_{31} = 0.159$, or $P_{31} = 0.16$

$\nu = 5 - 2 = 3$ (number of degrees of freedom); $t_{3,95} = 3.182$

Correlation

$y_c = 0.02 + 1.04x \pm S_{yx}\ t_{3,95}$

$y_c = 0.02 + 1.04x \pm 0.50$ [V] (95%)

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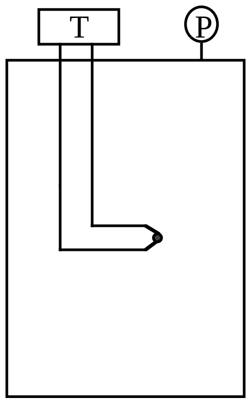
Example Data Reduction Error (Error Propagation)




Rigid Container

Pressure Measurements

Accuracy: 1%, $N_p = 20$
$\bar{p} = 2253.91$ psfa (lb/ ft$^2$ absolute)
$S_p = 167.21$ psfa

Temperature Measurements

Accuracy: 0.6 °R, $N_T = 10$
$\bar{T} = 560.4$°R, $S_T = 3.0$°R

Known: Ideal Gas $R = 54.7$ (ft-lb/lbm-°R)

Find: Density

Solution:

$$\text{Ideal Gas EOS } \bar{\rho} = \frac{\bar{p}}{R\bar{T}} = 0.074 \text{ lbm/ft}^3$$

It is noted that the uncertainty in the evaluation of the gas constant is on the order of ± 0.06 ft-lb/lbm-˚R (or ± 0.33 J/kg - K) due to the uncertainty of molecular weight. This error is neglected.

Pressure

$(B_{21})_p = (0.01)\ (2253.91) = 22.5$ psfa (instrument bias error)

$$(P_2)_p = \frac{S_p}{20^{1/2}} = 37.4 \text{ psfa}$$

$\nu_p = 20 - 1 = 19$ (number of degrees of freedom)

Temperature

$(B_2)_T = 0.6$ ˚R

$(P_2)_T = 0.9$ ˚R $\left(\frac{S_T}{10^{1/2}}\right)$

$\nu_T = 10 - 1 = 9$

Error Propagation

$R' = \bar{R} \pm u_R$ R': true reading; $\bar{R}$ : sample averaged reading

$$\bar{R} = f_1 = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_L)$$

$$u_R = f_2\ (B_{x1}, B_{x2}, \ldots, B_{xL};\ P_{x1}, P_{x2}, \ldots, P_{xL})$$

$$P_R = \pm\sqrt{\sum_{i=1}^{L}(\theta_i P_{xi})^2} \quad \text{Resultant Precision Index}$$

$$B_R = \pm \sqrt{\sum_{i=1}^{L} (\theta_i B_{xi})^2} \quad \text{Resultant Bias Limit}$$

$$\boxed{u_R = \sqrt{B_R^2 + (t_{\nu,95} P_R)^2}} \quad (95\%)$$

where the resultant number of degrees of freedom is

$$\boxed{\nu_R = \frac{\sum_{i=1}^{L} (\theta_i P_{xi})^2}{\sum_{i=1}^{L} (\theta_i P_{xi})^4/\nu_{xi}}}$$

$$\theta_i = \left(\frac{\partial R}{\delta x_i}\right)_{\overline{x}_i}$$

$$\left(\frac{\partial \rho}{\partial T}\right)^2 = \left(-\frac{p}{RT^2}\right)^2 = (1.3112 \text{ x } 10^{-4})^2 = 1.72 \text{ x } 10^{-8}$$

$$\left(\frac{\partial \rho}{\partial P}\right)^2 = \left(\frac{1}{RT}\right)^2 = (3.26 \text{ x } 10^{-5})^2 = 1.06 \text{ x } 10^{-9}$$

$$p = \sqrt{\left(\frac{\partial \rho}{\partial T} (P)_T\right)^2 + \left(\frac{\partial \rho}{\partial P} P_P\right)^2} = 0.0012 \text{ lbm/ft}^3; \quad (P_T = 0.9) \quad (P_P = 37.4)$$

$$B = \sqrt{\left(\left(\frac{\partial \rho}{\partial T}\right) B_T\right)^2 + \left(\left(\frac{\partial \rho}{\partial P}\right) B_P\right)^2} = 0.0007 \text{ lbm/ft}^3$$

$$\nu = \frac{\left[\left(\left(\frac{\partial \rho}{\partial T}\right) P_T\right)^2 + \left(\left(\frac{\partial \rho}{\partial P}\right) P_P\right)^2\right]^2}{\left(\frac{\partial \rho}{\partial T} P_T\right)^4 / \nu_T + \left[\left(\frac{\partial \rho}{\partial P}\right) P_P\right]^4 / \nu_P}$$

$$\nu = \frac{(0.0012)^4}{\frac{(1.312 \text{ x } 10^{-4} \text{ x } 0.9)^4}{9} + \frac{(3.26 \text{ x } 10^{-5} \text{ x } 37.4)^4}{19}} = 17.83 = 18$$

$$t_{18,95} = 2.101$$

$$u_p = \sqrt{B^2 + (t_{18,95}P)^2} = 0.0026 \text{ lbm/ft}^3$$

$$\rho' = 0.074 \pm 0.0026 \text{ lbm/ft}^3 \qquad (95\%)$$

II.8 ASME/ANSI 1986 Procedure for Estimation of Overall Uncertainty

1. Define the Measurement Process

   objectives; independent parameters and their nominal values; functional relationship; test results

2. List All of the Elemental Errors

   calibration; data acquisition; data reduction

3. Estimate the Elemental Errors

   bias limits; precision index (use the same confidence level)

4. Calculate the Bias and Precision Error for Each Measured Variable

   RSS

5. Propagate the Bias Limits and Precision Indices All the Way to the Result

   RSS (Example of Density Calculation)

6. Calculate the Overall Uncertainty of the results

   RSS

GUIDELINE FOR ASSIGNING ELEMENTAL ERROR

| Error | Error type |
|---|---|
| Accuracy | Bias |
| Common-mode voltage | Bias |
| Hysteresis | Bias |
| Installation | Bias |
| Linearity | Bias |
| Loading | Bias |
| Spatial | Bias |
| Repeatability | Precision |
| Noise | Precision |
| Resolution/scale/quantization | Precision |
| Thermal stability (gain, zero, etc.) | Precision |

**LECTURE NOTES II-- SENSORS**

## 1. Introduction

Transducers - electromechanical devices that convert a change in a mechanical quantity such as displacement or force into a change in electrical quantity. Many sensors are used in transducer design, e.g., potentiometer, differential transformers, strain gages, capacitor sensors, piezoelectric elements, piezoresistive crystals, thermistors, etc. We will cover the metrology in the lecture and followed by the discussion of sensors.

## 2. Metrology

The science of weights and measures, referring to the measurements of lengths, angles, and weights, including the establishment of a flat plane reference surface.

### 2.1 Linear Measurement

Line Standard defined by the two marks on a dimensionally stable material.

End Standard the length of end standards is the distance between the flat parallel end faces.

Gauge Block length standards for machining purposes.
Federal Accuracy Grade; combination of gauge blocks yields a range of length from 0.100 to 12.000 in., in 0.001 in. increments.

Vernier Caliper

Consult Figs. 12.2-12.4, Textbook

Micrometer

Consult Fig. 12.5, Textbook

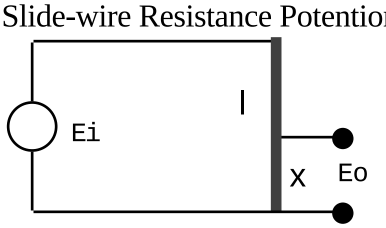
Tape Measure measuring tape up to 100 ft, uncertainty as low as 0.05%; hand measuring tools are commonly used for length measurements.

## 3. Displacement Sensor

Potentiometer, Differential Transformer, Strain Gage, Capacitance, Eddy Current

### 3.1 Potentiometer

Slide-wire Resistance Potentiometer:




$$E_c = \frac{x}{l} E_i \quad \text{or} \quad X = \frac{E_c}{E_i}\, l$$

Displacement can be measured from the above equation. Different potentiometers are available to measure linear as well as angular displacement. Potententiometers are generally used to measure large displacements, e.g., $\geq$ 10 mm of linear motion and $\geq$ 15 degrees of angular motion. Some special potentiometers are designed with a resolution of 0.001 mm.

Differential Transformer

LVDT (Linear Variable Differential Transformer) is a popular transducer which is based on a variable-inductance principle for displacement measurements. The position of the magnetic core controls the mutual inductance between the center of the primary coil and the two outer of secondary coils. The imbalance in mutual inductance between the center location, and an output voltage develops. Frequency applied to the primary coil can range from 50 to 25000 Hz. If the LVDT is used to measure dynamic displacements, the carrier frequency should be 10 times greater than the highest frequency component in the dynamic signal. In general, highest sensitivities are attained at frequencies of 1 to 5 kHz. The input voltages range from 5 to 15 V. Sensitivities usually vary from 0.02 to 0.2 V/mm of displacement per volt of excitation applied to the primary coil. The actual sensitivity depends on the design of each LVDT. The stroke varies in a range of ± 150 mm (low sensitivity). There are two other commonly used differential transformers: DCDT-- Direct Current Differential Transformer and RVDT-- Rotary Variable Differential Transformer (range of linear operation is ± 40 degrees). Consult Figs. 12.9 and 12.11 of Textbook for typical schematic diagrams of LVDT and Fig. 12.12 for that of RVDT.

LVDT and RVDT are known for long lifetime of usage and no overtravel damage.-

3.2 Resistance-type stain gage

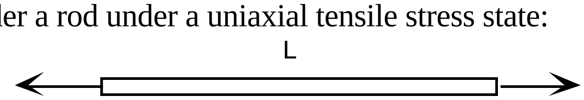
Lord Kelvin observed the strain sensitivity of metals (copper and iron) in 1856. The effect can be explained in the following analysis.

$$R = \frac{\rho L}{A} \qquad \text{(uniform metal conduction)}$$

where R = resistance, ρ = specific resistance, L = length of the conductor, A = cross-sectional area of the conductor

$$\frac{dR}{R} = \frac{d\rho}{\rho} + \frac{dL}{L} - \frac{dA}{A}$$

Consider a rod under a uniaxial tensile stress state:




$$\varepsilon_a = \frac{dL}{L} \quad , \quad \varepsilon_t = -\nu\varepsilon_a = -\nu \frac{dL}{L}$$

where $\varepsilon_a$ = axial strain, $\varepsilon_t$ = transverse strain, $\nu$ = Poisson ratio (note that $\nu_p$ was used in Textbook)

$$d_f = d_o\left(1 - \nu \frac{dL}{L}\right)$$

where $d_o$ = initial diameter, $d_f$ = diameter after the rod is strained

$$\frac{dA}{A} = -2\nu \frac{dL}{L} \quad ; \quad \frac{dA}{A} = -2\nu \frac{dL}{L}$$

Substituting $\frac{dA}{A}$ into the resistance equation, one obtains

$$\frac{dR}{R} = \frac{d\rho}{\rho} + \frac{dL}{L} + 2\nu\frac{dL}{L} \quad , \quad \text{or} \quad \frac{dR}{R} = \frac{d\rho}{\rho} + (1 + 2\nu)\frac{dL}{L} \ ; \ \varepsilon_a = \frac{dL}{L}$$

Thus, the sensitivity of the conductor $S_A$ becomes

$$S_A = \frac{dR/R}{\varepsilon_a} = \frac{d\rho/\rho}{\varepsilon_a} + (1+2\nu)$$

(change in ρ) (change in dimension)

$\nu = 0.3$ for most materials used.

$\pi_1 = \frac{1}{E_m} \frac{d\rho/\rho}{dL/L}$ $E_m$ : Young's modulus; $\pi_1$: Piezoresistance Coefficient

$\sigma_a = E_m \ \varepsilon_a$

$S_a = (1+2\nu) + E_m \pi_1$

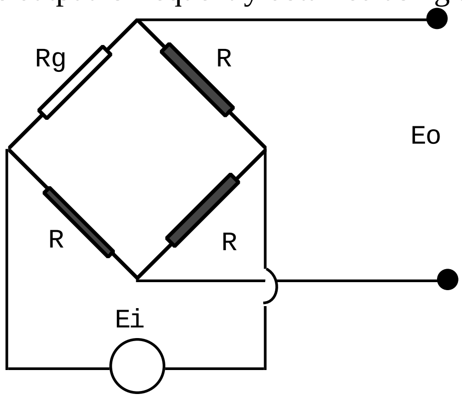
For advance alloy, $\frac{\Delta R/R}{R}$ is linearly proportional to ε.

Cooper - nickel alloy known as Advance of Constantan is a common material for strain gage.
Typical values of $S_A$ are summarized:

| Material | Composition (%) | $S_A$ |
|---|---|---|
| Advance or Constantan | 45 Ni, 55 Cu | 2.1 |
| Nichrome V | 80 Ni, 20 Cr | 2.1 |
| Isoelastic | 36 Ni, 8 Cr, 3 Al, 3 Fe | 3.6 |
| Platinum-Tungsten | 92 Ni, 8 W | 4.0 |

$$\frac{\Delta R}{R} = S_g \varepsilon = GF \ \varepsilon$$

$S_g$ = gage factor (note that $S_g < S_A$ as a result of the grid configuration); or GF is used
Voltage output is frequently obtained using a Wheatstone bridge:

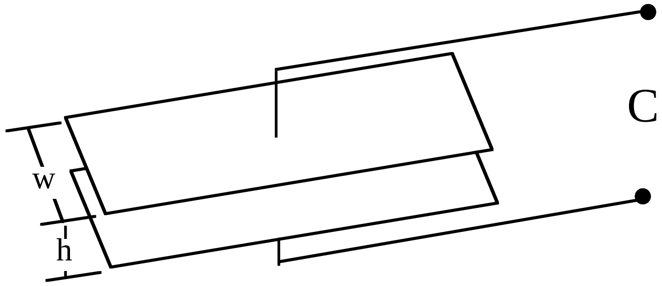



$$E_o = \frac{E_i}{4} \frac{\Delta Rg}{R}$$

$$E_o = \frac{1}{4} E_i S_g \varepsilon$$

The input voltage is controlled by the gage size and the initial resistance. The output voltage ($E_o$) usually ranges between 1 and 10 μv/microunit of strain (μm/m or μin/in).

## 3.3 Capacitance Sensor




Consider two metal plates separated by an air gap, h, the capacitance between terminals is given by the expression:

$$C = \frac{k\kappa A}{h}$$

C: capacitance in picofarads (pF or pf)
κ: dielectric constant for the medium between the plates
A: overlapping area of the two plates
k: proportionality constant (k = 0.225 for dimensions in inches, k = 0.00885 for dimensions in millimeters)

If the plate separation is changed by Δh, while A is kept unchanged; then

$$\frac{\partial C}{\partial h} = -\frac{k\kappa A}{h^2} * \frac{1}{h}$$ For varying h and fixed A, sensitivity, $S \equiv \frac{\Delta C}{\Delta h}$ is

$$\mathbf{S = -\frac{k\kappa A}{h^2}}$$

A = l w, where l is overlapping distance

If the overlapping area is changed, while h is fixed, then

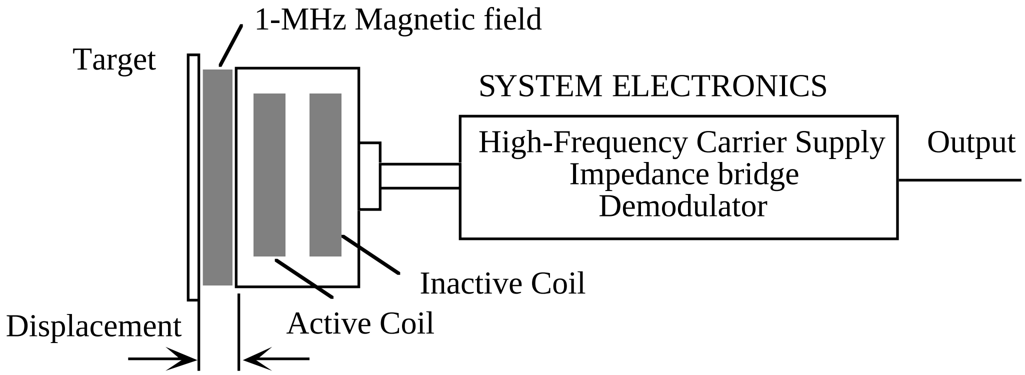
$$C = k\kappa lw/h$$

$$\mathbf{S} = \frac{\partial C}{\partial l} = \mathbf{\frac{k\kappa w}{h}}$$ (sensitivity for varying A, fixed h)

Typical sensitivity for a sensor of w = 10 mm, h = 0.2 mm, is 0.4425 pF/mm

## 3.4 Eddy Current Sensor

An eddy current sensor measures distance between the sensor and an electrically conducting surface:




Impedance bridge is used to measure changes in eddy currents. Typical sensitivity of the eddy current sensor with an aluminum target is about 100 mV/mil of 4 V/mm. Temperature variation generally has small or negligible effects especially for the sensing element with dual coils which is temperature compensated.

Eddy current sensors are often used for automatic control of dimensions in fabrication processes. They are also applied to determine thickness of organic coatings.

## 4. Load Cell (Force Measurements)

### 4.1 Introduction

$F = ma; \; W = mg$

Weight depends on local gravitational acceleration. It is known that g = 9.80665 $m/s^2$ is referred to the "standard" gravitational acceleration which corresponds to the value at sea level and 45° latitude. The deviation from the standard value can be calculated following:

$$g = 978.049\,(1 + 0.0052884 \sin^2 \phi - 0.0000059 \sin^2 2\phi)\ \text{cm/s}^2$$

The correction for altitude h is

$$g_c = -(0.00030855 + 0.00000022 \cos 2\phi)\, h + 0.000072 \left(\frac{h}{1000}\right)^2 \ \text{cm/s}^2$$

where h is in meters

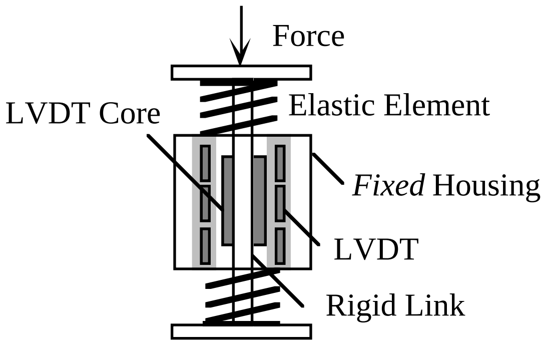
"Dead weight" is computed based on accurately known mass and the local g value. This is generally done by NIST.

<u>Methods of Force Measurement</u>

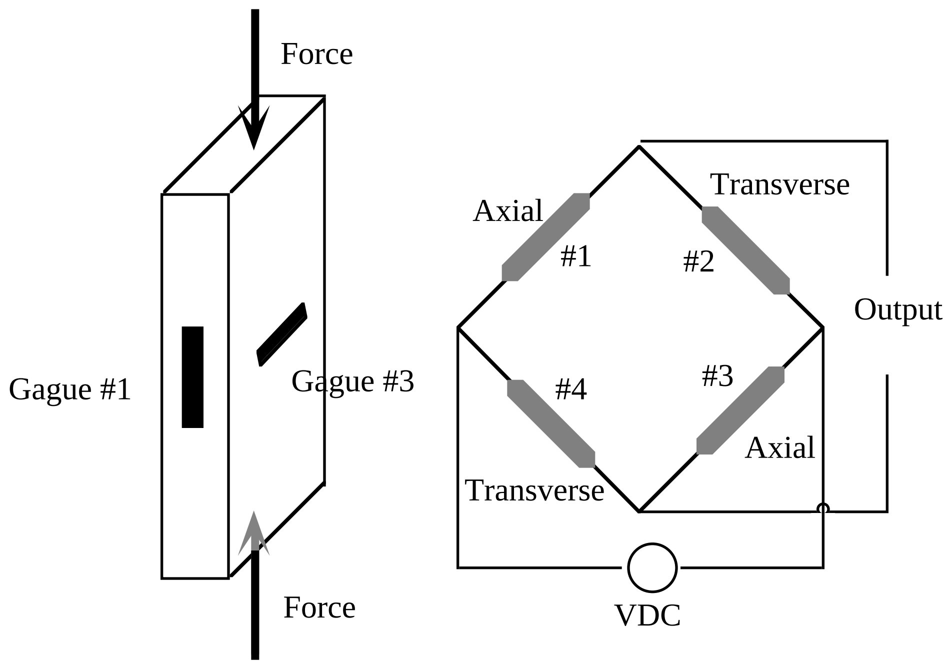
1. Balancing it against the known gravitational force on a standard mass.
2. Measuring the acceleration of a body of known mass to which unknown force is applied.
3. Balancing it against magnetic force developed by interaction of a current-carrying coil and a magnet.
4. Transducing the force to a fluid pressure then measuring the p.
5. Applying the force to some elastic member and measuring the resulting deflection.
6. Measuring the change in precision of a gyroscope caused by an applied torque related to the measured force.
7. Measured the change in natural frequency of a wire tensioned by the force.

## 4.2 Link-Type Load Cell

<u>LVDT Sensor</u>




<u>Strain Gauge Sensor</u>




Load cells utilize elastic members and relate deflections to forces; uniaxial link-type load cells shown above use strain gages as the sensor.

The axial and transverse strains resulting from the load P are:

$$\varepsilon_a = \frac{P}{AE} \qquad \varepsilon_t = -\frac{\nu P}{AE}$$

where A: cross-sectional area, E: Young's modulus, ν: Poisson's ratio
The responses of the gages are:

$$\frac{\Delta R_1}{R_1} = \frac{\Delta R_4}{R_4} = S_g\,\varepsilon_a = S_g\ P/(AE);\ Sg = (dRg/R)/\ \varepsilon_a$$

$$\frac{\Delta R_2}{R_2} = \frac{\Delta R_3}{R_3} = S_g\,\varepsilon_\tau = -\nu\,\frac{S_g P}{AE}$$

The output voltage $E_o$ from a wheatstone bridge having four identical arms ($R_1 = R_2 = R_3 = R_4$)

$$E_o = \frac{R_1R_2}{(R_1 + R_2)^2}\left(\frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2} - \frac{\Delta R_3}{R_3} + \frac{\Delta R_4}{R_4}\right) E_i$$

$$E_o = \frac{1}{2}\,\frac{S_gP(1 + \nu)\,E_i}{AE}$$

Let's define $C \equiv \dfrac{2\,AE}{S_g\,(1 + \nu)E_i}$ (calibration constant) , then $E_o = \dfrac{P}{C}$
sensitivity of the load cell is

$$S \equiv \frac{\partial E_o}{\partial P} = \frac{1}{C} \quad \text{or} \quad S = \frac{1}{C} = \frac{S_g\,(1 + \nu)E_i}{2\,AE}$$

<u>Remarks:</u>

a. $E_o$ is linearly proportional to the load P.

b. The range of the load cell is

$P = S_fA$ where $S_f$ is the fatigue strength.

This implies that high sensitivity is associated with low capacity and vice versa.

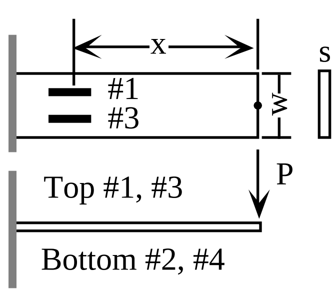
c. $$\left(\frac{E_o}{E_i}\right)_{max} = \frac{S_g(1 + \nu)\,E_i\,P_{max}}{2\,AE\,E_i} = \frac{S_g\,S_f\,(1 + \nu)}{2\,E}$$

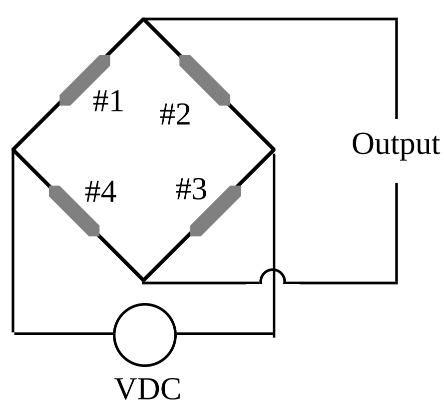
Most load-cell links are fabricated from AISI4340 steel ($E = 3 \times 10^7$ psi, $\nu = .30$, $S_f \approx 8 \times 10^4$ psi) and the ratio equals $\dfrac{E_o}{E_i} = 3.47$ mV/V; $S_g \approx 2$
Typical load cells are rated with $(E_o/E_i) = 3$mV/V at the full-scale value of the load ($P_{max}$). With this full-scale specification, the load P can be obtained from

$$\frac{(E_o/E_i)}{(E_o/E_i) * P_{max}}$$

Typically $E_i$ has a value of 10 V; therefore, $E_o$ is in the range of 30 mV.

## 4.3 Beam-Type Load Cell



Beam-type load cells are commonly used for measuring low-level loads where the link-type load cell is not effective.

$$\varepsilon_1 = -\varepsilon_2 = \varepsilon_3 = -\varepsilon_4 = \frac{6M}{Ebh^2} = \frac{6\ PX}{Ebh^2}$$

The response of the strain gage is (recall $\Delta R/R = S_g\ \varepsilon$)

$$\frac{\Delta R_1}{R_1} = -\frac{\Delta R_2}{R_2} = \frac{\Delta R_3}{R_3} = -\frac{\Delta R_4}{R_4} = \frac{6\ S_g\ PX}{Ebh^2}$$

If the four gages are identical, then the output is

$$P = \frac{(Ebh)^2}{6 S_g\ XE_i} E_o = C\ E_o \quad \text{and the calibration constant C} \quad C = \frac{(Ebh)^2}{6\ S_g X E_i}$$

The sensitivity of the load cell, S, can be determined from $\dfrac{\partial\ E_o}{\partial\ P}$

$$S = \frac{1}{C} = \frac{6\ S_g X E_i}{Ebh^2}$$

The maximum load $P_{max} = S_f bh^2/(6x)$ and $(E_o/E_i)_{max} = S_g S_f/E$
Typical beam-type load cells have ratings of $(E_o/E_i)^*$ between 4 and 5 mV/V at full-scale load.

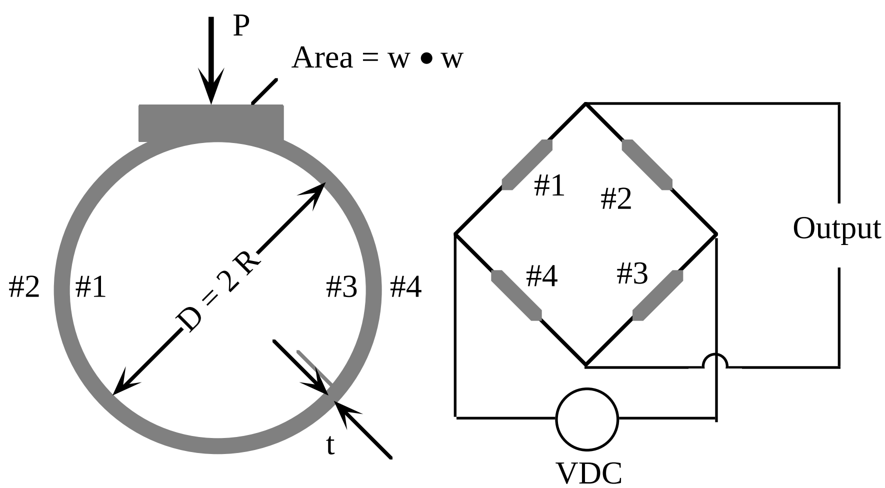
4.4 Shear-Web-Type Load Cell (Low-Profile or Flat Load Cells)

This type of load cell is compact and stiff and can be used in dynamic measurements.

4.5 Ring-Type Load Cell

$$\delta = 1.79\ \frac{PR^3}{Ewt^3}, \quad E_o = S\ \delta\ E_i \ \text{(S = sensitivity)}$$

$S_t = 1.79\ S\ R^3\ E_i / Ewt^3$, and $E_o/E_i \approx 300$ mV/V



# 5. Torque and Power Measurements

## 5.1. Torque Measurements

For a circular cylinder:

$$\tau_{max} = T\, R_o/J$$

where $\tau_{max}$ is the maximum shearing stress, T the applied torque, J the polar moment of inertia ($\pi R_o^4/2$ for a solid cylinder)

## 5.2. Power Measurements

$$\mathbf{P_s} = \omega \times \mathbf{T}$$

$$P_s = \omega\, T$$

where $P_s$ is the shaft power, $\omega$ the rotational speed, and T the applied torque

### 5.2.1 Prony Brake

The Prony brakes apply a well-defined load to, for example, an engine. The power is determined from the force applied to the torque arm and the rotational speed. (e.g., see Fig. 12.34 of Textbook)

### 5.2.2 Cradled Dynamometers

The cradled dynamometer measures the rotational speed of the power transmission shaft, and the reaction torque (to prevent movement of the stationary part of the prime mover).

ASME Performance Test Code lists sources of the overall uncertainty with the cradled dynamometers measurements to be

- trunnion bearing friction
- force measurement uncertainty
- moment arm-length measurement uncertainty
- rotational speed measurement uncertainty
- static unbalance of dynamometer

Types of dynamometers:

- eddy current dynamometer
- AC and DC generator
- waterbrake dynamometer (Fig. 12.36 in Textbook)

# 6. Temperature Sensor

## 6.1 Introduction

<u>Definition</u>
Temperature is a physical quantity which is related to the energy level of molecules, or the energy level of a system. Temperature is a thermodynamic property.

Review

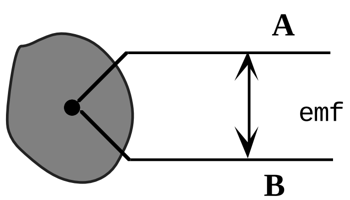
(1) Thermal equilibrium (Zeroth Law of Thermodynamics)

$T_A = T_B$ , $T_B = T_C \rightarrow T_A = T_C$

(2) Temperature scale
SI Units

6.2 Thermocouples (TC)

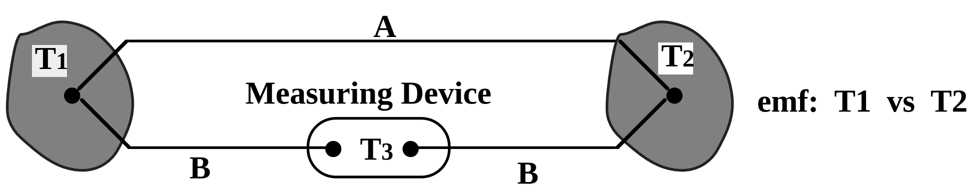
T.J. Seebeck (1821) discovered emf (electromotive force) exists across a junction formed of dissimilar metals at the junction temperature (Peltier effect), and the temperature gradient (Thomson effect). It should be noted that the Thomson effect is generally negligible when it is compared to the Peltier effect.




(1) Application Laws (P.H. Dike, 1954)

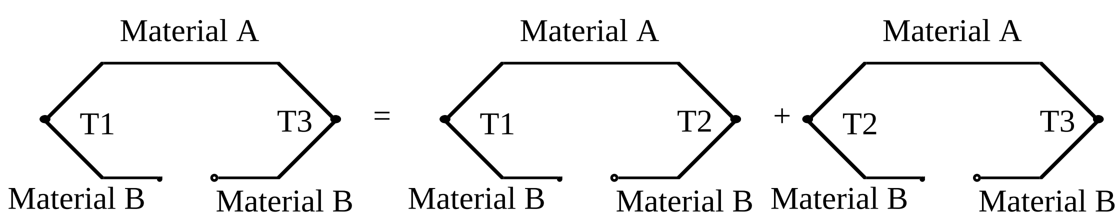
Law of intermediate metals

Insertion of an intermediate metal into a TC circuit will not affect the net emf, provided that the two junctions introduced by the third metal are maintained at an identical T.




Law of intermediate temperatures

If a TC circuit develops an emf $E_1$, with its junctions are at $T_1$ and $T_2$, and $E_2$ with $T_2$ and $T_3$, the TC will develop an emf $E_1 + E_2$ with its junctions maintained at $T_1$ and $T_3$.

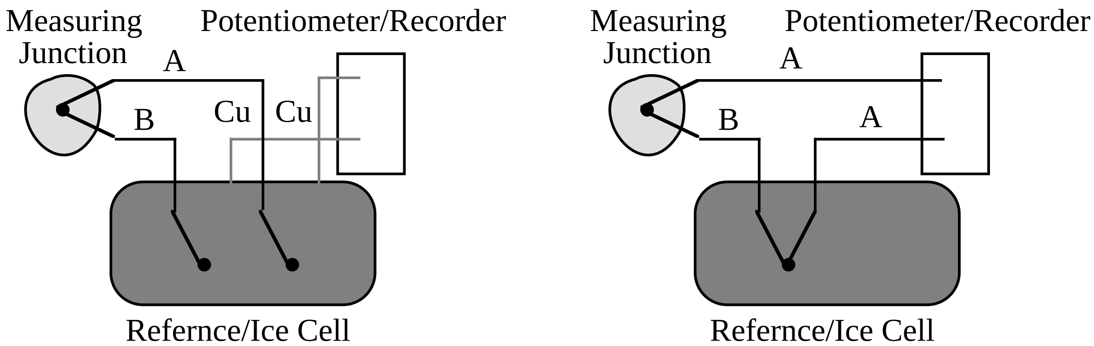



(2) TC Materials

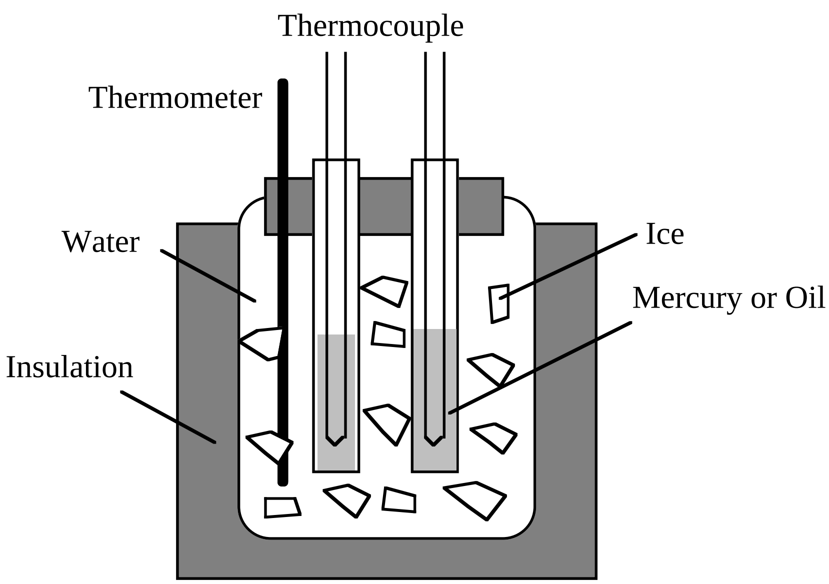
| Type | Materials |
|---|---|
| E | Chromel-Constantan |
| J | Iron-Constantan |
| T | Copper-Constantan |
| K | Chromel-Alumel |
| R | Platinum-Platinum/13% rhodium |
| S | Platinum-Platinum/10% rhodium |

Consult Fig. 8.18 of Textbook for thermocouple voltage output (Type E, J, K, E, R, S) as functions of temperature.

(3) Basic Circuit




0° C (32°F) is usually selected as the reference temperature




(4) emf Output

$$E = AT + 1/2\ BT^2 + 1/3\ CT^3 \qquad \text{(based on 0°C reference junction)}$$

$$S \equiv \frac{dE}{dT} = A + BT + CT^2 \qquad \text{(sensitivity)}$$

or

$$E = \Sigma\, c_i T^i \qquad \text{where } i = 0 \text{ to } n;$$

consult Table 8.7 of Textbook for polynomial coefficients for Type J and Type T.

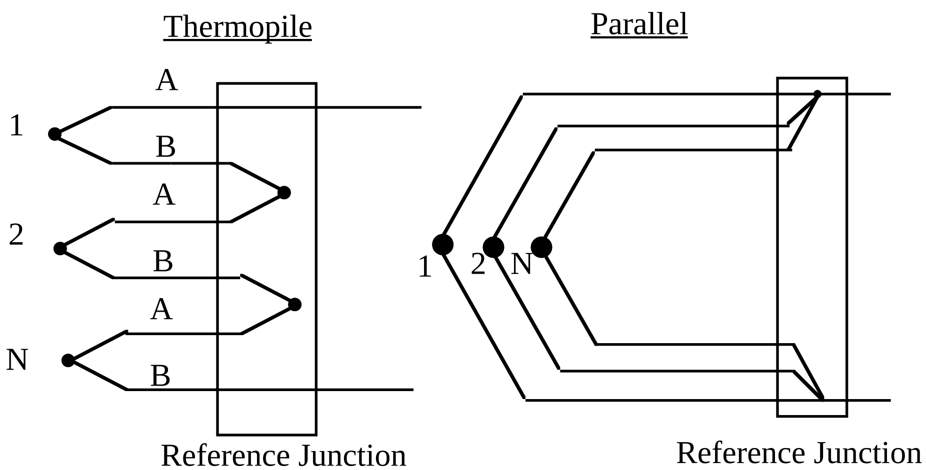
(5) Extension Wires (to minimize expensive wire length)

It should be noted that special formulated wires are available for each type of TC to minimize effects of small temperature variation at intermediate junctions.

(6) Thermocouple circuits

Thermopile
Output will be equal to the sums of the individual emf's, therefore, the sensitivity increases.

Parallel thermocouple
Output equals the average of $T_1$, $T_2$ and $T_3$ for the following arrangement:



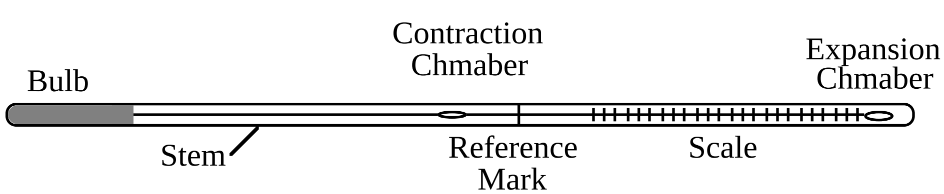

5.3 Expansion Thermometer

The expansion and contraction nature of materials when they are exposed to a temperature change is applied to temperature measurements. The phenomenon can be expressed as

$$\frac{\Delta l}{l} = \alpha \, \Delta T$$

where $\Delta l$ is the variation per unit length, and $\alpha$ is the thermal coefficient of expansion.

Liquid-in-Glass Thermometers

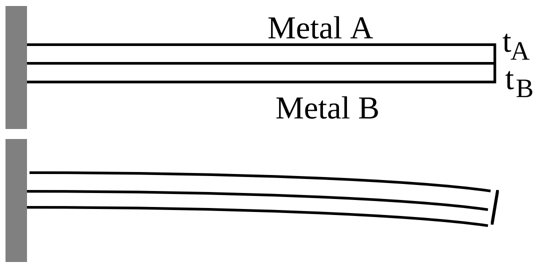



This type of thermometer utilizes the differential expansion between two different materials to measure temperature changes, for examples, mercury thermometer, alcohol thermometers, etc.

| Type | T range ( ° C) |
|---|---|
| Hg filled | -32 to 320 |
| Pressure filled | -35 to 530 |
| Alcohol | -75 to 129 |

Note that stem corrections are needed for "bench mark" measurements.

<u>Bimetallic Temperature Sensors</u>
This popular temperature sensor thermostat is based on the difference in thermal coefficient of expansion of two different metals which are brazed together.




$$r = \frac{[\,3(1 + m)^2 + (1 + mn)\,(m^2 + 1/mn)\,]t}{6(\alpha_A - \alpha_B)\,(1 + m)^2\,(T - T_o)}$$

where r is the radius of curvature

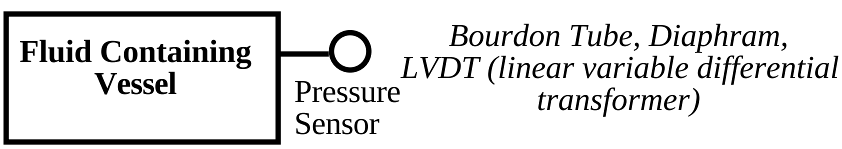
$m = \frac{t_B}{t_A}$ thickness ratio

$n = \frac{E_B}{E_A}$ modulus ratio, E = modulus of elasticity

$\alpha_A$: high coeff. of expansion, e.g. copper-based alloy

$\alpha_B$: low coeff. of expansion, e.g. Invar (nickel steel) $T_o$: initial bonding T

<u>Pressure Thermometer</u>
Fluid expansion characteristic is applied to T measurements.




Liquid-filled: completely filled with liquids; gas-filled: completely filled with gases; vapor-filled : liquid-vapor combination

5.4. Resistance Thermometer

Electrical resistance of most materials varies with temperature, this provides a basis for temperature measurements. Two types of resistance thermometer have been widely used.

| | RTD | Thermistor |
|---|---|---|
| Resistance | increases as T increases | decreases as T increases |
| T vs R | linear | nonlinear |
| T range | -250° to 1000°C | -100° to 250°C |

<u>RTD</u>
RTD denotes resistance temperature detector; usually, such metals as nickel, copper, platinum or silver, are used as sensor elements; consult Figure 8.5 of Textbook for relative resistance.

<u>Thermistor</u>
Semiconducting materials having negative resistance coefficients are used as sensor elements, e.g.

combination of metallic oxides of cobalt, magnesium, and nickel. An example of thermistor resistance variation with temperature is given by Figure 8.8 of Textbook.

RTD

Resistance-temperature relationship assumes a general form:

$$R = R_o (1 + \nu_1 T + \nu_2 T^2 + ....... + \nu_n T^n)$$

$\nu$'s : temperature coefficients of resistivity

$R_o$ : resistance at $T_o$

For practical purposes, second order polynomials are usually sufficient to correlate R with T:

$$R = R_o (1 + aT + bT^2)$$

Sensitivity of a linear T-dependence resistance element is:

$$R = R_o [1 + \alpha [T - T_o]]$$

$$S \equiv \frac{dR}{dT} = \alpha R_o$$

Electrical bridges are normally used to measure the resistance change in RTD.

Thermistor

Resistance-temperature function:

$$R = R_o \exp [\beta (\frac{1}{T} - \frac{1}{T_o})]$$

R : resistance at T[K]

$R_o$ : resistance at $T_o$[K]

$\beta$ : constant, (=350 - 4600 K)

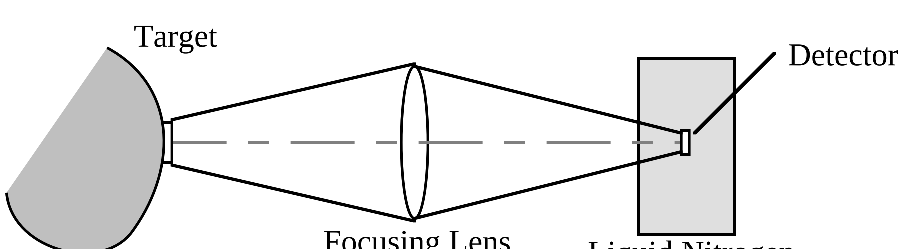
Thermistors have high sensitivities than RTD

$$S \equiv \frac{dR}{dT} = R_o \exp [\beta (\frac{1}{T} - \frac{1}{T_o})] * \frac{-\beta}{T^2}$$

6.5 Pyrometer

Electromagnetic radiation is measured by three distinct instruments -- total radiation, optical pyrometer and infrared pyrometer. The emissive power of a blackbody follows the Planck's Law (cf., Figure 8.25 of Textbook, 1995).

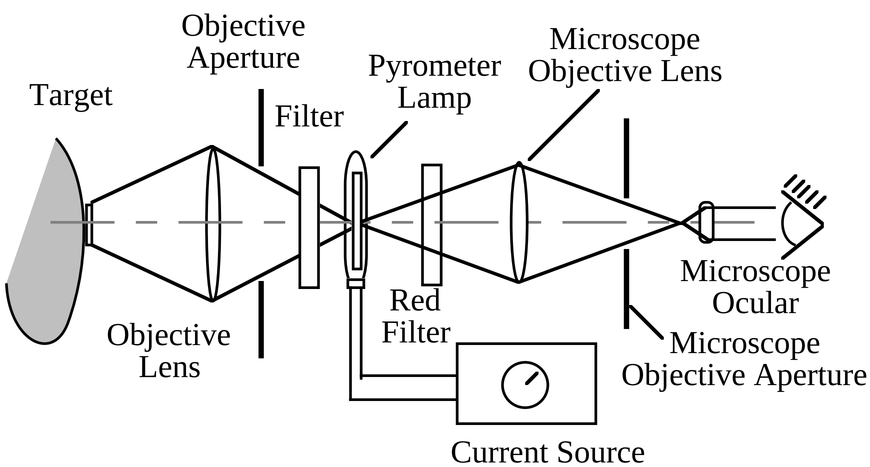
6.6 Total-Radiation Pyrometry




(1) Applicable to T > 550° C, although some devices may be used to measure lower T's.
(2) Object of measurements should approach the B.B. conditions.
(3) Materials of windows and lens are important.

| $\lambda$ ($\mu$m) | Material |
|---|---|
| 0.3-2.7 | pyrex glass |
| 0.3-3.8 | fused silica |
| 0.3-10 | calcium fluoride |

Optical Pyrometry
Matching the brightness of a filament and unknown T source, e.g., see illustration below and Figure 8.28 or Textbook. It can be used in the range of 700° C to 4000° C.




Infrared (IR) Pyrometry
IR instrument employs a photo cell, e.g. photo conductive, photovoltaic, photo electromagnetic, to detect photon flux. T range -40° C to 4000° C.

6.7 Heat Flux Sensor-- An Application of Temperature Sensor

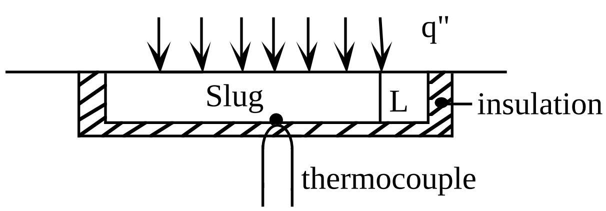
Introduction

$$q'' = \frac{q}{A}\left[\frac{W}{m^2}\right]$$

$q''$: heat flux , $q$: heat transfer rate [W]

Slug-Type Sensor

Consider a control volume as shown below:




From the first law of thermodynamics, one obtains

$$\dot{Q} = \left(\frac{dE}{dt}\right)_{c.v.} \quad , \text{ or } q'' A_s = \left(\frac{dE}{dt}\right)_{c.v.}$$

$$\int_{t_1}^{t_2} q'' A\, dt = m\, c\, (T_2 - T_1)$$

In differential form, one obtains

$$\boxed{q'' = \frac{m\,C}{A}\frac{dT}{dt}}$$

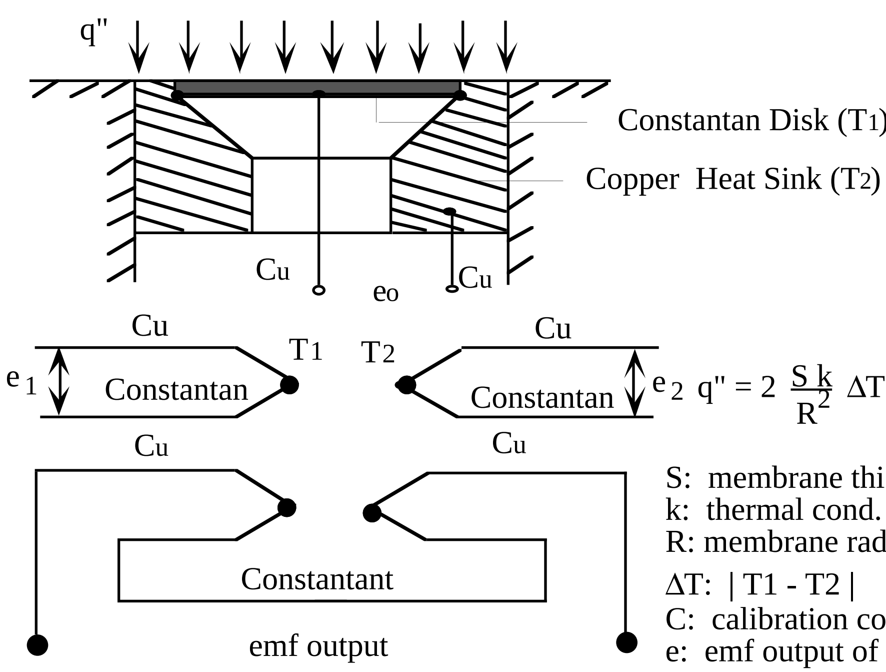
It should be noted that the above equation neglects heat loss from insulation and thermocouple wires, and it assumes a uniform temperature of the slug. To account for these effects, a heat loss coefficient may be introduced and one-dimensional or multi-dimensional heat condition analysis may be employed.

$$\boxed{q'' = \frac{m\,C}{A}\frac{dT}{dt} + U\, \Delta T}$$

where U is the overall heat transfer coefficient to account for heat loss to surroundings, and $\Delta T$ is the temperature difference.

<u>Steady-State or Asymptotic Sensor (Gardon Gage)</u>

A differential thermocouple between the disk center and its edge is formed when the thin constantan disk is exposed to heat flux, and an equilibrium temperature difference is established. The temperature difference is proportional to the heat flux (cf., Gardon, R., *Review of Scientific Instrument*, p. 366, May, 1953).



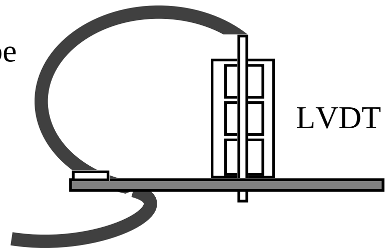

$$q'' = 2\,\frac{S\,k}{R^2}\,\Delta T = C\,e$$

S: membrane thickness
k: thermal cond. of the membrane
R: membrane radius
$\Delta T$: $|T1 - T2|$
C: calibration constant
e: emf output of t.c.

## 7. Pressure Sensor

Pressure transducers convert pressure into an electrical signal, through displacement, strain, or piezoelectric response.

7.1 Displacement-Type Pressure Measurement (Bourdon Tube)
A c-shape pressure vessel either a flat oval cross section which straightens as an internal pressure is applied. Good for static measurements, frequency response only 10 Hz.




7.2 Diaphram-Type Pressure Transducer
A diaphram or hollow cylinder serves as the elastic element, and a strain gage serves as the sensor. The sensitivity S is proportional to $(R_o/t)^2$ where t is the thickness and $R_o$ the radius.

Maximum $(R_o/t)$ value is determined by diaphram deflection, rather than yield strength. Diaphram-type pressure transducers can be used for static as well as dynamic measurements. Frequency response can reach 10 kHz.

7.3 Piezoelectric-Type Pressure Transducers
Piezoelectric pressure transducers can be used in very-high-pressure (say 100,000 psi ) and high temperatures (say 350° C) measurements. High frequency response is another distinct feature. Charge amplifiers are used as signal conditioner for the measurement system.

<u>Dual Sensitivity</u>
Output voltage is due to a primary quantity, e.g. load, torque, pressure, etc., and secondary quantities, e.g. temperature or secondary load.

Temperature-- temperature compensation is required for accurate measurements.
Secondary Load-- bending moments may be applied to a link-type load cell. Proper placement of the strain gage generally can eliminate the effects resulting from the secondary load.

7.4 Piezoelectric Sensors for Force and Pressure measurements

Piezoelectric material is a material is a material that produces an electric charge when subject to force or pressure. Single-crystal quartz of polycrystalline barium titanate has been used for piezoelectric sensors. When pressure is applied, the crystal deforms, and there is a relative displacement of the positive and negative charges within the crystal. The displacement of internal charges produces external charges of opposite signs on two surface of the crystal.

$$q = E_o C$$

q: charge develops on two surfaces,
C: capacitance of the piezoelectric crystal, or $q = S_q A p$
$S_q$: charge sensitivity of the piezoelectric crystal
A: area of electrode

where $C = \dfrac{k\kappa A}{h}$ for capacitance sensor. Substituting C and q into the $E_o$ equation:

$$E_o = \frac{q}{C} = \frac{S_q A p}{k\kappa A} h \text{ , or } E_o = \frac{q}{k\kappa} hP$$

The voltage sensitivity can be derived form $E_o = S_E$ hp:

$$S_E = \frac{S_q}{k\kappa}$$

Typical sensitivities are

| Material | Orientation | $S_q$(pC/N) | $S_E$(Vm/N) |
|---|---|---|---|
| Quartz | X-cut | 2.2 | 0.055 |
| Barium titanate | Parallel to orientation | 130 | 0.011 |

Piezoelectric sensors have high frequency response which is a distinct advantage of using these sensors.

## 7.5 Piezoresistive Sensors

Piezoresisitive materials exhigit a chang in resistance when subject to pressure. These sensors are fabricated from semiconductive materials, e.g., silicon containing boron as the trace imputity for the P-type material and arsenic as the trace imputity for the N-type material. The resistivity of the semiconducting materials is

$$\rho = \frac{1}{eN\mu}$$

where

- e: electron charge
- N: number of charge carriers
- $\mu$: mobility of charge carriers

This equation indicates that the resistivity changes when the piezoresistive sensor is subjected to either stress of strain known as piezoresistive effect:

$$\rho_{ij} = \delta_{ij} P + \pi_{\iota jkl} \tau_{kl}$$

where subscripts i, j, k, andl range from 1 to 3

- $\pi_{ijkl}$: 4th rank piezoresistrive tensor
- $\tau_{kl}$: stress tensor
- $\delta_{ij}$: Kroneker delta; $\delta_{ij} = 0$ for $I \neq j$ and $\delta_{ij} = 1$ for $i = j$

For cubic crystal (1, 2, 3 identify axes fo the crystal):

$$\rho_{11} = \rho[1 + \pi_{11}\sigma_{11} + \pi_{12}(\sigma_{22} + \sigma_{33})]$$
$$\rho_{22} = \rho[1 + \pi_{11}\sigma_{22} + \pi_{12}(\sigma_{33} + \sigma_{11})]$$
$$\rho_{33} = \rho[1 + \pi_{11}\sigma_{33} + \pi_{12}(\sigma_{11} + \sigma_{22})]$$
$$\rho_{12} = \rho\pi_{44}\tau_{12} \qquad \rho_{23} = \rho\pi_{44}\tau_{23} \qquad \rho_{31} = \rho\pi_{44}\tau_{31}$$

The implications of the above equations are the anisotropic resistivity.

$$E_i' = \rho_{ij} I_j'$$

where E = potential gradient, and I = current density.

$$\frac{E_1'}{\rho} = I_1' \left[1 + \pi_{11}\sigma_{11} + \pi_{12}(\sigma_{22} + \sigma_{33})\right] + \pi_{44}(I_2' \tau_{12} + I_3' \sigma_{31})$$

$$\frac{E_2'}{\rho} = I_2' \left[1 + \pi_{11}\sigma_{22} + \pi_{12}(\sigma_{33} + \sigma_{11})\right] + \pi_{44}(I_3' \tau_{23} + I_1' \tau_{12})$$

$$\frac{E_3'}{\rho} = I_3' \left[1 + \pi_{11}\sigma_{33} + \pi_{12}(\sigma_{11} + \sigma_{22})\right] + \pi_{44}(I_1' \tau_{31} + I_2' \tau_{23})$$

# 8 . Flow Measurements

## 8.1 Pressure Probes

### 8.1.1 Total Pressure

For a steady incompressible flow, the Bernoulli equation states that along the same stream line

$$P_1 + \frac{\rho u_1^2}{2} = P_2 + \frac{\rho u_2^2}{2}$$

Lets define the stagnation state is that $u_2 = 0$. For example, the stagnation pressure $P_t$ can be defined as

$$P_t = P_1 + \frac{\rho u_1^2}{2} = (P_2)_s + \frac{0^2}{2}$$

P is reached by bringing the flow to rest at a point by an isentropic process. Let's consider flow over a cylinder, with a uniform velocity approaching the cylinder:

$$u_1 = u_3,\ P_1 = P_3$$

where $u_1$ and $u_3$ are known as free-stream velocity, $P_1$ and $P_3$ are free-stream pressure. At location 4 (Fig. 9.18, Figliola and Beasley, 1995) , $u_3 \neq u_4$, therefore, $P_3 \neq P_4$. $P_4$ is known as the local static pressure. To measure the total pressure, impact cylinder, pitot tube and kiel probe are commonly used. Sketches of these probes are shown in Fig. 9.18 of Figliola and Beasley (1995).

Alignment of the impact port with flow direction is important in the total pressure measurements. The Kiel probe utilizes a converging section to force the flow to align with the impact port.

### 8.1.2 Static Pressure-Prandtl tube

Figure 9.20, Figliola and Beasley (1995).

8.1.3 Pitot-Static Pressure Probe

From the stagnation (total) and static pressure measurements, one can obtain velocity from their difference, or the dynamic pressure:

$$P_t = P_x + \frac{1}{2}\rho u^2$$
$$P_\upsilon = P_t - P_x$$
$$P_\upsilon = \frac{1}{2}\rho u^2$$
$$u = \sqrt{\frac{2(P_t - P_x)}{\rho}}$$

Pitot-static pressure probes have a low velocity limit due to the viscous effects; generally it requires that

$$Re_d = \frac{\rho u d}{u} > 1000 \ .$$

Correction factor is used; for $20 < Re_d < 1000$

$$C_\upsilon = 1 + \frac{8}{Re_d},$$

and $P_\upsilon = C_\upsilon \ P_i$ where $P_i$ is the indicated pressure difference.

8.2 Thermal Anemometer

A thermal anemometer utilizes an RTD sensor, and correlates the heat transfer rate to flow velocity; cf. King, L.V., Phil. Tans. Roy Soc. London, 214 (14): 373, 1914.

$$q = (a + bu^{0.5})(T_\omega - T_\infty)$$

By supplying a current flow, one can obtain a constant temperature operation because the heat dissipated is

$$q = i^2 R_W = i^2 Ro(1 + \alpha(T_\omega - T_\infty))$$

$$i^2 Ro(1 + \alpha(T_\omega - To)) = (a + bu^{0.5})(T\omega - T\infty)$$

where To is the reference temperature.

8.3 Volumetric Flow Rate Measurements

8.3.1 Pressure Differential Meters

$Q \propto (P_1 - P_2)^n$

Q, volumetric flow rate

n = 1 (laminar) or 2 (turbulent), e.g., for laminar flow

$Q = \frac{\pi d^2}{L} \frac{\Delta P}{128\mu}$ where d is the diameter, and L the length

*Laminar Flow Element*

<u>Obstruction Meters</u>

- Continuity Equation

$$u_1 A_1 = u_2 A_2 \quad (\rho = \text{constant})$$
$$u_1 = u_2 \frac{A_2}{A_1}$$

- Energy Equation and Momentum Equation

$$\frac{P_1}{\rho} + \frac{u_1^2}{2} = \frac{P_2}{\rho} + \frac{u_2^2}{2} + h_{L_{1-2}}$$

$h_{L_{1-2}}$ : the head losses due to the friction effects between 1 and 2.

$$\frac{u_2^2}{2} - \frac{u_1^2}{2} = \frac{P_1 - P_2}{\rho} + h_{L_{1-2}}$$

$$\frac{u_2^2}{2} - \frac{u_2^2}{2} \frac{A_2^2}{A_1^2} = \frac{P_1 - P_2}{\rho} + h_{L_{1-2}}$$

$$u_2 = \frac{1}{\sqrt{1 - \left(\frac{A_2}{A_1}\right)^2}} \sqrt{\frac{2(P_1 - P_2)}{\rho} + 2h_{L_{1-2}}}$$

$$Q(= u_2 A_2) = \frac{A_2}{\sqrt{1 - \left(\frac{A_2}{A_1}\right)^2}} * \sqrt{\frac{2(P_1 - P_2)}{\rho} + 2h_{L_{1-2}}}$$

The volumetric flow rate can be related to the pressure drop across an obstruction. There are three common types of obstructions: orifice plates, long radius nozzle, and venturi. When the flow approaches the obstruction, the flow area changes. Furthermore, the effective flow area changes due to the boundary layer effects, flow separation, and three-dimensional flow (secondary flow) in the vicinity of the obstruction. Figure 10.4 of Figliola and Beasley (1995) illustrates the effective area change around the obstruction. To account for the effective area change, an contraction coefficient can be introduced to replace $A_2$ and $A_o$.

$$Cc = \left(\frac{A_o}{A_2}\right)^{-1}; Cc = \left(\frac{A_2}{A_0}\right)$$

$$Q = \frac{C_c}{\left[1-\left(C_c - \frac{A_0}{A_1}\right)^2\right]^{\frac{1}{2}}} \sqrt{\frac{2\Delta P}{\rho} + 2h_{L_{1-2}}}$$

$$Q = \frac{C_c A_0}{\sqrt{1-\left(C_c \frac{A_0}{A_1}\right)^2}} \sqrt{\frac{2\Delta P}{\rho}} \sqrt{1+\frac{2h_{L_{1-2}}}{\left(\frac{2\Delta P}{\rho}\right)}}$$

The frictional head loss can be expressed as

$$C_f = \sqrt{1+\frac{2h_{L_{1-2}}}{\frac{2\Delta P}{\rho}}}$$

$$Q = \frac{C_f C_e A_0}{\sqrt{1-\left(C_c \frac{A_0}{A_1}\right)^2}} \sqrt{\frac{2\Delta P}{\rho}}$$

$$Q = CEA\sqrt{\frac{2\Delta P}{\rho}}$$

C: discharge coefficient

E: velocity of approach factor

$$E = \frac{1}{\sqrt{1-\left(\frac{A_0}{A_1}\right)^2}} = \frac{1}{\sqrt{1-\beta^4}}$$

where $\beta = d_0/d_1$, and $K_o = CE$ can be introduced

$$Q = K_o A \sqrt{2 \frac{\Delta P}{\rho}}$$

When the compressibility effects need to be considered, an adiabatic expansion coefficient Y can be used to account for the compressibility effects on the flow rate.

$$Q = YCEA \sqrt{2 \frac{\Delta P}{\rho}}$$

where Y is given in Fig. 10.7 for $k = C_p/C_v = 1.4$

*Orifice Mete*

*Venturi Flowmeter*

*Long-Radius Nozzle*

8.3.2 Drag - Rotameters

8.3.3 Choked Flow - Sonic Nozzle

Choke flow properties can be used to obtain the mass flow meter by measuring the upstream pressure, Pi. The critical pressure ratio is

8.3.4 Vortex Shedding Frequency - Vortex Shedding Meter

8.3.5 Electromagnetic Flow Meter

8.3.6 Turbine Flow Meter

# LECTURE NOTES III-- SYSTEM RESPONSE

## 1. Introduction

Amplitude response
Frequency response
Phase response
Rise time, or delay
Slew rate--the maximum rate of change that the system can follow

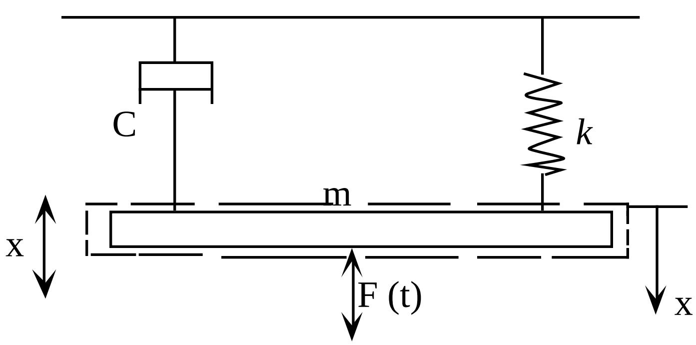
## 2. Simplified Physical Systems

### 2.1 Mechanical Systems

Mass
Spring force
Damping

### 2.2 Dynamic Characteristics of Simplified Mechanical Systems

Most simplified measuring systems can be approximated as a single degree of freedom system, linear restoring force, and viscous damping.




$$m\frac{d^2x}{dt^2} = F(t) - kx - c\frac{dx}{dt}\text{ , or}$$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

<u>Zero-Order System</u>
Output = Constant x Input (spring and damper removed)

<u>First-Order System</u>
Setting m = 0, one obtains: $c\frac{dx}{dt} + kx = F(t)$

<u>Second-Order System</u>

m assumes a non-zero value.

## 3. Second-Order System

### 3.1 General Description

$$a_2 \frac{d^2 q_o}{dt^2} + a_1 \frac{dq_o}{dt} + a_o q_o = b_o q_i \qquad a_2 \neq 0$$

Rearranging the second-order ODE, one obtains

$$\left(\frac{a_2}{a_o}\right)\frac{d^2 q_o}{dt^2} + \left(\frac{a_1}{a_o}\right)\frac{dq_o}{dt} + q_o = \left(\frac{b_o}{a_o}\right) q_i$$

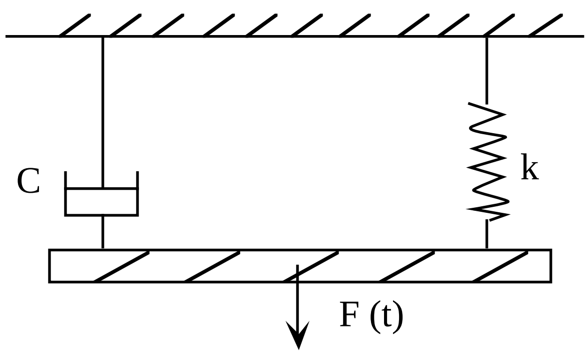
Define

$$P_s = \frac{b_o}{a_o} = \text{static sensitivity}$$

$$\omega_n = \left(\frac{a_o}{a_2}\right)^{1/2} = \text{(undamped) natural frequency}$$

$$\zeta = \frac{a_1}{2(a_o a_2)^{1/2}} = \text{damping ratio}$$

Recall




$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

$$P_s = \frac{F(t)}{k}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{c}{2\sqrt{km}}$$

3.2 Free Vibration Second-Order System, $F(t) = 0$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \text{ ; I.C. } t = 0 \quad x = x(0) \qquad \dot{x} = \dot{x}(0)$$

(a) Undamped, $c = 0$

$$m\frac{d^2x}{dt^2} + kx = 0 \qquad x = A\cos\omega_n t + B\sin\omega_n t$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\boxed{x(t) = x(0)\cos(\omega_n t) + \frac{\dot{x}(0)}{\omega_n}\sin(\omega_n t)}$$

(b) Viscous Damping, $c \neq 0$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

$$x(t) = \exp[-\zeta\,\omega_n t]\,(A\cos(\omega_d t) + B\sin(\omega_d t))$$

$$x(0) = A$$

$$\dot{x} = -\zeta\,\omega_n A + B\omega_d, \quad B = \frac{\dot{x} + \zeta\,\omega_n\,x(0)}{\omega_d}$$

$$x(t) = \exp[-\zeta\,\omega_n t]\left(x(0)\cos(\omega_d t) + \left[\frac{\dot{x}(0) + \zeta\,\omega_n x(0)}{\omega_d}\right]\sin(\omega_d t)\right)$$

where $\omega_d = \sqrt{(1-\zeta^2)}\;\omega_n$, damped frequency

---

*Solution to the ODE*

$$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0}$$

$$x = e^{st}$$

$$(ms^2 + cs + k)\,e^{st} = 0;\;\; ms^2 + cs + k = 0;\; s = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

If $\left(\frac{c}{2m}\right)^2 - \frac{k}{m} < 0$, then oscillation in t.

$$\frac{c^2}{4m^2}\,\frac{m}{k} < 1, \text{ or } \left(\frac{c}{\sqrt{2mk}}\right)^2 < 1 \text{ and } \zeta < 1$$

$$s = -\frac{c}{2m} \pm \sqrt{\frac{k}{m}} \bullet \sqrt{(1-\zeta^2)}\;\; i$$

$$s = -\frac{c}{2m} \pm \sqrt{(1-\zeta^2)\left(\frac{k}{m}\right)}\quad i$$

$$\frac{c}{2m} = \zeta\sqrt{\frac{k}{m}} = \zeta\;\;\omega_n$$

$$(1 - \zeta^2)\left(\frac{k}{m}\right) = \omega_n \quad \sqrt{(1 - \zeta^2)} = \omega_d$$

Therefore, the solution is

$$x(t) = \exp(-\zeta\,\omega_n t) \bullet (A\cos(\omega_d t) + B\sin(\omega_d t))$$

I.C.:

$t = 0 \quad \boxed{x(0) = A}$

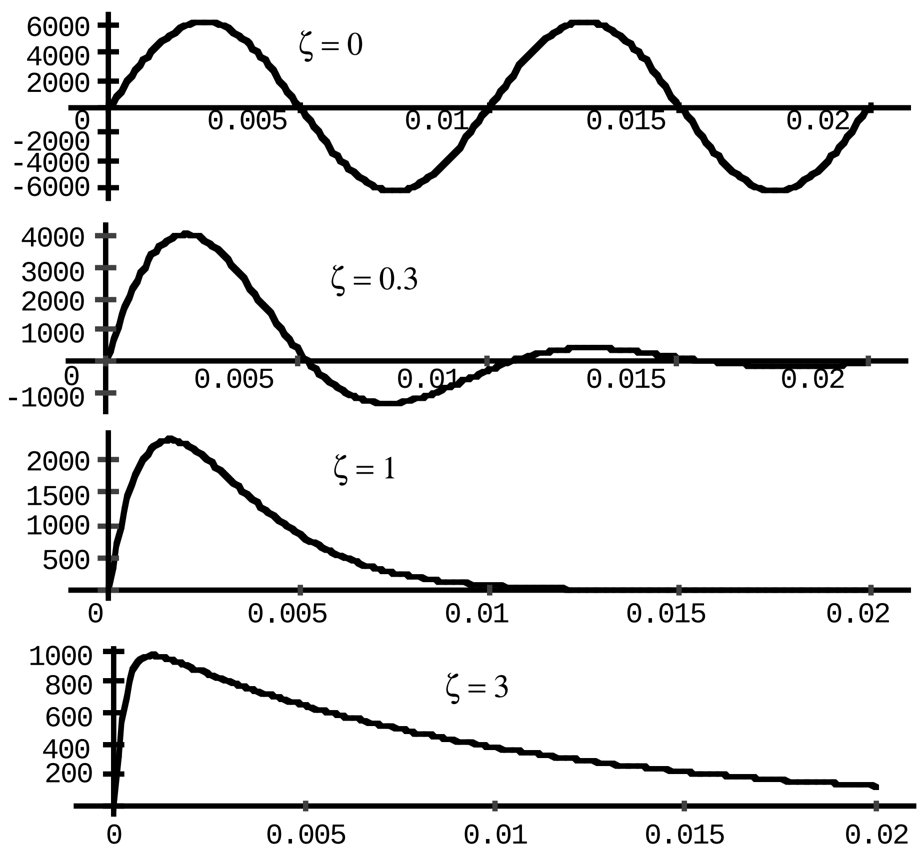
$$\dot{x}(0) = (-\zeta\,\omega_n)\exp(0) \bullet A + \exp(0)\omega_d \cos(0)$$

$$\dot{x}(0) = -\zeta\,\omega_n A + B\,\omega_d$$

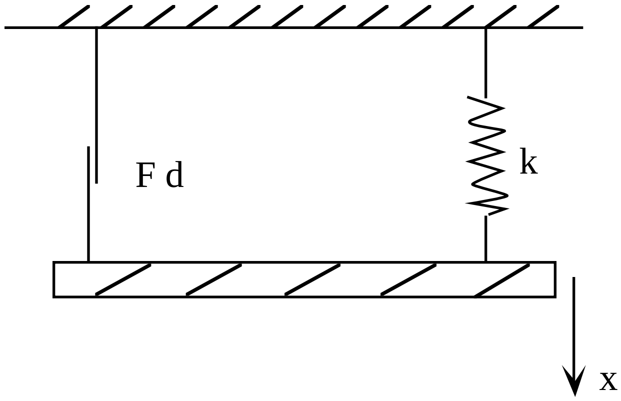
$$\boxed{B = \frac{\dot{x}(0) + \zeta\,\omega_n\,x(0)}{\omega_d}}$$

$$\boxed{x(t) = \exp(-\zeta\,\omega_n t) \bullet \left(x(0)\cos(\omega_d t) + \frac{\dot{x}(0) + \zeta\,\omega_n\,x(0)}{\omega_d}\sin(\omega_d t)\right)}$$

<u>Examples of second-order systems</u>



(c) <u>Coulomb Damping</u> (dry friction)




$$\boxed{m\ddot{x} + kx \pm F_d = 0}$$

For downward motion, the equation of motion is

$$m\ddot{x} + kx - F_d = 0, \text{ or}$$

$$\boxed{m\ddot{x} + k\,(x - F_d/k) = 0}$$

Let $\tilde{x} = x - F_d/k$, thus $m\ddot{\tilde{x}} + k\tilde{x} = 0$

$\tilde{x} = A_1 \cos \omega_n t + B_1 \sin \omega_n t$

$x = (A_1 \cos \omega_n t + B_1 \sin \omega_n t) + F_d/k$

I.C.: $t = 0$; $x = x_0$ and $\dot{x} = 0$

$A_1 = x_0 - F_d/k, \qquad B_1 = 0$

$$\boxed{x = \left(x_0 - \frac{F_d}{k}\right)\cos \omega_n t + \frac{F_d}{k}} \qquad \text{(A)}$$

When the motion is reversed,

$$\boxed{m\ddot{x} + k\,(x + F_d/k) = 0}$$

$x = (A_2 \cos \omega_n t + B_2 \sin \omega_n t) - F_d/k$

The initial condition is obtained by setting $t = \dfrac{\pi}{\omega_n}$ to Eq. (A)

I.C. $t = \dfrac{\pi}{\omega_n}$ ;

$x\left(\dfrac{\pi}{\omega_n}\right) = -x_0 + 2\dfrac{F_d}{k}$

$\dot{x}\left(\dfrac{\pi}{\omega_n}\right) = 0$

Thus $A_2 = x_0 - \frac{3F_d}{k}$ , $B_2 = 0$

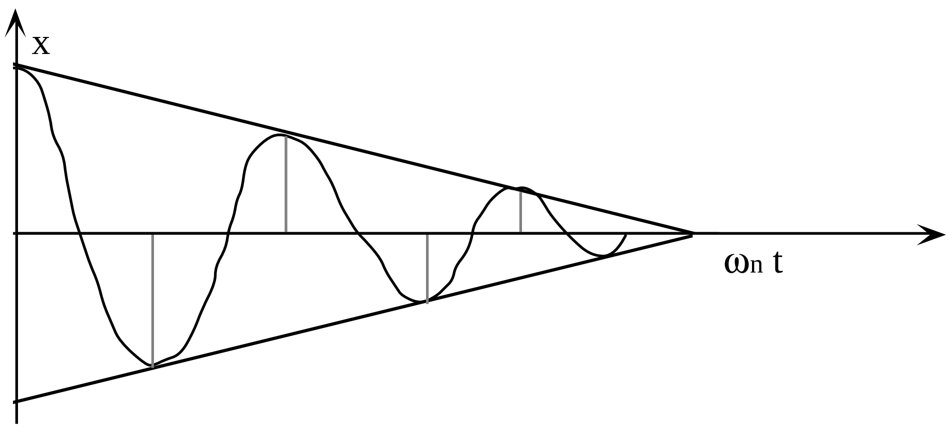
$$\boxed{x = \left(x_0 - \frac{3F_d}{k}\right)\cos(\omega_n t) - \frac{F_d}{k}} \qquad \text{(B)}$$

Thus, the decrement of the peak amplitude is

$$x_{n+1} - x_n = 4\,\frac{F_d}{k}$$

The following equation is an approximation to the solution:

$$x(t) = x_0\left[1 - \frac{4F_d}{x_0 k}\,\frac{t}{T}\right]\sin(\omega_n t)$$




**4 . First-Order System:** $a_1 \frac{dq_0}{dt} + a_0 q_0 = b_0 q_i$ (i.e., with $a_2 = 0$)

4.1 General Solution for Systems with a constant forcing

$F(t) = F_0 = \text{constant}$

$$c\,\frac{dx}{dt} + kx = F$$

$$c\,\frac{dx}{dt} + kx = F_o$$

I.C.: $t = 0$; $x = x_0$

$$c\,\frac{dx}{dt} = F_0 - kx$$

$$\frac{c}{k}\,\frac{dkx}{dt} = F_0 - kx$$

$$\frac{dkx}{F_0 - kx} = \frac{k}{c}\,dt$$

$$\left[\ln(F_0 - kx)\right]_{x_0}^{x} = -\left[\frac{kt}{c}\right]_0^t$$

$$\ln \frac{F_0 - kx}{F_0 - kx_0} = -\frac{k}{c}\, t$$

$$\frac{F_0 - kx}{F_0 - kx_0} = \exp\left[-\frac{k}{c}\, t\right]$$

Define time constant, $\tau$, as

$$\tau \equiv \frac{c}{k}$$

$$\boxed{\frac{F_0 - kx}{F_0 - kx_0} = \exp\left[-\frac{t}{\tau}\right]}$$

Let's examine the asymptotic solution as t approaches $\infty$:

$F_0 - kx_\infty = 0$, or $x_\infty = F_0/k$

Rearranging the solution, one obtains

$$\frac{F_0/k - x}{F_0/k - x_0} = \exp\left[-\frac{t}{\tau}\right]$$

$$\frac{x_\infty - x}{x_\infty - x_0} = \exp\left[-\frac{t}{\tau}\right] \quad , \text{ or}$$

$$\boxed{x = x_\infty + (x_0 - x_\infty) \exp\left[-\frac{t}{\tau}\right]}$$

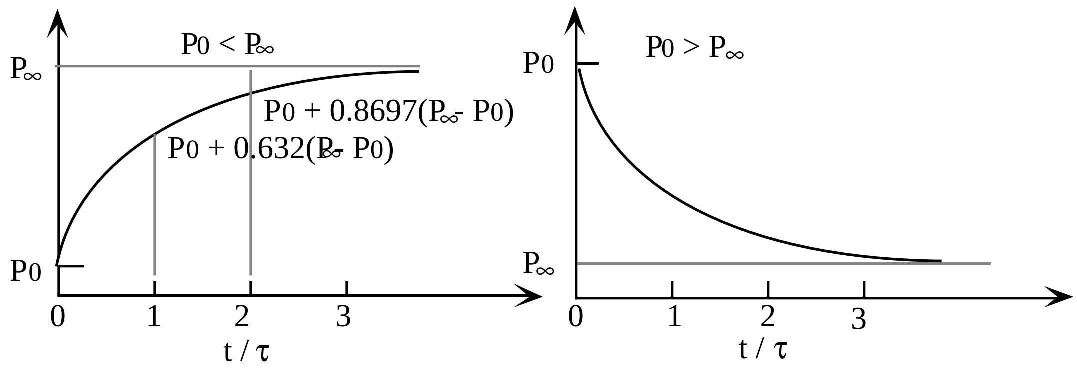
Grouping $x_\infty$, the solution becomes:

$$\boxed{x = x_\infty(1 - e^{-t/\tau}) + x_o\, e^{-t/\tau}}$$

The solution can be generalized by replacing "x" with "P" which represents the process variable, e.g. temperature in Lab. No. 3.

$$\boxed{P = P_\infty + (P_0 - P_\infty) \exp\left(-\frac{t}{\tau}\right)}$$

$$\boxed{P = P_\infty \left(1 - \exp\left[-\frac{t}{\tau}\right]\right) + P_o \left(\exp\left(-\frac{t}{\tau}\right)\right)}$$

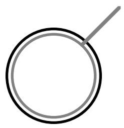



In the above figure, $\tau$ represents the time required to complete 63.2% of the dynamic process. It is often assumed that the process is completed during five time constants.

## 4.2 Examples

### 4.2.1 Lumped Capacitance Method for Transient Heat Transfer

Control Volume



$$\dot{Q} - \dot{W} = \left(\frac{dE}{dt}\right)_{c.v.}$$

$\dot{Q} = - h\, A_s\, (T_s - T_\infty)$ Newton's law of cooling

$- hA_s\, (T_s - T_\infty) = \rho\, cV \dfrac{dT}{dt}$ rigid system

$h = \dfrac{\rho cV}{A_s}\, \dfrac{(dT\,/\,dt)}{T_s - T_\infty}\ ;\quad T = T_s$ uniform T

$$h = \frac{\rho cV}{A_s}\frac{(dT\,/\,dt)}{T - T_\infty}$$

$$\frac{dT}{T - T_\infty} = - \frac{hA_s}{\rho cV}\, dt$$

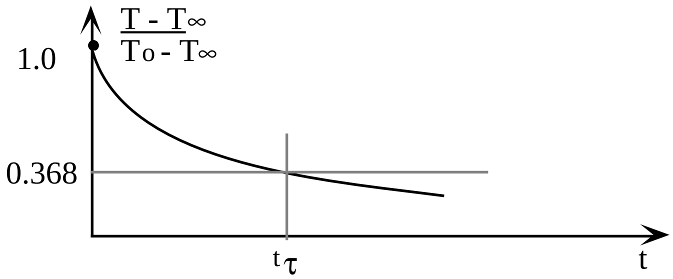
$\ln\,(T - T_\infty) = - \dfrac{hA_s}{\rho cV}\, t + K_1$ h = constant

$$T - T_\infty = K_2 \exp\left[- \frac{hA_s}{\rho cV}\, t\right]$$

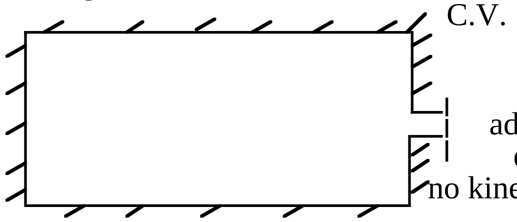
I.C.: $t = 0\ ;\ T = T_o$

$$\frac{T - T_\infty}{T_o - T_\infty} = \exp\left[- \frac{hA_s}{\rho cV}\, t\right] = \exp\left[- \frac{t}{\tau_t}\right]$$

where the time constant is defined as $\tau_t = \dfrac{\rho c V}{h A_s}$




4.2.1 Discharge Process




adiabatic, rigid control volume,
equilibrium inside the tank,
no kinetic and potential energy changes.

$$\dot{Q} - \dot{W} + \sum_{in} \dot{m}_i h_i - \sum_{out} \dot{m}_e h_e = (\dot{E})_{c.v.}$$

$$\dot{Q} = \dot{W} = \dot{m}_i = 0$$

$$\left(\frac{dE}{dt}\right)_{c.v.} = - \dot{m}_e h_e$$

$$\frac{d}{dt}(mu)_{c.v.} = \left(\frac{dm}{dt}\right)_{cv} \bullet h_e; \int \frac{d}{dt}(mu)_{cv} = \int \left(\frac{dm}{dt}\right) he$$

where u is the internal energy and h is the enthalpy

$$d(mu)_{c.v.} = h\, dm \text{ or}$$

$$h\, dm = m\, du + u\, dm$$

$$(h - u)\, dm = m\, du$$

$$\boxed{\frac{du}{h - u} = \frac{d\,m}{m}}$$

$$\frac{dm}{m} = - \frac{V\,dv}{V\,v} = - \frac{dv}{v} \Rightarrow \left(\frac{dm}{m} = \frac{d(V/v)}{(V/v)} = \frac{V}{V}\,\frac{d\,1\,/\,v}{Vv} = - v \bullet \frac{1}{v^2} \bullet dv = - \frac{d\,v}{v}\right)$$

h - u = pv; where V is the total volume and v is the specific volume

$$\frac{du}{pv} = - \frac{dv}{v}$$

or $\boxed{du + p\ dv = 0}$

It is also known (from Gibbs Equation) that T ds = du + p dv, s is the entropy; thus,

$$\boxed{ds = 0}$$

The isentropic process follows

$$p_1 v_1^k = p_2 v_2^k \quad \text{where k is the specific heat ratio} \equiv \frac{Cp}{Cv}$$

$$\frac{p_2}{p_1} = \left(\frac{v_1}{v_2}\right)^k = \left(\frac{R\ T_1/p_1}{R\ T_2/p_2}\right)^k$$

$$\frac{p_2}{p_1} = \left(\frac{T_1}{T_2}\right)^k \left(\frac{p_2}{p_1}\right)^k, \text{ or}$$

$$\boxed{\left(\frac{T_2}{T_1}\right) = \left(\frac{p_2}{p_1}\right)^{(k - 1) / k}}$$

$$\frac{m_2}{m_1} = \left(\frac{V/v_2}{V/v_1}\right) = \left(\frac{v_1}{v_2}\right) = \left(\frac{p_2}{p_1}\right)^{1/k}$$

$$\boxed{\frac{m_2}{m_1} = \left(\frac{p_2}{p_1}\right)^k}$$

<u>Example</u> Discharge process of Experiment No. 3


It was shown earlier that the discharge process is an isentropic process; conversely,

$$\frac{P_t}{P_0} = \left(\frac{v_0}{v_t}\right)^k = \left(\frac{\rho_t}{\rho_0}\right)^k$$

where "t" and "0" denote the tank pressure at time t and initial pressure. Recall the IG EOS:

$$P_t = \rho_t RT_t$$

Thus, one may relate the system mass to its thermodynamic state.

$$\boxed{\frac{dm}{dt} = - \dot{m}_e = V \frac{d\rho_t}{dt} = V \frac{d\,(P_t/RT_t)}{dt}}$$

where V is the volume of the tank. Substituting the isentropic relationship into the above equation, one obtains

$$\dot{m}_e = - V \frac{d\left(\frac{p_t}{p_0}\right)^{1/k} \bullet \rho_0}{dt}$$

$$\dot{m}_e = - V\rho_0 \left(\frac{p_t}{p_0}\right)^{1/k - 1} \frac{1}{k} \frac{d\,(p_t/p_0)}{dt}$$

$$\boxed{\dot{m}_e = - \frac{V\rho_0}{k} \left(\frac{p_t}{p_0}\right)^{(1 - k)/k} \frac{d(p_t/p_0)}{dt}} \qquad \text{(A)}$$

Let's define the discharge coefficient:

$$\dot{m}_e = C_D \, \dot{m}_{e,\,i}$$

where $c_D$ is the discharge coefficient and $\dot{m}_{e,\,i}$ is the ideal mass flow rate. For a converging nozzle, the ideal flow rate is the choked mass flow if the tank pressure is higher than the critical pressure, P∗. The critical pressure, P∗, can be determined from

• $$\frac{P_a}{P_*} = \left(\frac{2}{k + 1}\right)^{k/(k-1)}$$

As an example, $P_a$ = 14.7 psia, k = 1.4 for air, $P_*$ = 27.8 psia ($P_a / P_*$ = 0.5283 ; $P_*$ = 1.8929 $P_a$)

The choked flow condition has a mass flow rate

$$\boxed{\dot{m}_* = \frac{A\ P_t}{\sqrt{R\ T_t}} \sqrt{k} \left(\frac{2}{k + 1}\right)^{(k + 1)/[2(k - 1)]}} \qquad \text{(B)}$$

where A is the throat or nozzle opening area. This equation is valid when $P_t > P_*$. When $P_t / P_a < P_* / P_a$, it can be shown that

$$\boxed{\dot{m} = \frac{A\ P_t}{\sqrt{R\ T_t}} \sqrt{\frac{2k}{k - 1}\left[\left(\frac{P_a}{P_t}\right)^{2/k} - \left(\frac{P_a}{P_t}\right)^{(k + 1)/k}\right]}} \qquad \text{(C)}$$

Substituting the choked flow equation into the mass flow rate expression, i.e., combining Eqs. (B) & (A), one obtains

$$-\frac{V\rho_o}{k}\left(\frac{P_t}{P_o}\right)^{(1-k)/k}\frac{d(P_t/P_o)}{dt} = \dot{m}_e = C_D\frac{A\ P_t}{\sqrt{R\ T_t}}\sqrt{k}\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

$$\left(\frac{P_t}{P_o}\right)^{(1-3k)/2k}\frac{d(P_t/P_o)}{dt} = -C_D\frac{A\ k\sqrt{kR\ T_o}}{V}\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

The solution to the above equation with an I.C. of $P_t = P_o$ at $t = 0$ is

<u>Choked Flow</u>

$$-\frac{V\rho_o}{k}\left(\frac{P_t}{P_o}\right)^{(1-k)/k}\frac{d(P_t/P_o)}{dt} = C_D\frac{A\ P_t}{\sqrt{R\ T_t}}\sqrt{k}\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

$$\left(\frac{P_t}{P_o}\right)^{(1-k)/k}\frac{d(P_t/P_o)}{dt} = -C_D\frac{A\sqrt{k}}{V}\frac{k}{\rho_o}\frac{P_t}{\sqrt{R\ T_t}}\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

$$= -C_D\frac{A\sqrt{k}}{V}\bullet\frac{P_t}{P_o}\bullet\frac{\rho_o R\ T_o}{\rho_o\sqrt{R\ T_t}}\bullet k\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

$$= -C_D\frac{Ak\sqrt{k}}{V}\bullet\frac{P_t}{P_o}\bullet\sqrt{R\ T_o}\sqrt{\frac{T_o}{T_t}}\bullet\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

$$\frac{P_t}{P_o}\bullet\sqrt{\frac{T_o}{T_t}} = \frac{P_t}{P_o}\bullet\left(\frac{P_o}{P_t}\right)^{(k-1)/2k} = \left(\frac{P_t}{P_o}\right)^{(k+1)/2k}$$

$$\left(\frac{P_t}{P_o}\right)^{(1-3k)/2k}\bullet\frac{d(P_t/P_o)}{dt} = -C_D\frac{A\ k\sqrt{kR\ T_o}}{V}\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

$$\frac{2k}{1-k}\bullet\frac{d(P_t/P_o)^{(1-k)/2k}}{dt} = -C_D\frac{A\ k\sqrt{kR\ T_o}}{V}\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

Applying the initial condition $t = 0$, $P_t = P_0$, one obtains

$$\left(\frac{P_t}{P_o}\right)^{(1-k)/2k} - 1 = -C_D\frac{A\ \cancel{k}\sqrt{kR\ T_o}}{V}\bullet\frac{1-k}{2\cancel{k}}\bullet\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}\bullet t$$

$$\left(\frac{P_o}{P_t}\right)^{(k-1)/2k} = 1 + C_D\frac{A\sqrt{kR\ T_o}}{V}\bullet\frac{k-1}{2}\bullet\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}\bullet t$$

• $$\boxed{\frac{P_t}{P_o} = \left[1 + C_D\frac{A}{V}\sqrt{kR\ T_o}\bullet\frac{k-1}{2}\bullet\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]} t\right]^{2k/(1-k)}}$$

Also, one obtains

$$\boxed{t = \frac{V}{C_D\ A\sqrt{kR\ T_o}}\left(\frac{2}{k-1}\right)\left(\frac{k+1}{2}\right)^{(k+1)/[2(k-1)]}\left[\left(\frac{P_o}{P_t}\right)^{(k-1)/2k} - 1\right]}$$

which can be used to obtain the time to reach P*, i.e. t* by setting $P_t$ = P*. For t> t*, the mass flow rate can be obtained by considering Eqns. (A) and (C):

<u>Unchoked Flow</u>

$$\frac{\frac{d(P_t/P_a)}{dt}}{\left(\frac{P_t}{P_a}\right)^{(k-1)/k}\sqrt{\left(\frac{P_t}{P_a}\right)^{(k-1)/k} - 1}} = -\frac{C_D\ Ak\sqrt{kR\ T_o}}{V}\left(\frac{2}{k-1}\right)^{1/2}\left(\frac{P_a}{P_o}\right)^{(k-1)/2k}$$

A closed form solution was obtained by Owczarek (1964) using the following transformation:

$$x = \sqrt{\left(\frac{P_t}{P_a}\right)^{(k-1)/k} - 1}$$

$$t - t_* = \frac{2k}{k-1}\left[\frac{V}{C_D\ A}\left(\frac{k-1}{2}\right)^{1/2}\frac{(P_o / P_a)^{(k-1)/2k}}{k\sqrt{kR\ T_o}}\right]$$

$$\cdot\left[0.492 - \frac{x}{8}(2x^2 + 5)\sqrt{x^2+1} - \frac{3}{8}\ln\left(x + \sqrt{x^2+1}\right)\right]$$

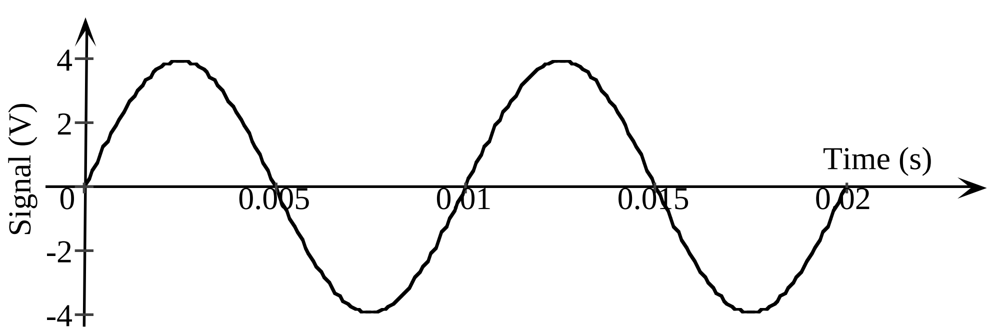
<u>Reference</u>

Owczarek, J.A., <u>Fundamentals of Gas Dynamics</u>, International Textbook Co., Scranton, Pa, 1964.

## 5. Frequency Domain Representation of Time-Series Data

### 5.1 Fourier Series Representation of Periodic Signals

Consider a periodic signal q(t) with a period T such as the undamped free vibration:




If the signal satisfies the Dirichlet conditions, then it can be expressed by a Fourier series. The

Dirichlet condition is as follows:

(a) $q_i$ must be defined, single-valued and piecewise continuous with a finite number of infinite discontinuities, or

(b) $q_i$ must satisfy $\int_T q^2(t)\,dt < \infty$

Condition (b) can be viewed to relate to energy. Thus, parameters of engineering interest satisfy this requirement. Therefore, $q_i$ can be expressed by a Fourier series.

$$\boxed{q(t) = A_o + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{2n\pi t}{T}\right) + B_n \sin\left(\frac{2n\pi t}{T}\right)\right]}$$

where $A_o = \frac{1}{T}\int_0^T q(t)\,dt$ (A)

$$A_n = \frac{2}{T}\int_0^T q(t) \cos\left(\frac{2n\pi t}{T}\right) dt \qquad \text{(B)}$$

$$B_n = \frac{2}{T}\int_0^T q(t) \sin\left(\frac{2n\pi t}{T}\right) dt \qquad \text{(C)}$$

The coefficients, $A_o$, $A_n$ and $B_n$ can be obtained by performing the integration over a period, e.g.

$$\int_0^T q(t)\,dt = A_o \bullet T + \sum_{n=1}^{\infty}\int_0^T\left(A_n \cos\frac{2n\pi t}{T} + B_n \sin\frac{2n\pi t}{T}\right)dt$$

Applying the orthogonality property of the trigonometry functions, i.e.,

$$\int_0^T \cos\frac{n\pi t}{T}\,dt = 0 \text{ and } \int_0^T \sin\frac{n\pi t}{T}\,dt = 0$$

one obtains

$$\boxed{A_o = \frac{1}{T}\int_0^T q(t)\,dt}$$

Multiplying $\cos\left(\frac{2n\pi t}{T}\right)$ to Eq. (A) and $\sin\left(\frac{2n\pi t}{T}\right)$, to Eq. (B) one obtains expressions for $A_n$ and $B_n$. These coefficients, $A_o$, $A_n$ and $B_n$, are the Fourier coefficients.

## 5.2 Fourier Integral

Instead of trigonometry functions, complex exponential functions can be used for Fourier Series:

$$q(t) = A_o + \sum_{n=1}^{\infty} \left( A_n \cos \frac{2n\pi t}{T} + B_n \sin \frac{2n\pi t}{T} \right)$$

$$\cos \frac{2n\pi t}{T} = \frac{e^{(2n\pi t / T)\, i} + e^{-(2n\pi t / T)\, i}}{2}$$

$$\sin \frac{2n\pi t}{T} = \frac{e^{(2n\pi t / T)\, i} - e^{-(2n\pi t / T)\, i}}{2\, i}$$

$$q(t) = A_o + \sum_{n=1}^{\infty} A_n \left( \frac{e^{(2n\pi t / T)\, i} + e^{-(2n\pi t / T)\, i}}{2} \right) + \sum_{n=1}^{\infty} B_n \left( \frac{e^{(2n\pi t / T)\, i} - e^{-(2n\pi t / T)\, i}}{2i} \right)$$

Let's define $C_o = A_o$ for use in a new series

$$C_n = \frac{A_n - i\, B_n}{2} \text{ , and } C_{-n} = \frac{A_n + i\, B_n}{2}$$

$$q(t) = C_o + \sum_{n=1}^{\infty} C_n\, e^{(2n\pi t / T)\, i} + \sum_{n=1}^{\infty} C_{-n}\, e^{-(2n\pi t / T)\, i}$$

$$\boxed{q(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{(2n\pi t / T)\, i}}$$

The coefficients in the complex exponential function series can also be obtained by

$$C_o = \frac{1}{T} \int_0^T q(t)\, dt$$

$$C_n = \frac{1}{T} \int_0^T q(t)\, e^{-(2n\pi t / T)\, i}\, dt$$

$$C_{-n} = \frac{1}{T} \int_0^T q(t)\, e^{-[(-2n\pi t) / T]\, i}\, dt = \frac{1}{T} \int_0^T q(t)\, e^{+(2n\pi t / T)\, i}\, dt$$

or $$\boxed{C_n = \frac{1}{T}\int_0^T q(t)\, e^{-(2n\pi t/T)i}\, dt}$$ ; where $-\infty < n < \infty$

Let's consider the transition from a periodic to an aperiodic function by allowing the period T to approach infinity. Conversely, the function will never repeat itself; thus, it becomes aperiodic. Substituting $C_n$'s into the series summation, one obtains

$$q(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T}\int_0^T q(\tau)\, e^{-(2n\pi\tau/T)i}\, d\tau\right] \bullet e^{(2n\pi t/T)i}$$

$$q(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T}\int_0^T q(\tau)\, e^{-(2n\pi\tau/T)i}\, d\tau\right] \bullet e^{(2n\pi t/T)i} \bullet \frac{\pi}{2T} \bullet \frac{2T}{\pi}$$

$$q(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\left[\int_0^T q(\tau)\, e^{-(2n\pi\tau/T)i}\, d\tau\right] \bullet e^{(2n\pi t/T)i} \bullet \left(\frac{2\pi}{T}\right)$$

Define $\omega_n = \frac{2n\pi}{T}$ , $\Delta\omega = \frac{2\pi}{T}$

$$q(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \bullet e^{i\omega_n t} \bullet \int_0^T q(\tau)\, e^{-i\omega_n \tau}\, d\tau\right] \Delta\omega$$

Let's consider $T \to \infty$ , or $\Delta\omega \to 0$

$$q(t) = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi}\, e^{i\omega_n t} \int_0^T q(\tau)\, e^{-i\omega_n \tau}\, d\tau\right] \Delta\omega$$

Applying a linear transformation from $\int_0^T$ to $\int_{-T/2}^{T/2}$, one obtains

$$q(t) = \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{2\pi}\left[\int_{-\infty}^{\infty} q(\tau)\, e^{-i\omega\tau}\, d\tau\right] d\omega$$

$$q(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i \omega t} Q(\omega)\, d\omega$$

$$\boxed{Q(\omega) = \int_{-\infty}^{\infty} q(\tau)\, e^{-i \omega \tau}\, d\tau}$$

<u>Remarks on Q(ω)</u>

(a) Q(ω) is the Fourier transform of q(t); it is also defined as

$$Q(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} q(\tau)\, e^{-i \omega \tau}\, d\tau \quad , \quad \text{and}$$

$$q(t) = \int_{-\infty}^{\infty} Q(\omega)\, e^{i \omega t}\, d\omega$$

(b) Q(ω) describes q(t) in terms related to amplitude and frequency contents; $\omega = 2\pi \bullet f$, where f is the frequency.

(c) $|Q(f)|^2$ or $|Q(\omega)|^2$ is a measure of the power contained within the dynamic portion of q(t)

$$|Q(f)|^2 = \text{Re}\,[Q(f)]^2 + \text{Im}\,[Q(f)]^2$$

where Re [Q(f)] and Im [Q(f)] are the real and imaginary parts of Q(f).

(d) It can be shown that Q(f) = A(f) - i B(f), and

$$|Q(f)|^2 = \frac{1}{2}\; C^2(f) \text{ and } C^2(f) = A^2(f) + B^2(f)$$

Thus, for periodic functions, the amplitude and phase shift are respectively

$$C(f) = \sqrt{2\; |Q(f)|^2}\;, \text{ and } \phi(f) = \tan^{-1} \frac{\text{Im } Q(f)}{\text{Re } Q(f)}$$

(e) Power Spectrum
<u>Signal power vs. frequency</u>

$$\frac{[q(t)]^2}{2} = \int_{0}^{\infty} Q(f)\, df$$

where the static portion of the energy has been removed.

$$\left.\frac{[q(t)]^2}{2}\right|_{f \pm \delta f/2} = \int_{f - \frac{\delta f}{2}}^{f + \frac{\delta f}{2}} Q(f)\, df$$

5.3 Discrete Fourier Transform

Recall the Fourier Series expressed in complex exponential function

$$q(t) = \sum_{n=-\infty}^{\infty} C_k \, e^{(2n\pi t / T) i}$$

$$C_k = \frac{1}{T}\int_0^T q(t)\, e^{-(2k\pi t / T) i}\, dt$$

Consider N discrete data at δt intervals

$$C_k = \frac{1}{N\delta t} \bullet \sum_{r=0}^{N-1} \left[ q\,(r\delta t)\; e^{-i\,(2k\pi\, r\, \delta t\, /\, N\delta t)} \right] \delta t$$

$$C_k = \frac{1}{N} \sum_{r=0}^{N-1} q(r\delta t)\, e^{-(2\pi rk)/N}$$

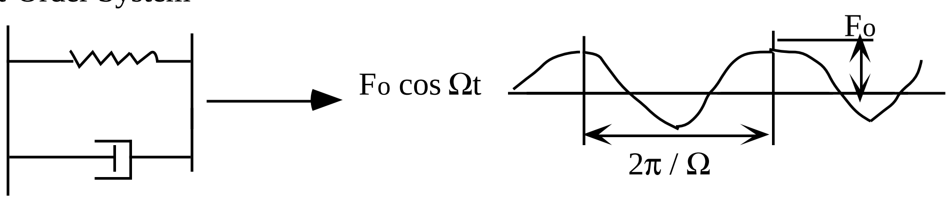
Resolution: $\frac{1}{N\delta t}$

Discrete frequency: $k = 0, 1, 2, \ldots, \frac{N}{2} - 1$; and $f_k = \frac{k}{N} \bullet \frac{1}{\delta t}$

An algorithm was developed by Cooley and Tukey to compute the discrete Fourier transform is known as the fast Fourier Transform (FFT). The first FFT algorithm was developed by Gauss. Instead of $N^2$ operation of discrete Fourier transform, the FFT algorithm takes only $N \log_2 N$ operations.

## 6. Forced System

6.1 First-Order System




$F(t) = F_o \cos \Omega t$ $F_o$: amplitude of the forcing function

$c\frac{dx}{dt} + kx = F_o \cos \Omega t$ Ω: frequency of the forcing function, radian/s

The solution of the equation is:

$$\boxed{x = A_1 \exp\left[-\frac{t}{\tau}\right] + \frac{F_o / k}{\sqrt{1 + (\tau\,\Omega)^2}} \cos(\Omega t - \phi)}$$

where the first term is the general solution and the second term the particular solution.

$A_1$: determined from the initial condition, i.e., $t = 0$, $x = x_o$, $A_1$ can be found

$\tau$: time constant, $\tau \equiv c / k$

$\phi$: phase lag, $\phi \equiv \tan^{-1}(\Omega c / k) = \tan^{-1}\left(\frac{2\pi\tau}{T}\right)$

T: the period of excitation in s, $T \equiv \frac{2\pi}{\Omega}$

The homogenous solution (the first part) of the above solution represents the transient response, while the particular solution denotes the steady-state relationship, as $t \gg \tau$. At steady state limit, the equation becomes

$$x = \frac{F_o / k}{\sqrt{1 + (\tau\,\Omega)^2}} \cos(\Omega t - \phi)$$

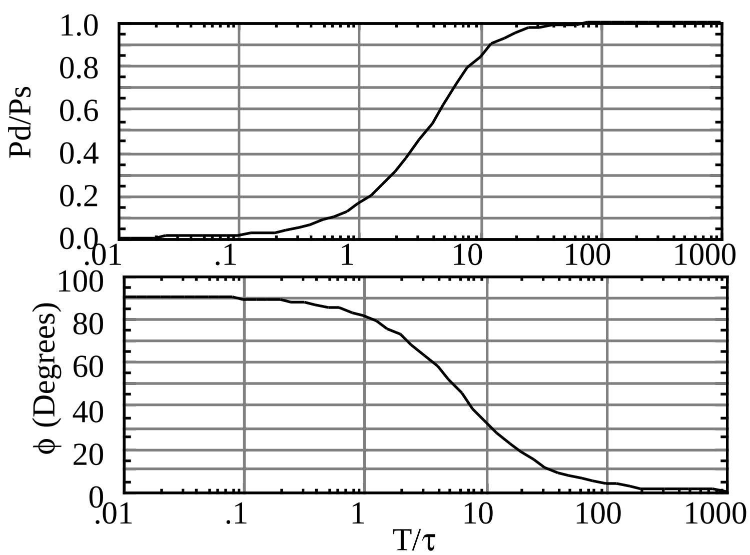
Define $x_s \equiv F_o / k$, the static response for a static input $F_o$ is

$$x = \frac{x_s}{\sqrt{1 + (\tau\,\Omega)^2}} \cos(\Omega t - \phi)$$

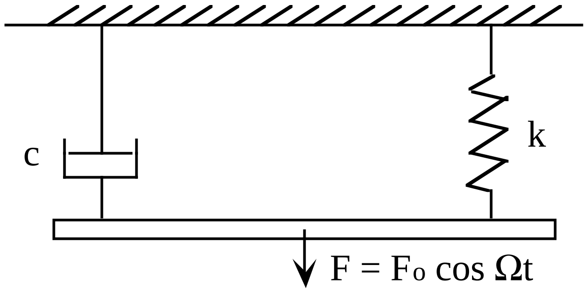
The maximum displacement is

$$x_d = \frac{x_s}{\sqrt{1 + (\tau\,\Omega)^2}} \qquad \text{(it occurs when } \Omega t - \phi = 0\text{)}$$

$$\frac{x_d}{x_s} = \frac{1}{\sqrt{1 + (\tau\,\Omega)^2}} = \frac{1}{\sqrt{1 + (2\pi\,\tau / T)^2}}$$



## 6.2 Second-Order System




$$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)} \quad ; F(t) = F_o \cos \Omega t$$

The solution is

$$x = \frac{(F_o / k)\cos(\Omega t - \phi)}{\{[1 - (\Omega / \omega_n)^2]^2 + [(c / c_c)(\Omega / \omega_n)]^2\}^{1/2}}$$

$\phi$: phase angle, $\phi \equiv \tan^{-1}\dfrac{2(c / c_c)(\Omega / \omega_n)}{1 - (\Omega / \omega_n)^2}$

$\omega_n$: natural frequency, $\omega_n \equiv \sqrt{\dfrac{k}{m}}$

$c_c$: critical damping coefficient, $c_c \equiv 2\sqrt{mk}$

Consider a general system and define $P_d$ and $P_s$ similar to the first-order system:

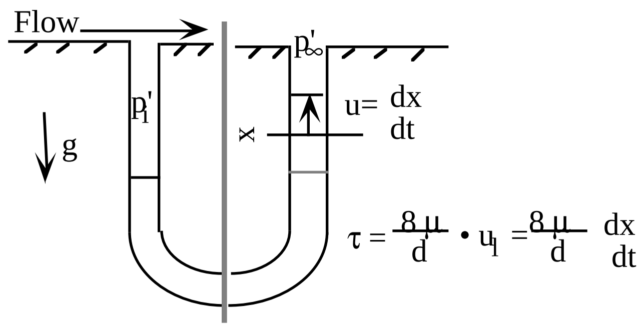
$$P_d = \frac{(F_o / k)}{\{[1 - (\Omega / \omega_n)^2]^2 + [(c / c_c)(\Omega / \omega_n)]^2\}^{1/2}} \quad , \text{or}$$

$$\boxed{\frac{P_d}{P_s} = \frac{1}{\{[1 - (\Omega / \omega_n)^2]^2 + [2\zeta(\Omega / \omega_n)]^2\}^{1/2}}} \quad \text{and } P_s \equiv \frac{F_o}{k}$$

where $\zeta \equiv c / c_c$, critical damping ratio

<u>Example</u> Oscillation of Liquid Column

Negligible gas density; no surface tension; viscous force following the laminar pipe flow; tube diameter: d; liquid column length: L




Let's use $\rho = \rho_f$; $\mu = \mu_f$; $P_i = P'_i - P'_\infty$ (i.e., gage P.)

$$\frac{\pi d^2}{4}\,\rho L\,\frac{d^2x}{dt^2} \;=\; \frac{\pi d^2}{4}\,P_i \;-\; 2g\left(\frac{\pi d^2}{4}\rho\right) x \;-\; (\pi d L)\,\frac{8\mu}{d}\,\frac{dx}{dt}$$

acceleration  pressure  unbalanced gravity force  viscous force

$$\frac{\pi d^2}{4}\,\rho L\,\frac{dx^2}{dt^2} + (\pi d L)\left(\frac{8\mu}{d}\right)\frac{dx}{dt} + 2g\left(\frac{\pi d^2}{4}\,\rho\right) x = \frac{\pi d^2}{4}\,P_i$$

$$\left(\frac{L}{2g}\right)\frac{d^2x}{dt^2} + \frac{16\mu L}{d^2\rho g}\,\frac{dx}{dt} + x = \frac{1}{2\rho g}\,P_i; \qquad \frac{a_2}{a_o}\,\ddot{x} + \frac{a_1}{a_o}\,\dot{x} + \frac{a_o}{a_o}\,x = \left(\frac{b_o}{a_o}\right) x_i$$

Undamped Natural Frequency

$$\omega_n = \sqrt{\frac{2g}{L}} \qquad\qquad \left(\sqrt{\frac{a_o}{a_2}}\right)$$

Damping Ratio

$$\zeta = \frac{8\mu}{\rho d^2}\sqrt{\frac{2L}{g}} \qquad\qquad \left(\frac{a_1}{2\sqrt{a_o a_2}}\right)$$

## 7. Numerical Solvers

### 7.1 Maple for Second-Order ODE-- 100 Hz undamped forced oscillation

```
• read `ODE.m`;  (Followed by Enter Key in Mac)
```

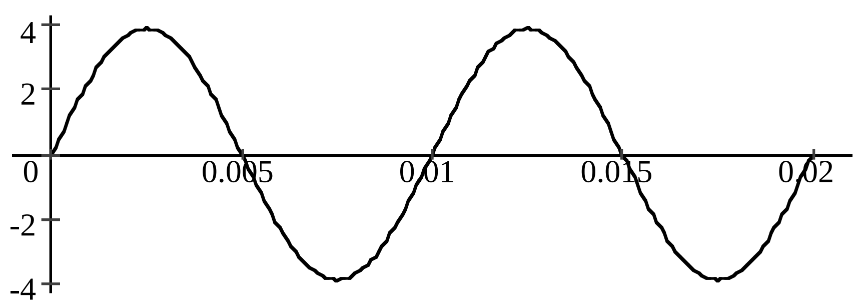
---

```
• f:=(t,x,z) -> z;
g:=(t,x,z) -> -c/m*z-k/m*x+p*a/m;
k:=39478.4176;
m:=.1;
c:=0;
p:=3447400;
a:=.000071256;
dpinit:=[0,0,0];
dppts:=rungekutta([f,g],dpinit,.0001,200);
plot ({makelist(dppts,1,3)});  (Followed by Enter Key)
```

```
                        f := (t,x,z) -> z
                                  c z    k x    p a
                g := (t,x,z) -> - --- - --- + ---
                                   m      m      m
                         k := 39478.4176
                             m := .1
                             c := 0
                          p := 3447400
                         a := .000071256
                       dpinit := [0, 0, 0]
      dppts := array(0 .. 200,, [
                   0 = [0, 0, 0]
                   1 = [.0001, .00001227835598, .2454863045]
                   2 = [.0002, .00004906496154, .4900037869]

                   197 = [.0197, .0001102170366, -.7325936405]
                   198 = [.0198, .00004906674309, -.4900100610]
                   199 = [.0199, .00001227951232, -.2454926365]
                                              -9             -5
                   200 = [.0200, .52660*10   , -.63650*10   ]
               ])
```

---




---

## 7.2 Matlab Example- fft Analysis

```
% This is for 1 kHz, for 10 kHz, need to change dt and m2
%   draw Power spectral density of given data set.
%   data file name should be dat1.dat
%             does not need be power of 2.

    load dat1.dat
    dt=0.001
    y=dat1(:,1);
    n=size(y);
    x=0:dt:(n-1)*dt;
    ymean=mean(y);
    y=y-ymean;
    m=n(1,1);
    xi=x(1);
    xf=x(m);
    ts=(xf-xi)/(m-1);
    fs=1/ts;
    Y=fft(y);
    Pyy=Y.*conj(Y);
    n=size(Pyy);
    m=n(1,1);
    m2=m/20;
    f=fs/m*(0:m2-1);
    plot(f,Pyy(1:m2))
%    f=fs/m*(0:m-1);
%    plot(f,Pyy)
    title('Power spectral density')
    xlabel('Frequency (Hz)')
```

Sample FFT Plot using the data file dat1.dat: