Fluids Kinematics (Acceleration) and Reynolds Transport Theorem (RTT)

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Lagrangian vs. Eulerian Viewpoint

- **Lagrangian**: Keeps track of individual fluid particles,

$$\underline{V_P}(t) = \frac{d\underline{r_P}(t)}{dt} = u_P(t)\hat{\boldsymbol{\imath}} + v_P(t)\hat{\boldsymbol{\jmath}} + w_P(t)\hat{\boldsymbol{k}}$$

$$\underline{a_P}(t) = \frac{d\underline{V_P}(t)}{dt} = \frac{du_P(t)}{dt}\hat{\boldsymbol{\imath}} + \frac{dv_P(t)}{dt}\hat{\boldsymbol{\jmath}} + \frac{dw_P(t)}{dt}\hat{\boldsymbol{k}}$$

- **Eulerian**: Focuses on a fixed point $\underline{x} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ in space,

$$\underline{V}(\underline{x}, t) = u(\underline{x}, t)\hat{\boldsymbol{\imath}} + v(\underline{x}, t)\hat{\boldsymbol{\jmath}} + w(\underline{x}, t)\hat{\boldsymbol{k}}$$

$$\underline{a}(\underline{x}, t) = a_x(\underline{x}, t)\hat{\boldsymbol{\imath}} + a_y(\underline{x}, t)\hat{\boldsymbol{\jmath}} + a_z(\underline{x}, t)\hat{\boldsymbol{k}}$$

Acceleration in the Eulerian Approach

- For a simple 1D flow,

$$a_x = \lim_{\Delta t \to 0} \frac{u(x+\Delta x, t+\Delta t) - u(x,t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{u(x+\Delta x, t+\Delta t) - u(x, t+\Delta t) + u(x, t+\Delta t) - u(x,t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} + \lim_{\substack{\Delta t \to 0 \\ (\Delta x \to 0)}} \frac{u(x+\Delta x, t+\Delta t) - u(x, t+\Delta t)}{\Delta x} \cdot \frac{\Delta x}{\Delta t}$$

$$\therefore a_x = \frac{\partial u(x,t)}{\partial t} + u(x,t) \cdot \frac{\partial u(x,t)}{\partial x}$$

- For a general 3D flow,

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$
$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$
$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

or,

$$\underline{a} = \frac{D\underline{\underline{V}}}{Dt} = \underbrace{\frac{\partial \underline{\underline{V}}}{\partial t}}_{\text{local acc.}} + \underbrace{\underline{V} \cdot \nabla \underline{V}}_{\text{convective acc.}}$$

Note:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{V} \cdot \nabla$$

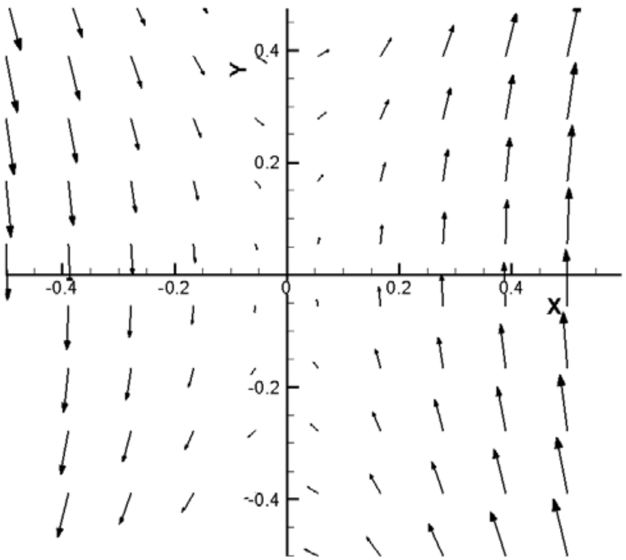
where,

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

Referred to as the *material derivative* or *total derivative* or *substantial derivative*.

Example: Acceleration

- An incompressible 2D flow has the velocity components u = 2y and v = 8x. Find (a) the acceleration and (b) the pressure distribution along a streamline that passes through the origin (x,y) = (0,0) where the pressure is p_0. Assume incompressible and irrotational flow.



Velocity vector field $\underline{V} = 2y\hat{\boldsymbol{\imath}} + 8x\hat{\boldsymbol{\jmath}}$.

(a)

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0 + (2y)(0) + 8x(2) = 16x$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0 + (2y)(8) + (8x)(0) = 16y$$

$$\therefore \underline{a} = a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} = 16x\hat{\boldsymbol{\imath}} + 16y\hat{\boldsymbol{\jmath}}$$

Note:

$$a = |\underline{a}| = \sqrt{a_x^2 + a_y^2} = \sqrt{(16x)^2 + (16y)^2}$$

$$\therefore a = 16\sqrt{x^2 + y^2}$$

Example: Acceleration – Contd.

- Euler equation*

$$\nabla p = -\rho \underline{a}$$

or

$$\frac{\partial p}{\partial x} = -\rho a_x = -16\rho x \quad (1)$$
$$\frac{\partial p}{\partial y} = -\rho a_y = -16\rho y \quad (2)$$

Integrate (1) w.r.t x,

$$\int \frac{\partial p}{\partial x} dx = \int (-16\rho x) dx = -8\rho x^2 + f(y) = p \quad (3)$$

Differentiate (3) w.r.t y, then by using (2),

$$\frac{\partial p}{\partial y} = 0 + \frac{df(y)}{dy} = -16\rho y$$

or

$$f(y) = \int (-16\rho y) dy = -8\rho y^2 + C \quad (4)$$

*In the horizontal plane without gravity

Example: Acceleration – Contd.

Combining (3) and (4),

$$p = -8\rho x^2 - 8\rho y^2 + C$$

Since $p = p_0$ at (x, y) = (0,0),

$$p_0 = 0 + 0 + C$$

or

$$C = p_0$$

Finally,

$$\therefore p = p_0 - 8\rho(x^2 + y^2) \quad (5)$$

Example: Acceleration – Contd.

Streamline equation

$$\frac{dx}{u} = \frac{dy}{v}$$

or

$$\frac{dx}{2y} = \frac{dy}{8x} \quad (6)$$

Integrate (6) by using separation of variables, then

$$\int 2y dy = \int 8x dx$$

or

$$y^2 = 4x^2 + C$$

For the streamline through the origin $(x,y) = (0,0)$, $C = 0$. Thus,

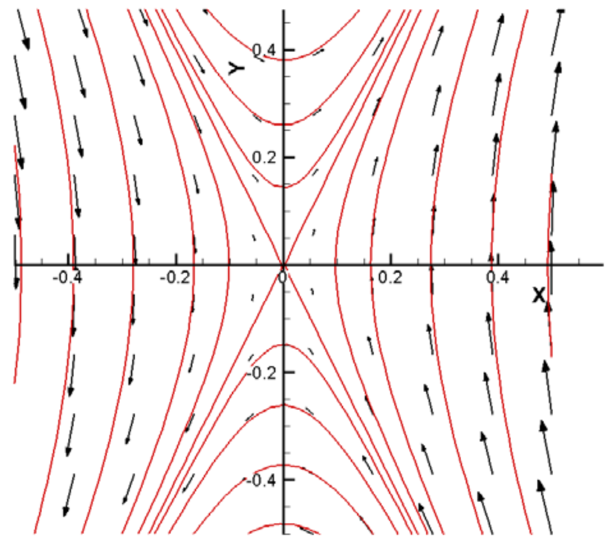
$$\therefore y^2 = 4x^2 \quad (7)$$

By plugging (7) into (5),

$$p = p_0 - 8\rho(x^2 + 4x^2)$$

Thus,

$$\therefore \boldsymbol{p = p_0 - 40\rho x^2}$$



Streamlines for the velocity field $\underline{V} = 2y\hat{\boldsymbol{i}} + 8x\hat{\boldsymbol{j}}$.

Example: Acceleration – Contd.

Alternatively, by applying the Bernoulli equation,

$$p + \frac{1}{2}\rho V^2 = p_0 + \frac{1}{2}\rho V_0^2$$

or

$$p = p_0 + \frac{1}{2}\rho(V_0^2 - V^2)$$

where,

$$V = \left|\underline{V}\right| = \sqrt{u^2 + v^2} = \sqrt{4y^2 + 64x^2}$$

Along the streamline $y^2 = 4x$,

$$V = \sqrt{4(4x^2) + 64x^2} = \sqrt{80x^2}$$

and

$$V_0 = V)_{x=0,y=0} = \sqrt{4(0)^2 + 64(0)^2} = 0$$

Thus,

$$\therefore p = p_0 + \frac{1}{2}\rho\left[(0)^2 - \left(\sqrt{80x^2}\right)^2\right] = \boldsymbol{p_0 - 40\rho x^2}$$

Laws of Mechanics

1. **Conservation of mass**: $$\frac{dm}{dt} = 0$$

2. **Conservation of linear momentum**: $$\underline{F} = m\underline{a} = \frac{d}{dt}\left(m\underline{V}\right)$$

3. Conservation of angular momentum: $$\underline{M} = \frac{d\underline{H}}{dt}$$

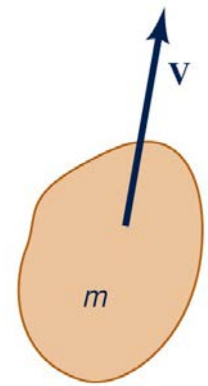
4. **Conservation of Energy**: $$\frac{dE}{dt} = \dot{Q} - \dot{W}$$

- The laws apply to either solid or fluid systems
- Ideal for solid mechanics, where we follow the same system
- For fluids, the laws need to be rewritten to apply to a specific region in the neighborhood of our product (i.e., CV)

Extensive vs. Intensive Property

Governing Differential Equations (GDE's):

$$\frac{d}{dt}\big(\underbrace{m, m\underline{V}, E}_{B}\big) = \big(0, \underline{F}, \dot{Q} - \dot{W}\big)$$



- B = The amount of m, $m\underline{V}$, or E contained in the total mass m; **Extensive property** – Dependent on mass
- β (or b) = The amount of B per unit mass; **Intensive property** – Independent on mass

$$\beta \equiv \frac{dB}{dm}$$

$$B = \int_{\mathcal{V}} \beta \underbrace{\rho dV}_{=dm}$$

If homogeneous,

$$\beta = \frac{B}{m} \quad \text{and} \quad B = \beta m$$

B	$b = B/m$
m	1
$m\mathbf{V}$	$\mathbf{V}$
E	e

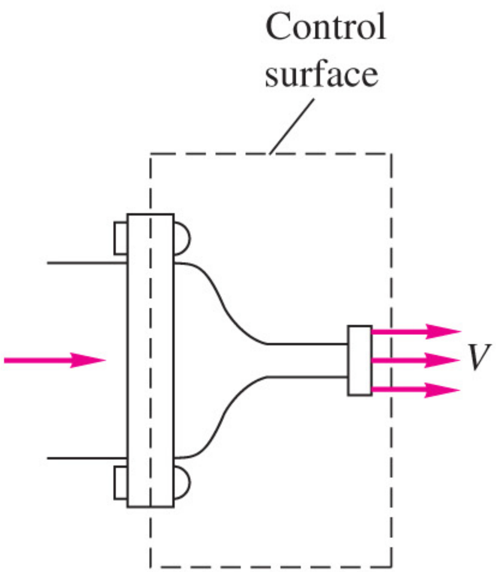
System vs. Control Volume

- **System**: A collection of matter of fixed identity
  - Always the same atoms or fluid particles
  - A specific, identifiable quantity of matter

- **Control Volume** (CV): A volume in space through which fluid may flow
  - A geometric entity
  - Independent of mass

Examples of CV

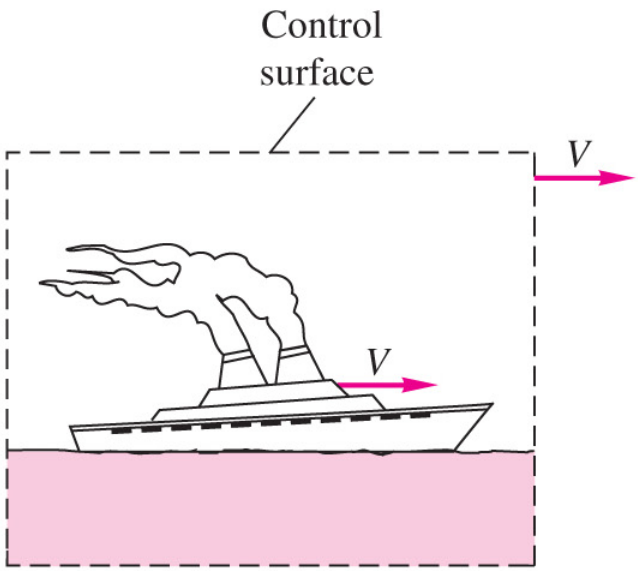
<u>Fixed CV</u>



(a)

CV fixed at a nozzle

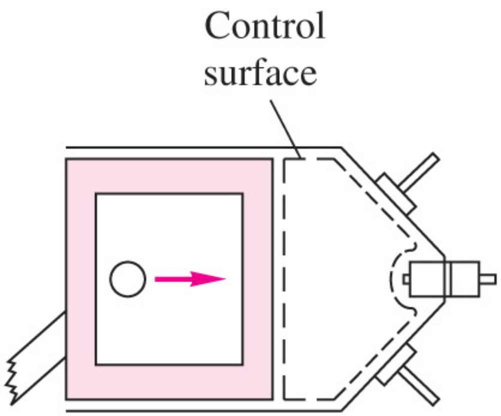
<u>Moving CV</u>



(b)

CV moving with ship

<u>Deforming CV</u>



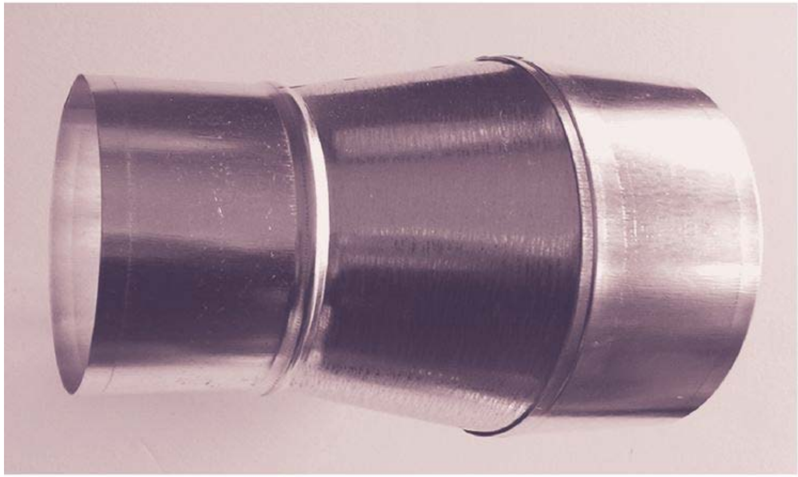
(c)

CV deforming within cylinder

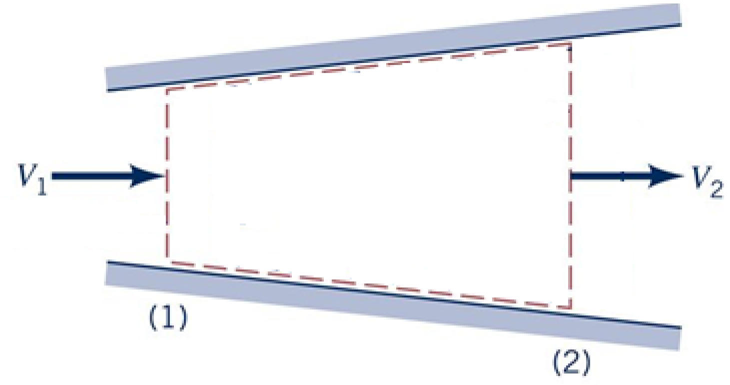
Reynolds Transport Theorem (RTT)

- An analytical tool to shift from describing the *laws governing fluid motion* using the system concept to using the control volume concept

RTT for a Simple Fixed CV



Variable area duct

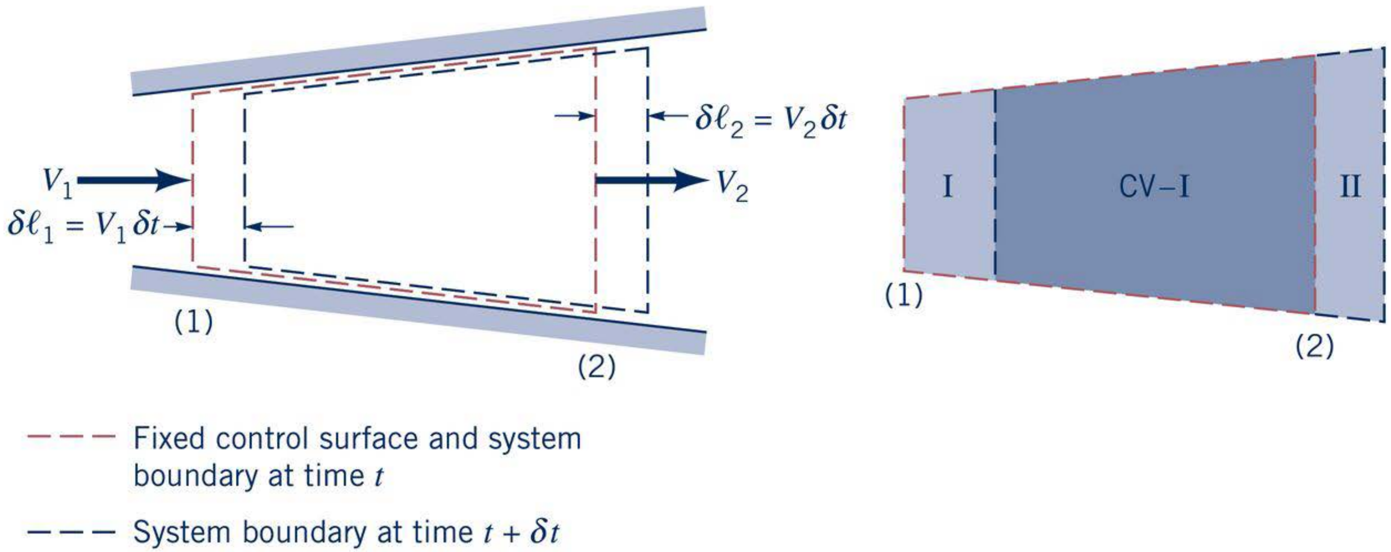


At time t:

SYS = CV

$$B_{\text{sys}}(t) = B_{\text{CV}}(t)$$

RTT for a Simple Fixed CV



At time $t + \delta t$:

SYS = (CV – I) + II

$$B_{\mathrm{sys}}(t + \delta t) = B_{\mathrm{CV}}(t + \delta t) - \delta B_I + \delta B_{II}$$

RTT for a Simple Fixed CV – Contd.

- Time Rate of Change of B_{sys}

$$\frac{DB_{\text{sys}}}{Dt} = \lim_{\delta t \to 0} \frac{B_{\text{sys}}(t+\delta t) - B_{\text{sys}}(t)}{\delta t}$$

Since $B_{\text{sys}}(t) = B_{\text{CV}}(t)$ and $B_{\text{sys}}(t+\delta t) = B_{\text{CV}}(t+\delta t) - \delta B_I + \delta B_{II}$

$$\frac{DB_{\text{sys}}}{Dt} = \lim_{\delta t \to 0} \frac{\left\{B_{\text{CV}}(\underline{x}, t+\delta t) - \delta B_I + \delta B_{II}\right\} - B_{\text{CV}}(\underline{x}, t)}{\delta t}$$

$$\therefore \underbrace{\frac{DB_{\text{sys}}}{Dt}}_{\substack{\text{Time rate of}\\ \text{change of } B\\ \text{within the}\\ \text{system}}} = \underbrace{\lim_{\delta t \to 0} \frac{B_{\text{CV}}(\underline{x}, t+\delta t) - B_{\text{CV}}(\underline{x}, t)}{\delta t}}_{\substack{\text{1) Change of } B\\ \text{within CV over } \delta t}} + \underbrace{\lim_{\delta t \to 0} \frac{\delta B_{II}}{\delta t}}_{\substack{\text{2) Amount of } B\\ \text{flowing out}\\ \text{through CS}\\ \text{over } \delta t}} - \underbrace{\lim_{\delta t \to 0} \frac{\delta B_I}{\delta t}}_{\substack{\text{3) Amountt of } B\\ \text{flowing in}\\ \text{through CS}\\ \text{over } \delta t}} \quad \text{Eq. (1)}$$

RTT for a Simple Fixed CV – Contd.
- The first term of RHS of Eq.(1)

$$\lim_{\delta t \to 0} \frac{B_{\mathrm{CV}}\left(\underline{x}, t + \delta t\right) - B_{\mathrm{CV}}(\underline{x}, t)}{\delta t} = \frac{\partial B_{\mathrm{CV}}}{\partial t}$$

In a general form,

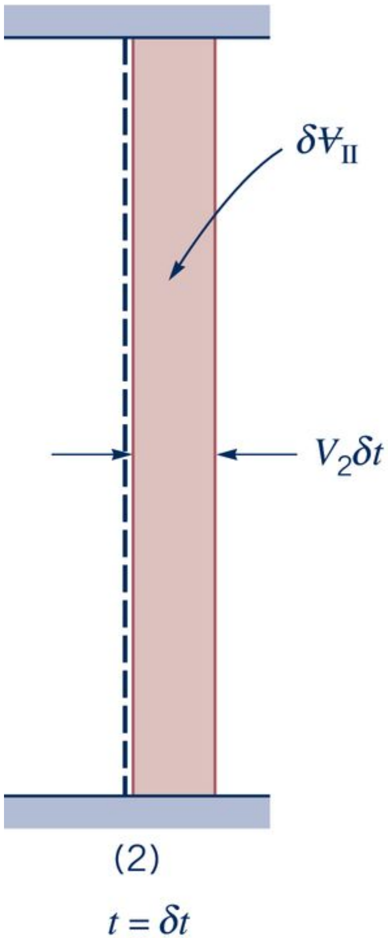
$$B_{\mathrm{CV}} = \int_{\mathrm{CV}} \underbrace{\beta \underbrace{\rho dV}_{=dm}}_{=dB}$$

Thus,

$$\frac{\partial B_{\mathrm{CV}}}{\partial t} = \frac{\partial}{\partial t} \int_{\mathrm{CV}} \beta \rho dV$$

RTT for a Simple Fixed CV – Contd.

- The 2nd term of RHS of Eq.(1)



$$\delta \forall_{II} = A_2 \cdot V_2 \delta t$$

and

$$\delta m_{\mathrm{II}} = \rho \delta \forall_{II} = \rho V_2 A_2 \delta t$$

The amount of B flowing out of CV through A_2 over a short time δt:

$$\therefore \delta B_{\mathrm{II}} = \beta \delta m_{\mathrm{II}} = \beta \rho V_2 A_2 \delta t$$

Thus,

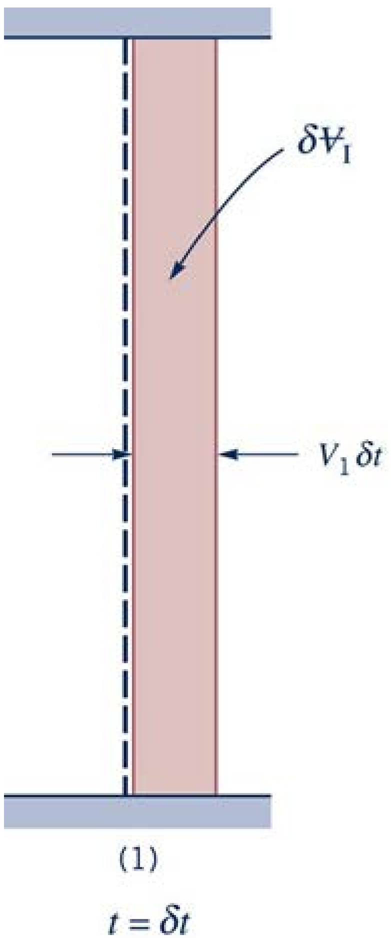
$$\lim_{\delta t \to 0} \frac{\delta B_{II}}{\delta t} = \beta \rho V_2 A_2 \equiv \dot{B}_{\mathrm{out}}$$

In a general form (see Appendix),

$$\dot{B}_{\mathrm{out}} = \int_{CS_{\mathrm{out}}} \beta \rho \underline{V} \cdot \underline{n} dA$$

RTT for a Simple Fixed CV – Contd.

- The 3rd term of RHS of Eq.(1)



$$\delta \forall_I = A_1 \cdot V_1 \delta t$$

and

$$\delta m_{\mathrm{I}} = \rho \delta \forall_I = \rho V_1 A_1 \delta t$$

The amount of B flowing in to CV through A_1 over a short time δt:

$$\therefore \delta B_{\mathrm{I}} = \beta \delta m_{\mathrm{I}} = \beta \rho V_1 A_1 \delta t$$

Thus,

$$\lim_{\delta t \to 0} \frac{\delta B_I}{\delta t} = \beta \rho V_1 A_1 \equiv \dot{B}_{\mathrm{in}}$$

In a general form (see Appendix),

$$\dot{B}_{\mathrm{in}} = -\int_{CS_{\mathrm{in}}} \beta \rho \underline{V} \cdot \underline{n} dA$$

RTT for a Simple Fixed CV – Contd.

Consequently, the relationship between the time rate of change of B for the system and that for the CV is given by,

$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{\text{CV}} \beta\rho d\text{V} + \underbrace{\int_{CS_{\text{out}}} \beta\rho \underline{V}\cdot\hat{\boldsymbol{n}}dA}_{\dot{B}_{\text{out}}} - \underbrace{\left(-\int_{CS_{\text{in}}} \beta\rho \underline{V}\cdot\hat{\boldsymbol{n}}dA\right)}_{\dot{B}_{\text{in}}}$$

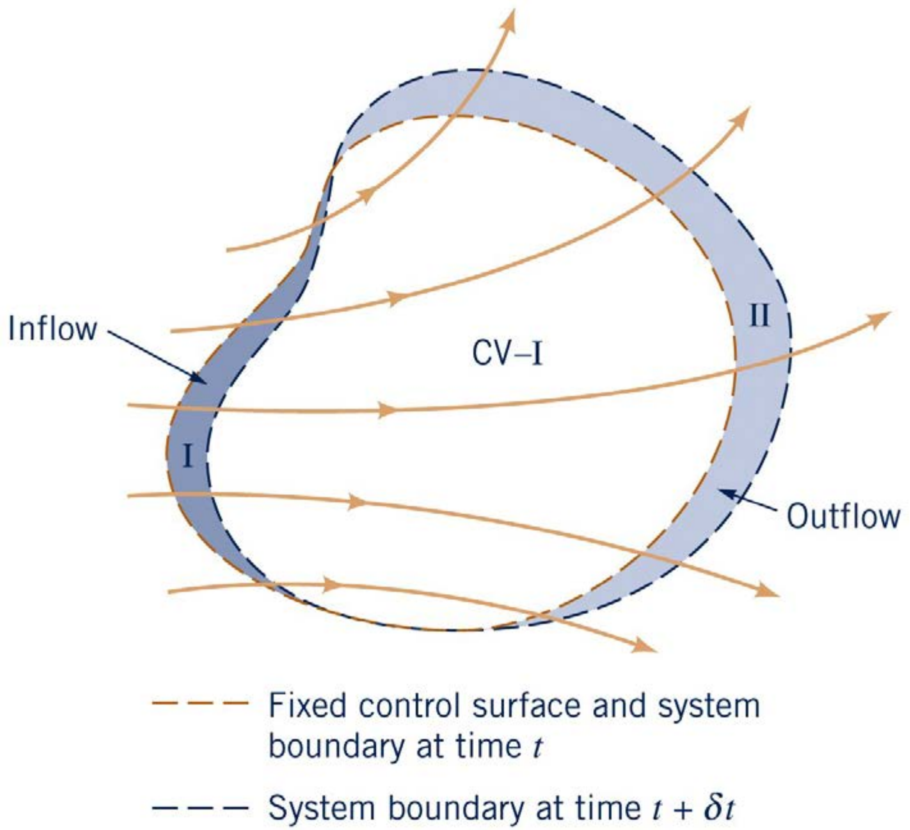
With the fact that $CS = CS_{\text{out}} + CS_{\text{in}}$,

$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{\text{CV}} \beta\rho dV + \int_{\text{CS}} \beta\rho \underline{V}\cdot\hat{\boldsymbol{n}}dA$$

Time rate of change of B within a system	=	Time rate of change of B within CV	+	Net flux of B through CS = $\dot{B}_{\text{out}} - \dot{B}_{\text{in}}$

Appendix: RTT for a Fixed CV

RTT for a Fixed CV



At time t: SYS = CV

$$B_{sys}(t) = B_{CV}(t)$$

At time $t + \delta t$: SYS = (CV – I) + II

$$B_{sys}(t + \delta t)$$
$$= B_{CV}(t + \delta t) - dB_I + dB_{II}$$

RTT for a Fixed CV – Contd.
- Time Rate of Change of B_{sys}

$$\frac{DB_{sys}}{Dt} = \lim_{\delta t \to 0} \frac{B_{sys}(t+\delta t) - B_{sys}(t)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\{B_{CV}(\underline{x}, t+\delta t) - \delta B_I + \delta B_{II}\} - B_{CV}(\underline{x}, t)}{\delta t}$$

$$\therefore \underbrace{\frac{DB_{sys}}{Dt}}_{\substack{\text{Time rate of} \\ \text{change of } B \\ \text{within the} \\ \text{system}}} = \underbrace{\lim_{\delta t \to 0} \frac{B_{CV}(\underline{x}, t+\delta t) - B_{CV}(\underline{x}, t)}{\delta t}}_{\substack{\text{1) Change of } B \\ \text{within CV over } \delta t}} + \underbrace{\lim_{\delta t \to 0} \frac{\delta B_{II}}{\delta t}}_{\substack{\text{2) Amount of } B \\ \text{flowing out} \\ \text{through CS} \\ \text{over } \delta t}} - \underbrace{\lim_{\delta t \to 0} \frac{\delta B_I}{\delta t}}_{\substack{\text{3) Amountt of } B \\ \text{flowing in} \\ \text{through CS} \\ \text{over } \delta t}} \quad \text{Eq. (1)}$$

RTT for a Fixed CV – Contd.

- The first term of RHS of Eq.(1)

$$\lim_{\delta t \to 0} \frac{B_{CV}\left(\underline{x}, t+\delta t\right) - B_{CV}\left(\underline{x}, t\right)}{\delta t} = \underbrace{\frac{\partial B_{CV}}{\partial t} = \frac{\partial}{\partial t}\int_{CV} \beta \rho dV}_{\substack{\text{Time rate of change of} \\ B \text{ within CV}}}$$

RTT for a Fixed CV – Contd.

- The 2nd term of RHS of Eq.(1)

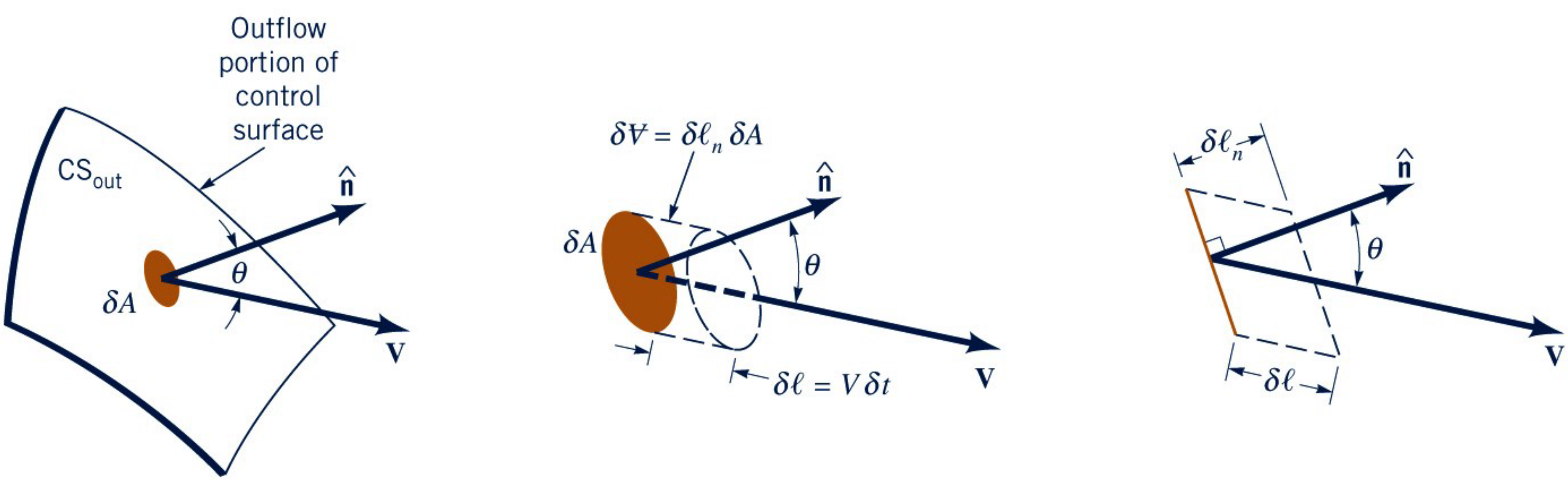
$$\delta m_{\text{out}} = \rho \delta \forall$$

and

$$\delta \forall = \delta A \cdot \delta \ell_n = \delta A \cdot \Big(\underbrace{\delta \ell}_{=V\delta t} \cos\theta \Big) = \delta A \cdot (V \delta t \cos\theta)$$

Thus, the amount of B flowing out of CV through δA over a short time δt:

$$\therefore \delta B_{\text{out}} = \beta \delta m_{\text{out}} = \beta \rho V \cos\theta \, \delta t \delta A$$



(a) (b) (c)

RTT for a Fixed CV – Contd.
- The 2nd term of RHS of Eq.(1) – Contd.

By integrating δB_{out} over the entire outflow portion of CS,

$$\delta B_{II} = \delta t \int_{CS_{\text{out}}} \beta\rho V \cos\theta \, dA$$

Thus,

$$\lim_{\delta t \to 0} \frac{\delta B_{II}}{\delta t} = \int_{CS_{\text{out}}} \beta\rho \underbrace{V \cos\theta}_{V_n} dA \equiv \dot{B}_{\text{out}}$$

i.e., Out flux of B through CS

Note that $V\cos\theta = \underline{V} \cdot \hat{\boldsymbol{n}}$,

$$\therefore \dot{B}_{\text{out}} = \int_{CS_{\text{out}}} \beta\rho \underline{V} \cdot \hat{\boldsymbol{n}} dA$$

RTT for a Fixed CV – Contd.

- The 3rd term of RHS of Eq.(1)

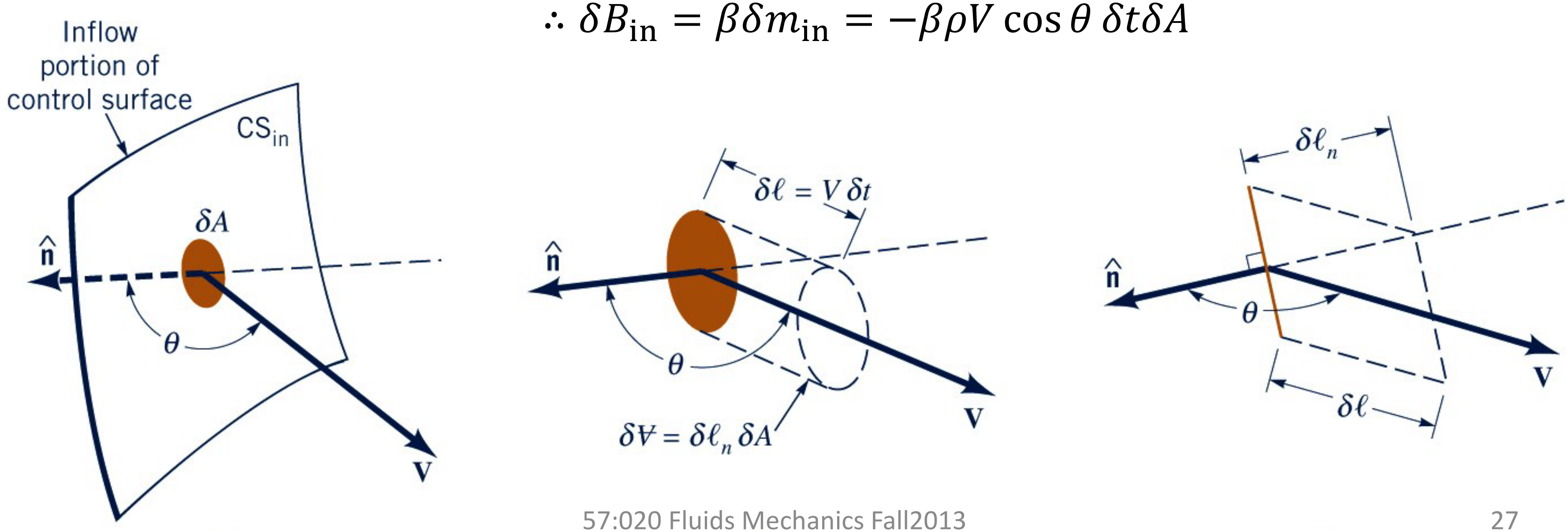
$$\delta m_{in} = \rho \delta \mathcal{V}$$

and

$$\delta \mathcal{V} = \delta A \cdot \delta \ell_n = \delta A \cdot \left(\underbrace{\delta \ell}_{=V\delta t} \left(- \underbrace{\cos\theta}_{<0} \right) \right) = \delta A \cdot (-V \delta t \cos\theta)$$

Thus, the amount of B flowing out of CV through δA over a short time δt:

$$\therefore \delta B_{\text{in}} = \beta \delta m_{\text{in}} = -\beta \rho V \cos\theta \, \delta t \delta A$$



(a) (b) (c)

RTT for a Fixed CV – Contd.
- The 3rd term of RHS of Eq.(1) – Contd.

By integrating δB_{out} over the entire outflow portion of CS,

$$\delta B_I = -\delta t \int_{CS_{\text{in}}} (\beta \rho V \cos\theta) dA$$

Thus,

$$\lim_{\delta t \to 0} \frac{\delta B_I}{\delta t} = -\int_{CS_{\text{in}}} (\beta \rho V \cos\theta) dA \equiv \dot{B}_{\text{in}}$$

i.e., influx of B through CS

Note that $V \cos\theta = \underline{V} \cdot \hat{\boldsymbol{n}}$,

$$\therefore \dot{B}_{\text{in}} = -\int_{CS_{\text{in}}} \beta \rho \underline{V} \cdot \hat{\boldsymbol{n}} dA$$

RTT for a Simple Fixed CV – Contd.

Consequently, the relationship between the time rate of change of B for the system and that for the CV is given by,

$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{\text{CV}} \beta\rho d\text{V} + \underbrace{\int_{CS_{\text{out}}} \beta\rho \underline{V}\cdot\hat{\boldsymbol{n}}dA}_{\dot{B}_{\text{out}}} - \underbrace{\left(-\int_{CS_{\text{in}}} \beta\rho \underline{V}\cdot\hat{\boldsymbol{n}}dA\right)}_{\dot{B}_{\text{in}}}$$

With the fact that $CS = CS_{\text{out}} + CS_{\text{in}}$,

$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{\text{CV}} \beta\rho dV + \int_{\text{CS}} \beta\rho \underline{V}\cdot\hat{\boldsymbol{n}}dA$$

Time rate of change of B within a system = Time rate of change of B within CV + Net flux of B through CS = $\dot{B}_{\text{out}} - \dot{B}_{\text{in}}$