Ex) An incompressible plane flow has the velocity components $u = 2y$, $v = 8x$, $w = 0$. (a) Find the acceleration components. (b) Find the pressure distribution $p(x, y)$ if the pressure at the origin is p_0. Assume frictionless flow.

Solution:

(a) Acceleration

$$\underline{a} = \underbrace{\frac{\partial \underline{V}}{\partial t}}_{\substack{local \\ acceleration}} + \underbrace{\nabla \cdot \nabla \underline{V}}_{\substack{convective \\ acceleration}}$$

where

$$\underline{V} = u\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\jmath}}$$

or

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$$

With $u = 2y$ and $v = 8x$,

$$a_x = 0 + (2y)(0) + (8x)(2) = 16x$$

$$a_y = 0 + (2y)(8) + (8x)(0) = 16y$$

$$\therefore \underline{a} = 16x\hat{\boldsymbol{\imath}} + 16y\hat{\boldsymbol{\jmath}}$$

Note:

$$|\underline{a}| = \sqrt{a_x^2 + a_y^2} = \sqrt{(16x)^2 + (16y)^2}$$

$$\therefore a = 16\sqrt{x^2 + y^2}$$

(b) Euler equation

$$\rho \underline{a} = -\nabla p$$

where

$$\nabla p = \frac{\partial p}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial p}{\partial y}\hat{\boldsymbol{\jmath}}$$

Thus,

$$\frac{\partial p}{\partial x} = -\rho a_x = -16\rho x$$

$$\frac{\partial p}{\partial y} = -\rho a_y = -16\rho y$$

Integrate $\partial p/\partial x$,

$$p = \int \frac{\partial p}{\partial x} dx = \int (-16\rho x) dx = -8\rho x^2 + f(y)$$

then differentiate,

$$\frac{\partial p}{\partial y} = 0 + \frac{\partial f}{\partial y} = -16\rho y$$

$$\frac{\partial f}{\partial y} = -16\rho y$$

Or

$$f(y) = \int (-16\rho y) dy = -8\rho y^2 + C$$

Thus,

$$p = -8\rho x^2 - 8\rho y^2 + C$$

$p = p_0$ at (x,y) = (0,0),

$$p|_{x=0,y=0} = 0 + 0 + C = p_0$$

or

$$C = p_0$$

Finally,

$$\therefore p = p_0 - 8\rho(x^2 + y^2)$$

Note: Pressure along streamline

Streamline

$$\frac{dx}{u} = \frac{dy}{v}$$

or

$$\frac{dx}{2y} = \frac{dy}{8x}$$

Integrate,

$$\int 2y dy = \int 8x dx$$

or

$$y^2 = 4x^2 + C$$

For streamline that goes through the origin $x = y = 0$, $C = 0$

$$\therefore y^2 = 4x^2$$

From the Euler equation

$$p = p_0 - 8\rho(x^2 + 4x^2)$$

Thus,

$$\therefore p = p_0 - 40\rho x^2$$

Alternate approach:

Bernoulli equation along the streamline

$$p + \frac{1}{2}\rho V^2 = p_0 + \frac{1}{2}\rho V_0^2$$

where

$$V^2 = u^2 + v^2 = (2y)^2 + (8x)^2 = 4(16x^2 + y^2) = 4(16x^2 + 4x^2) = 80x^2$$

and

$$V_0 = 0$$

at $x = y = 0$. Thus,

$$p = p_0 - \frac{1}{2}\rho V^2 = p_0 - \frac{1}{2}\rho(80x^2)$$

$$\therefore p = p_0 - 40\rho x^2$$