(5) Vorticity Theorems

The incompressible flow momentum equations focus attention on $\underline{V}$ and p and explain the flow pattern in terms of inertia, pressure, gravity, and viscous forces. Alternatively, one can focus attention on $\underline{\omega}$ and explain the flow pattern in terms of the rate of change, deforming, and diffusion of $\underline{\omega}$ by way of the vorticity equation. As will be shown, the existence of $\underline{\omega}$ generally indicates the viscous effects are important since fluid particles can only be set into rotation by viscous forces. Thus, the importance of this topic (for potential flow) is to demonstrate that under most circumstances, an inviscid flow can also be considered irrotational.

1. Vorticity Kinematics

$$\underline{\omega} = \nabla \times \underline{V} = (w_y - v_z)\hat{i} + (u_z - w_x)\hat{j} + (v_x - u_y)\hat{k}$$

$$\omega_i = \varepsilon_{ijk}\frac{\partial u_j}{\partial x_k} = \left(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k}\right)$$

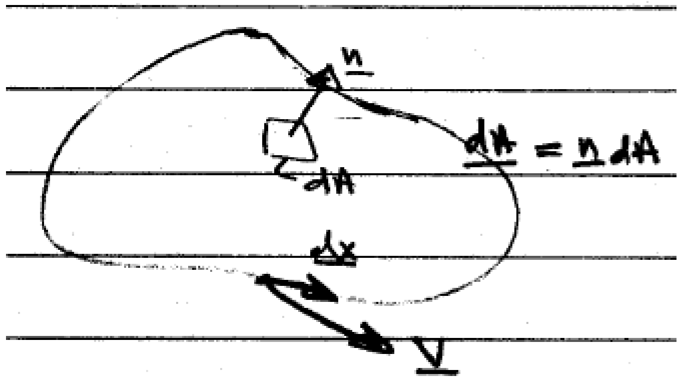
$$\underbrace{\begin{aligned}&\varepsilon_{123} = \varepsilon_{321} = \varepsilon_{231} = 1\\ &\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1\\ &\varepsilon_{ijk} = 0 \ otherwise\end{aligned}}_{alternating\ \ tensor}$$

= 2 $\times$ *the angular velocity of the fluid element*

(i, j, k cyclic)

A quantity intimately tied with vorticity is the circulation:

$$\Gamma = \oint \underline{V} \cdot \underline{dx}$$



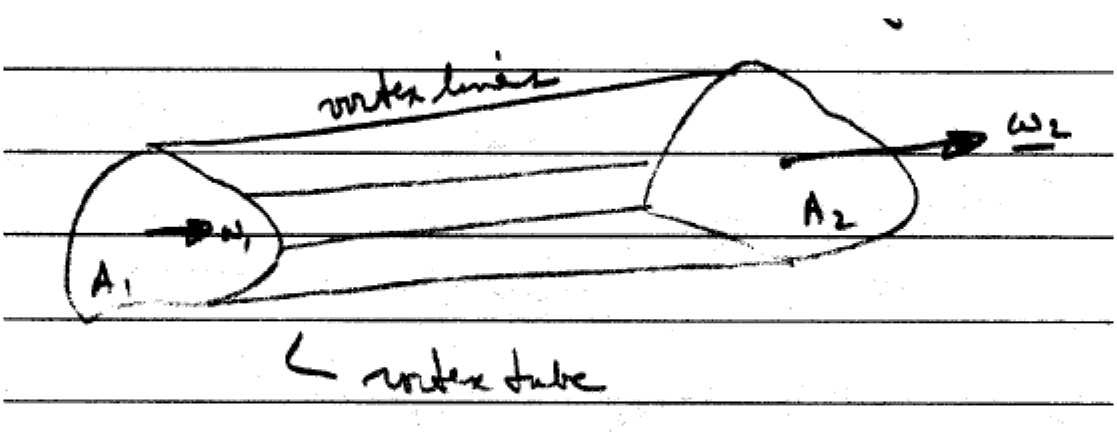
Stokes Theorem:

$$\oint \underline{a} \cdot \underline{dx} = \int_A \nabla \times \underline{a} \cdot \underline{dA}$$

$$\therefore \Gamma = \oint \underline{V} \cdot \underline{dx} = \int_A \nabla \times \underline{V} \cdot \underline{dA} = \int_A \underline{\omega} \cdot \underline{n} dA$$

Which shows that if $\underline{\omega}$ =0 (i.e., if the flow is irrotational, then Γ = 0 also.

Vortex line = lines which are everywhere tangent to the vorticity vector.



Next, we shall see that vorticity and vortex lines must obey certain properties known as the Helmholtz vorticity theorems, which have great physical significance.

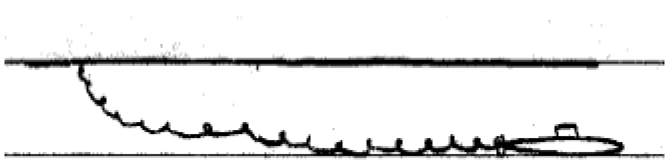
The first is the result of its very definition:

$$\underline{\omega} = \nabla \times \underline{V}$$

$$\nabla \cdot \underline{\omega} = \nabla \cdot (\nabla \times \underline{V}) = 0 \qquad \text{Vector identity}$$

i.e. the vorticity is divergence-free, which means that there can be no sources or sinks of vorticity within the fluid itself.

Helmholtz Theorem #1: a vortex line cannot end in the fluid. It must form a closed path (smoke ring), end at a boundary, solid or free surface, or go to infinity.



Propeller vortex is known to drift up towards the free surface

The second follows from the first and using the divergence theorem:

$$\int_{\forall} \nabla \cdot \underline{\omega}\, d\forall = \int_{A} \underline{\omega} \cdot \underline{n}\, dA = 0$$

Application to a vortex tube results in the following

$$\underbrace{\int_{A1} \underline{\omega} \cdot \underline{n}\, dA}_{-\Gamma_1} + \underbrace{\int_{A2} \underline{\omega} \cdot \underline{n}\, dA}_{\Gamma_2} = 0$$

Minus sign due to outward normal → $-\Gamma_1$

Or $\Gamma_1 = \Gamma_2$

<u>Helmholtz Theorem #2</u>:

The circulation around a given vortex line (i.e., the strength of the vortex tube) is constant along its length.

This result can be put in the form of a simple one-dimensional incompressible continuity equation. Define ω_1 and ω_2 as the average vorticity across A_1 and A_2, respectively

$$\omega_1 A_1 = \omega_2 A_2$$

which relates the vorticity strength to the cross sectional area changes of the tube.

2. <u>Vortex dynamics</u>

Consider the substantial derivative of the circulation assuming incompressible flow and conservative body forces

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt} \oint \underline{V} \cdot \underline{dx}$$

$$= \oint \frac{D\underline{V}}{Dt} \cdot dx + \oint \underline{V} \cdot \frac{D}{Dt} \underline{dx}$$

From the N-S equations we have

$$\frac{D\underline{V}}{Dt} = \frac{1}{\rho}\underline{f} - \frac{\nabla p}{\rho} + \nu\nabla^2\underline{V}$$
$$= -\nabla\left(F + {p}/{\rho}\right) + \nu\nabla^2\underline{V}$$

Define $\underline{f} = -\nabla F$ for the gravitational body force F=ρgz.

Also, $\frac{D}{Dt}d\underline{x} = d\frac{D\underline{x}}{Dt} = d\underline{V}$

$$\frac{D\Gamma}{Dt} = \underbrace{\oint\left[-\nabla\left(F + p/\rho\right)\right]\cdot d\underline{x}}_{-\oint dF - \oint \frac{dp}{\rho}} + \oint\left[\nu\nabla^2\underline{V}\right]\cdot d\underline{x} + \underbrace{\oint \underline{V}\cdot d\underline{V}}_{\frac{1}{2}\oint d(\underline{V}\cdot\underline{V})}$$

$$= \underbrace{\oint\left[-dF - \frac{dp}{\rho} + \frac{1}{2}dV^2\right]} + \nu\oint\nabla^2\underline{V}\cdot d\underline{x}$$

=0 since integration is around a closed contour and F,p, & V are single valued!

$$\frac{D\Gamma}{Dt} = \nu\oint\nabla^2\underline{V}\cdot dx = -\nu\oint\nabla\times\underline{\omega}\cdot dx$$

$$\nabla\times\underbrace{\left(\nabla\times\underline{V}\right)}_{\underline{\omega}} = \nabla\underbrace{\left(\nabla\cdot\underline{V}\right)}_{=0} - \nabla^2\underline{V}$$

Implication: The circulation around a material loop of particles changes only if the net viscous force on those particles gives a nonzero integral.

If $\nu = 0$ or $\underline{\omega} = 0$ (i.e., inviscid or irrotational flow, respectively) then

$$\frac{D\Gamma}{Dt} = 0$$

The circulation of a material loop never changes

Kelvins Circulation Theorem: for an ideal fluid (i.e. inviscid, incompressible, and irrotational) acted upon by conservative forces (e.g., gravity) the circulation is constant about any closed material contour moving with the fluid, which leads to:

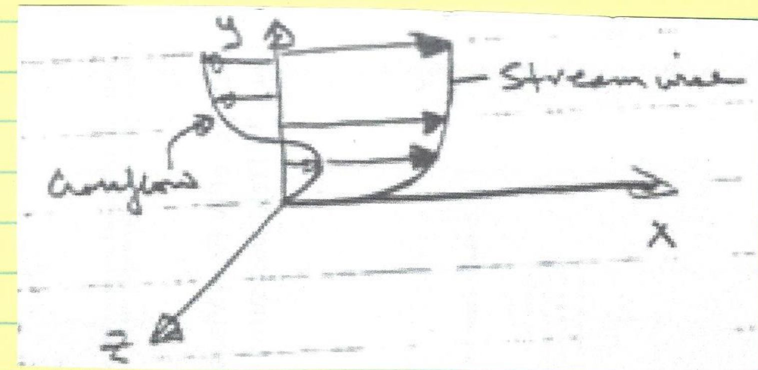
Helmholtz Theorem #3: No fluid particle can have rotation if it did not originally rotate. Or, equivalently, in the absence of rotational forces, a fluid that is initially irrotational remains irrotational. In general, we can conclude that vortices are preserved as time passes. Only through the action of viscosity can they decay or disappear.

Kelvins Circulation Theorem and Helmholtz Theorem #3 are very important in the study of inviscid flow. The important conclusion is reached that a fluid that is initially irrotational remains irrotational, which is the justification for ideal-flow theory.

Production of Vorticity at Walls

A solid wall produces vorticity that emanates into the fluid due to the no slip condition / shear stress

1) Limiting ψ and ω lines on a solid wall



$u_x + v_y + w_z = 0 \Big|_{y=0}$ Assume flat wall but also valid curved surface

$v_y \Big|_{y=0} = 0$

Taylor series y direction

$u = 0 + u_{y_0} y + \cdots$

$v = 0 + 0 + v_{yy_0} y^2/2 + \cdots$

$w = 0 + w_{y_0} y + \cdots$

$$dz/dx \Big|_{\psi_0} = \tan\theta = \lim_{y \to 0} \frac{w}{u} = \frac{w_{y_0}}{u_{y_0}}$$

θ = angle ψ_0 with x axis

since v/u and $v/w|_0 = 0$ ψ lies on surface $y = 0$

Since $\nabla \cdot \underline{V} = 0$ γ cannot end in fluid
ie no sources or sinks $\underline{V}$ in fluid ie
γ cannot end in the fluid: it must
form closed path, end at a boundary
(solid or interface) or go to infinity.
However, γ can emanate from surface
at singular points such stagnation
point

Singular Points

Singular points in the pattern of skin-friction lines occur at isolated points on the surface where the skin friction (τ_{w_1}, τ_{w_2}) in Equation (3), or alternatively the surface vorticity (ω_1, ω_2) in Equation (4), becomes identically zero. Singular points are classifiable into two main types: nodes and saddle points. Nodes may be further subdivided into two subclasses: nodal points and foci (of attachment or separation).

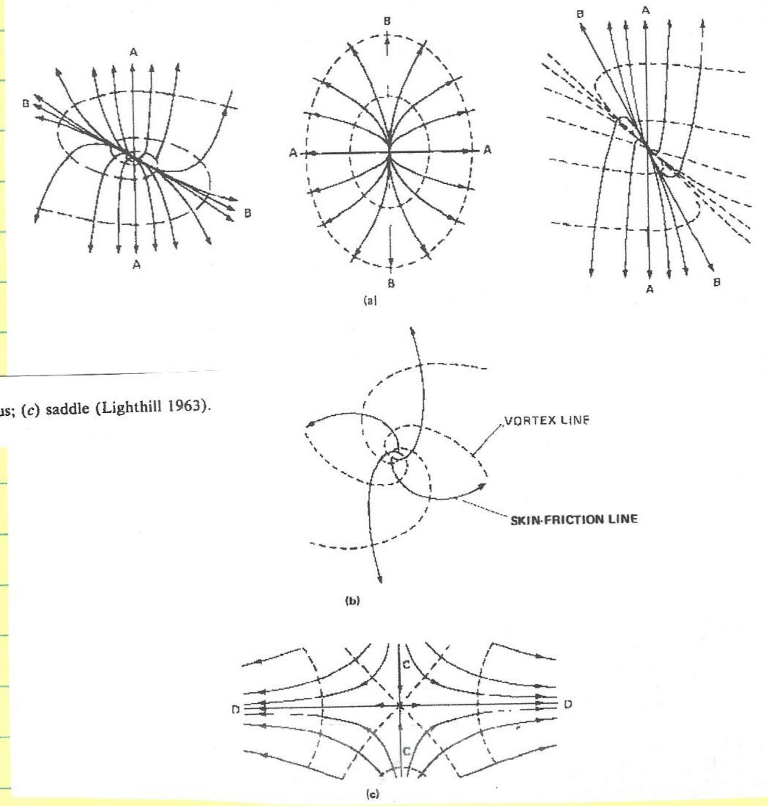


Figure 1 Singular points: (*a*) node; (*b*) focus; (*c*) saddle (Lighthill 1963).

Next consider $\underline{\omega}$ components on $y=0$

$$\omega_x = w_y - v_z = w_{y_0}$$

Since $w_y = 0$ $\underline{\omega}$ lies in wall

$$\omega_y = u_z - w_x = 0$$

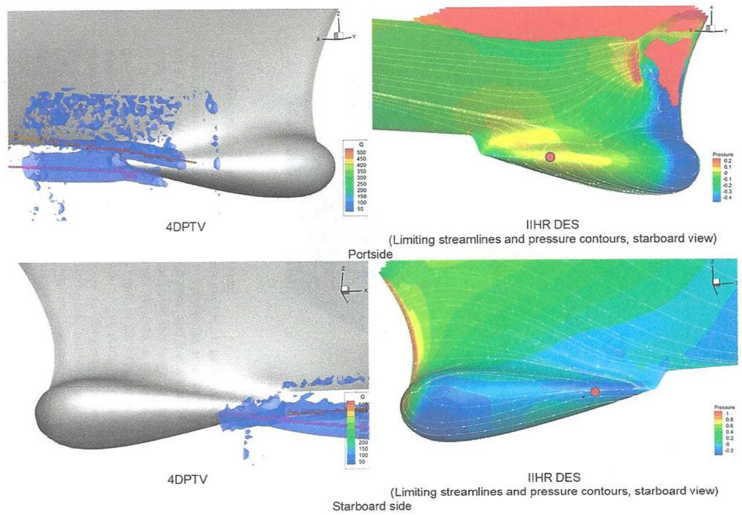
$$\omega_z = v_x - u_y = -u_{y_0}$$

$\left.\right\} y=0$

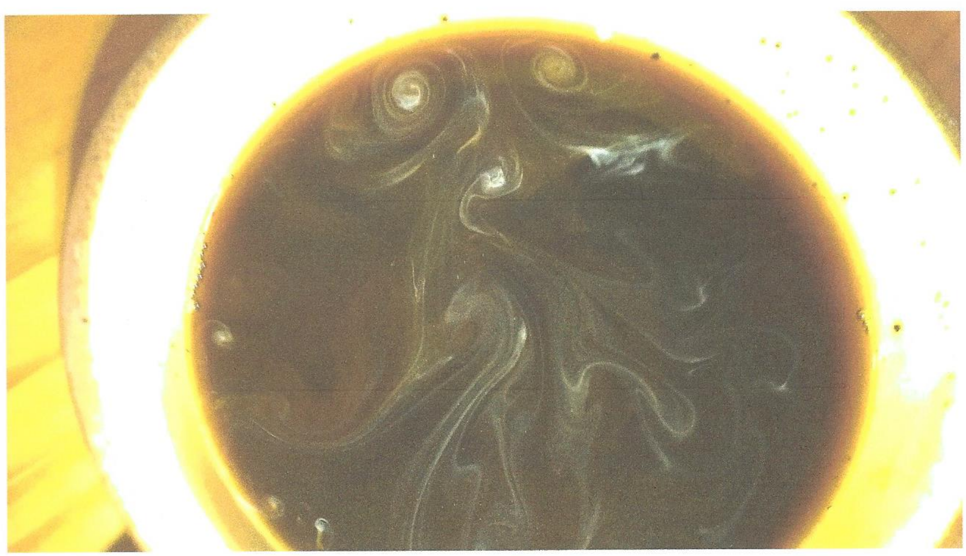
$$\left.\frac{dz}{dx}\right|_{\omega \text{ lines}} = \omega_z / \omega_x = \frac{-u_{y_0}}{w_{y_0}} = -1 \big/ \left.dz/dx\right|_{\tau}$$

vortex lines are perpendicular τ on $y=0$ but not necessarily for $y \neq 0$ ie in fluid volume

Since $\nabla \cdot \underline{\omega} = 0$ ω cannot end in the fluid with some conclusions reached for τ



Local Flow 4DPTV Measurement System in IIHR Towing Tank & CFDShip-Iowa DES

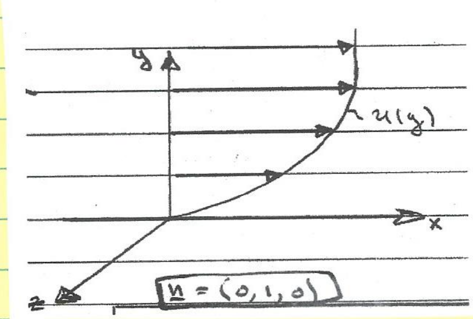


Vortices formed by milk when poured into a cup of coffee

2) Relationship ω and τ_w

$y=0 \quad u_x = 0, \; v_y = 0 \;$ and $\; w_z = 0$

Consider a 1-D flow near a wall



or x-axis along $x|_{\theta=0}$ such that $\omega = 0$ and $\omega_{y_0} = 0$

The viscous stresses are given by:

$\tau_{ij} n_j$ where $\tau_{ij} = \mu \varepsilon_{ij}$

$$\tau_{11} n_1 + \tau_{12} n_2 + \tau_{13} n_3 = \tau_x$$

$$\tau_{21} n_1 + \tau_{22} n_2 + \tau_{23} n_3 = \tau_y$$

$$\tau_{31} n_1 + \tau_{32} n_2 + \tau_{33} n_3 = \tau_z$$

$$\tau_{12} = \mu \varepsilon_{12} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \mu \frac{\partial u}{\partial y}$$

$$\tau_{22} = \mu \varepsilon_{22} = 2\mu \frac{\partial v}{\partial y} = 0$$

$$\tau_{32} = \mu \varepsilon_{32} = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) = 0$$

Which shows that

$$\tau_x = \mu \frac{\partial u}{dy} \qquad \tau_y = \tau_z = 0$$

However from the definition vorticity we also see that

$$\tau_x = \mu \frac{\partial u}{\partial y} = -\mu\omega_z$$

more generally,

$$\underline{F}_{viscous} = \tau_{ij} n_j$$

$$= -\mu\, \underline{n} \times \underline{\omega}$$

$$\tau_x = -\mu\omega_z = \mu u_{y_0}$$

$$\tau_z = \mu\omega_x = \mu w_{y_0}$$

ijkijk

$$\underline{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} \qquad \underline{n} \times \underline{\omega} = -\omega_x \hat{k} + \omega_z \hat{i}$$

$$\underline{n} = n_2 \hat{j}$$

The wall shear stress and vorticity are directly related.

Once vorticity is generated, its subsequent behavior is governed by the vorticity equation.

3) Vorticity Flux at a Solid Wall

In analogy heat flux, the vorticity flux vector is

$$\sigma_i = -n_j \frac{\partial \omega_i}{\partial x_j}$$

flux of $\underline{\omega}$ across plane with normal $\underline{n} = \hat{j}$

$$\underline{\sigma} = -\underline{n} \cdot \nabla \underline{\omega}$$

$\sigma_x = -\frac{\partial \omega_x}{\partial y}$, $\sigma_y = -\frac{\partial \omega_y}{\partial y}$, $\sigma_z = -\frac{\partial \omega_z}{\partial y}$

σ_x & σ_z can be related to p_x & p_z by evaluating the momentum equation on $y=0$

$$\rho\left[\frac{\partial \underline{V}}{\partial t} + \frac{1}{2}\nabla(\underline{V}\cdot\underline{V}) - \underline{V}\times\underline{\omega}\right] = -\nabla p - \mu \nabla \times \underline{\omega} \quad \text{neglect } g$$

$y=0$ $\quad \nabla p = -\mu \nabla \times \underline{\omega}$ $\quad$ since $\underline{V}(0) = 0$

pressure gradient along surface drives σ_z & σ_x along surface

$$\frac{\partial p}{\partial x} = -\mu \frac{\partial \omega_z}{\partial y} = \mu \sigma_z$$

$$\frac{\partial p}{\partial z} = \mu \frac{\partial \omega_x}{\partial y} = -\mu \sigma_x$$

pressure gradient $\perp$ surface related $\underline{\omega}(0)$ gradients along wall

$$\frac{\partial p}{\partial y} = \mu\left(\frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x}\right)$$

$\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\times$ $\quad \hat{i}\hat{j}\hat{k}\hat{i}\hat{j}\hat{k}$

$(\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k})$

$\frac{\partial \omega_y}{\partial x}\hat{k} - \frac{\partial \omega_z}{\partial x}\hat{j} - \frac{\partial \omega_x}{\partial y}\hat{k} + \frac{\partial \omega_z}{\partial y}\hat{i}$

$+ \frac{\partial \omega_x}{\partial z}\hat{j} - \frac{\partial \omega_y}{\partial z}\hat{i}$

$\left(\frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z}\right)\hat{i} + \left(\frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x}\right)\hat{j}$

$+ \left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y}\right)\hat{k}$ on $y=0$

using $\nabla \cdot \underline{\omega} = 0$

$$\sigma_y = -\frac{\partial \omega_y}{\partial y} = \frac{\partial \omega_x}{\partial x} + \frac{\partial \omega_z}{\partial z}$$

since $\omega_y = 0$

$\omega_y = 0$ but flux ω_y out of wall depends on ω_x & ω_z gradients along the wall

ROLE OF FREE-SURFACE BOUNDARY CONDITIONS AND NONLINEARITIES IN WAVE/BOUNDARY-LAYER AND WAKE INTERACTION

by

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A thesis submitted in partial fulfillment
of the requirements for the Doctor of
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Thesis supervisor: Associate Professor Frederick Stern

3.4 Vortex/Free-Surface Interaction

Vorticity can either be distributed as in a shear flow or concentrated as in elemental vortices (i.e., vortex rings and pairs and a wing-tip vortex). Vortex/free-surface interaction refers to investigations of interactions of elemental vortices with a free surface. The interactions are primarily controlled by Re, Fr, Weber number We (= $\rho U^2 L/T$, where L is the vortex-ring radius R or the spacing of the vortex pair D, and T is surface tension), and contamination number W (= $\frac{\Delta T\, R}{\mu \Gamma}$, where ΔT is the difference in surface tension between a clean and contaminated surface and Γ is the circulation of the vortex ring or pair) values. Complex interactions occur involving interrelated free-surface deformation, secondary-vorticity generation, and vorticity disconnection/reconnection. The free-surface deformations include gravity and capillary waves and scars, striation, and whirls. Secondary-vorticity generation is related both to the free-surface deformation and the vortex disconnection/reconnection process, i.e., segments of vorticity lines move toward and merge with the free surface leaving the open ends of the remaining vortex lines terminating at the free surface.

First, some basic aspects of vorticity production, flux, and transport will be discussed (e.g., Panton, 1984). Vorticity can not be generated (or destroyed) in the interior of a homogeneous fluid under normal conditions since by vector identity $\nabla \cdot \omega = 0$, which implies that there can be no sources or sinks of vorticity within the fluid. However, vorticity can be generated (i.e., produced/fluxed) at boundaries. Production/flux refers to specified values and gradients of vorticity, respectively, at the boundary. Vorticity is produced at a solid-wall boundary due to the no-slip condition where the wall vorticity is related to the wall-shear stress by

$$\mathbf{n} \cdot \tau_w = -\frac{1}{Re} \mathbf{n} \times \omega \qquad (3.4)$$

In analogy to heat flux, the vorticity flux **q** is defined as

$$q_i = - n_j \frac{\partial \omega_i}{\partial x_j} \qquad (3.5)$$

where q_i means the flux of i vorticity across a boundary with normal n_j. Positive and negative values of q_i correspond to vorticity sinks and sources, respectively. (3.5) can be equivilently expressed through vector idetity (Rood, 1993a,b) by

$$\begin{aligned} \mathbf{q} &= -\mathbf{n} \cdot \nabla \omega \\ &= \mathbf{n} \times \nabla \times \omega - (\nabla \omega) \cdot \mathbf{n} \end{aligned} \qquad (3.6)$$

The terms on the right-hand side of (3.6) can be further expanded through the use of the NS equation [i.e., (4.2) for laminar flow] and vector identity, respectively, to read

$$\mathbf{n} \times \nabla \times \omega = \mathbf{n} \times [- Re (\mathbf{a} + \nabla p)] \qquad (3.7)$$

$$(\nabla\omega) \cdot \mathbf{n} = \nabla(\omega \cdot \mathbf{n}) - \omega \cdot \nabla\mathbf{n} \qquad (3.8)$$

where $\mathbf{a} = \frac{\partial U_i}{\partial t} + \sum_{j=1}^{3} U_j \frac{\partial U_i}{\partial x_j}$ is the fluid acceleration and ∇p is the piezometric pressure gradient. (3.7) is a vector tangent to the free surface with magnitude proportional to the sum of the fluid acceleration and piezometric pressure gradient. (3.8) is the sum of the gradient of the normal component of vorticity and dot product of ω and $\nabla\mathbf{n}$, which is related to the surface curvature. Thus, the physical mechanism for $\mathbf{q}$ is a combination of acceleration, piezometric pressure gradient, gradient of normal component of vorticity, and dot product of ω and $\nabla\mathbf{n}$. Vorticity flux at solid-wall boundaries due to pressure gradients and acceleration were discussed by Lighthill (1963) and Morton (1984), respectively. Vorticity flux at a free surface has been discussed by Batchelor (1967), Lugt (1987), and Rood (1993a,b). Once generated, vorticity is governed by the vorticity-transport equation

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)V + \frac{1}{Re}\nabla^2\omega \qquad (3.8)$$

$\frac{D\omega}{Dt}$ represents the temporal and convective rate of change of ω. $(\omega \cdot \nabla)V$ represents the change in magnitude and redistribution from one component to another of ω by stretching and turning, respectively. $\frac{1}{Re}\nabla^2\omega$ represents the net rate of viscous diffusion of ω. For two-dimensional flow, $(\omega \cdot \nabla)V$ is zero. For three-dimensional flow, complex interactions occur due to stretching and turning. Note that (3.8) does not contain either pressure or vorticity generation terms.

Once vorticity is generated, its subsequent behavior is governed by the vorticity equation.

N-S $$\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} = -\nabla \left(p / \rho \right) + \nu \nabla^2 \underline{V}$$ *neglect* $\underline{f}$

Or $$\frac{\partial \underline{V}}{\partial t} + \nabla \left(\frac{1}{2} \underline{V} \cdot \underline{V} \right) - \underline{V} \times \underline{\omega} = -\nabla \left(p / \rho \right) + \nu \nabla^2 \underline{V}$$

The vorticity equation is obtained by taking the curl of this equation. (Note $\nabla \times (\nabla \theta) = 0$).

$$\underbrace{\frac{\partial \underline{\omega}}{\partial t}}_{\text{Rate of change of } \underline{\omega}} - \underbrace{\nabla \times \left(\underline{V} \times \underline{\omega} \right)} = \nu \nabla^2 \underline{\omega}$$

Rate of change of $\underline{\omega}$ =

$$= \underline{V}(\nabla \cdot \underline{\omega}) - \underline{\omega}(\nabla \cdot \underline{V}) - (\underline{V} \cdot \nabla)\underline{\omega} + (\underline{\omega} \cdot \nabla)\underline{V}$$

Therefore, the transport Eq. for $\underline{\omega}$ is

$$\underbrace{\frac{\partial \underline{\omega}}{\partial t} + (\underline{V} \cdot \nabla)\underline{\omega}}_{\frac{D\underline{\omega}}{Dt}} = \left(\underline{\omega} \cdot \nabla \right) \underline{V} + \nu \nabla^2 \underline{\omega}$$

Rate of deforming vortex lines

Rate of viscous diffusion of $\underline{\omega}$

$$\frac{\partial \underline{\omega}}{\partial t} + \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \underline{\omega} = \left(\omega_x \frac{\partial}{\partial x} + \omega_y \frac{\partial}{\partial y} + \omega_z \frac{\partial}{\partial z} \right) \underline{V} + \nu \nabla^2 \underline{\omega}$$

$$\frac{\partial \omega_x}{\partial t} + u \frac{\partial \omega_x}{\partial x} + v \frac{\partial \omega_x}{\partial y} + w \frac{\partial \omega_x}{\partial z} = \underbrace{\omega_x \frac{\partial u}{\partial x}}_{\text{Stretching}} + \underbrace{\omega_y \frac{\partial u}{\partial y} + \omega_z \frac{\partial u}{\partial z}}_{\text{turning}} + \nu \nabla^2 \omega_x$$

$$\frac{\partial \omega_y}{\partial t} + u\frac{\partial \omega_y}{\partial x} + v\frac{\partial \omega_y}{\partial y} + w\frac{\partial \omega_y}{\partial z} = \omega_x \frac{\partial v}{\partial x} + \omega_y \frac{\partial v}{\partial y} + \omega_z \frac{\partial v}{\partial z} + \nu\nabla^2\omega_y$$

$$\frac{\partial \omega_z}{\partial t} + u\frac{\partial \omega_z}{\partial x} + v\frac{\partial \omega_z}{\partial y} + w\frac{\partial \omega_z}{\partial z} = \omega_x \frac{\partial w}{\partial x} + \omega_y \frac{\partial w}{\partial y} + \omega_z \frac{\partial w}{\partial z} + \nu\nabla^2\omega_z$$

Note: (1) Equation does not involve p explicitly

(2) for 2-D flow $(\underline{\omega}\cdot\nabla)\underline{V} = 0$ since $\underline{\omega}$ is perp. to $\underline{V}$ and there can be no deformation of $\underline{\omega}$, ie

$$\frac{D\underline{\omega}}{Dt} = \nu\nabla^2\underline{\omega}$$

To determine the pressure field in terms of the vorticity, the divergence of the N-S equation is taken.

$$\nabla\cdot\left[\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V} = -\nabla\left(p/\rho\right) + \nu\nabla^2\underline{V}\right]$$

$$\nabla^2(p/\rho) = -\nabla\cdot\left[\underline{V}\cdot\nabla\underline{V}\right]$$ *Poisson Eq. for p*

$$= -\frac{1}{2}\nabla^2\left(\underline{V}\cdot\underline{V}\right) + \underline{V}\cdot\nabla^2\underline{V} + \underline{\omega}\cdot\underline{\omega}$$

does not depend explicitly on ν

Derivation of pressure Poisson equation:

Three vector identities to be used:

(1) $\mathbf{V}\cdot\nabla\mathbf{V} = \frac{1}{2}\nabla(\mathbf{V}\cdot\mathbf{V}) - \mathbf{V}\times(\nabla\times\mathbf{V})$

(2) $\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$

(3) $\nabla\times(\nabla\times\mathbf{a}) = -\nabla^2\mathbf{a} + \nabla(\nabla\cdot\mathbf{a})$

Pressure Poisson equation in vector form:

$$
\begin{aligned}
\nabla^2\left(\frac{p}{\rho}\right) &= -\nabla\cdot(\mathbf{V}\cdot\nabla\mathbf{V}) \\
&= -\nabla\cdot\left(\frac{1}{2}\nabla(\mathbf{V}\cdot\mathbf{V}) - \mathbf{V}\times(\nabla\times\mathbf{V})\right) \\
&= -\frac{1}{2}\nabla^2(\mathbf{V}\cdot\mathbf{V}) + \nabla\cdot(\mathbf{V}\times\boldsymbol{\omega}) \\
&= -\frac{1}{2}\nabla^2(\mathbf{V}\cdot\mathbf{V}) + \boldsymbol{\omega}\cdot(\nabla\times\mathbf{V}) - \mathbf{V}\cdot(\nabla\times\boldsymbol{\omega}) \\
&= -\frac{1}{2}\nabla^2(\mathbf{V}\cdot\mathbf{V}) + \boldsymbol{\omega}\cdot\boldsymbol{\omega} - \mathbf{V}\cdot(\nabla\times(\nabla\times\mathbf{V})) \\
&= -\frac{1}{2}\nabla^2(\mathbf{V}\cdot\mathbf{V}) + \boldsymbol{\omega}\cdot\boldsymbol{\omega} - \mathbf{V}\cdot\left[-\nabla^2\mathbf{V} + \nabla\left(\cancel{\nabla\cdot\mathbf{V}}\right)\right] \\
&= -\frac{1}{2}\nabla^2(\mathbf{V}\cdot\mathbf{V}) + \mathbf{V}\cdot\nabla^2\mathbf{V} + \boldsymbol{\omega}\cdot\boldsymbol{\omega}
\end{aligned}
$$

Pressure Poisson equation in tensor form:

$$\nabla^2\left(\frac{p}{\rho}\right) = -\frac{1}{2}\nabla^2(\mathbf{V}\cdot\mathbf{V}) + \mathbf{V}\cdot\nabla^2\mathbf{V} + \boldsymbol{\omega}\cdot\boldsymbol{\omega}$$

$$= -\frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_i}\left[\left(u_j e_j\right)\cdot\left(u_k e_k\right)\right] + \left(u_i e_i\right)\cdot\frac{\partial^2\left(u_k e_k\right)}{\partial x_j \partial x_j} + \left(\nabla\times\mathbf{V}\right)\cdot\left(\nabla\times\mathbf{V}\right)$$

$$= -\frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_i}\left(u_j u_k \delta_{jk}\right) + u_i \delta_{ik}\cdot\frac{\partial^2 u_k}{\partial x_j \partial x_j} + \left(\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}e_i\right)\cdot\left(\varepsilon_{lmn}\frac{\partial u_n}{\partial x_m}e_l\right)$$

$$= -\frac{1}{2}\frac{\partial^2\left(u_j u_j\right)}{\partial x_i \partial x_i} + u_i\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \left(\varepsilon_{ijk}\varepsilon_{lmn}\frac{\partial u_k}{\partial x_j}\frac{\partial u_n}{\partial x_m}\right)\left(e_i\cdot e_l\right)$$

$$= -\frac{1}{2}\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}\left(u_j u_j\right)\right) + u_i\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \left(\varepsilon_{ijk}\varepsilon_{lmn}\frac{\partial u_k}{\partial x_j}\frac{\partial u_n}{\partial x_m}\right)\delta_{il}$$

$$= -\frac{1}{2}\frac{\partial}{\partial x_i}\left(2u_j\frac{\partial u_j}{\partial x_i}\right) + u_i\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right)\frac{\partial u_k}{\partial x_j}\frac{\partial u_n}{\partial x_m}$$

$$= -\frac{\partial}{\partial x_i}\left(u_j\frac{\partial u_j}{\partial x_i}\right) + u_i\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \delta_{jm}\delta_{kn}\frac{\partial u_k}{\partial x_j}\frac{\partial u_n}{\partial x_m} - \delta_{jn}\delta_{km}\frac{\partial u_k}{\partial x_j}\frac{\partial u_n}{\partial x_m}$$

$$= -\left(\frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_i} + u_j\frac{\partial^2 u_j}{\partial x_i \partial x_i}\right) + u_i\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial u_k}{\partial x_j}\frac{\partial u_k}{\partial x_j} - \frac{\partial u_k}{\partial x_j}\frac{\partial u_j}{\partial x_k}$$

$$= -\frac{\partial u_k}{\partial x_j}\frac{\partial u_j}{\partial x_k}$$

Although $\underline{\omega}$ equation (and Ψ equation) do not involve p explicitly/directly we have shown that τ_w related ω_x and ω_z and $\underline{\sigma} = -\underline{n}\cdot\nabla\,\underline{\omega}$

Lastly consider momentum equation on the wall

$$0 = -\nabla p + \nabla\cdot\tau_{ij}$$

Shows τ_w related ∇p

Stream Function Vorticity Approach (restricted 2D)

$u = \psi_y \quad v = -\psi_x \quad \omega_z = v_x - u_y = \omega$

$\omega_t + u\omega_x + v\omega_y = \nu \nabla^2 \omega$

$\omega_t + \psi_y \omega_x - \psi_x \omega_y = \nu (\omega_{xx} + \omega_{yy})$ parabolic t

elliptic (x,y)

$\psi_{xx} + \psi_{yy} = -\omega$ Poisson

two equations two unknowns ω & ψ

$\nabla^2 p = 2\rho (u_x v_y - u_y v_x)$

$= 2\rho (-\psi_{yx} \psi_{xy} + \psi_{yy} \psi_{xx})$

$= 2\rho (\psi_{xx} \psi_{yy} - \psi_{xy}^2)$

vs primitive variable approach

$\nabla \cdot \underline{v} = 0$

$\frac{D\underline{v}}{Dt} = -\nabla (p/\rho) + \nu \nabla^2 \underline{v}$

$\nabla^2 (p/\rho) = -\frac{\partial u_k}{\partial x_i} \frac{\partial u_i}{\partial x_k}$

In both cases need appropriate initial & boundary conditions

3-46 Derive the two-dimensional Poisson relation for pressure, Eq. (3-256).

$$\nabla\cdot\left[\frac{\partial \underline{V}}{\partial t}+\underline{V}\cdot\nabla\underline{V}=-\nabla(\hat{p}/\rho)+\nu\nabla^2\underline{V}\right]$$

$$\nabla^2(\hat{p}/\rho)=-\nabla\cdot(\underline{V}\cdot\nabla\underline{V})$$

$$\underline{V}=u\hat{i}+v\hat{j}$$

$$\nabla=\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}$$

$$\nabla\cdot\left[(uu_x+vu_y)\hat{i}+(uv_x+vv_y)\hat{j}\right]$$

$$=\frac{\partial}{\partial x}(uu_x+vu_y)+\frac{\partial}{\partial y}(uv_x+vv_y)$$

$$=\overset{1}{u_x^2}+\overset{2}{uu_{xx}}+\overset{3}{v_xu_y}+\overset{4}{vu_{yx}}+\overset{5}{u_yv_x}+\overset{6}{uv_{xy}}+\overset{7}{v_y^2}+\overset{8}{vv_{yy}}$$

$$2+6=u\frac{\partial}{\partial x}(u_x+v_y)=0$$

$$4+8=v\frac{\partial}{\partial y}(u_x+v_y)=0$$

$$3=5=2v_xu_y$$

$$1+7=-v_yu_x+-u_xv_y=-2u_xv_y$$

$$\nabla^2\hat{p}=2\rho(u_xv_y-u_yv_x)=\nabla^2 p$$

To do this, write the x- and y-momentum (Navier-Stokes) equations in the forms

$$\frac{\partial p}{\partial x}=\cdots\quad(1)\qquad\qquad\frac{\partial p}{\partial y}=\cdots\quad(2)$$

Take $\partial/\partial x$(Eq. 1) and add it to $\partial/\partial y$(Eq. 2) to give $\nabla^2 p$. The gravity term vanishes (assuming that **g** is constant) and the viscous terms (assuming constant μ) vanish by virtue of the continuity equation. What remains is a string of 8 acceleration-related terms:

$$\nabla^2 p=-\rho\Big[\underset{1}{(\frac{\partial u}{\partial x})^2}+\underset{2}{u\frac{\partial^2 u}{\partial x^2}}+\underset{3}{\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}}+\underset{4}{v\frac{\partial^2 u}{\partial x\partial y}}+\underset{5}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}+\underset{6}{u\frac{\partial^2 v}{\partial x\partial y}}+\underset{7}{(\frac{\partial v}{\partial y})^2}+\underset{8}{v\frac{\partial^2 v}{\partial y^2}}\Big]$$

Now combine as follows: Terms 2 and 6, when $(u\,\partial/\partial x)$ is factored out, vanish due to continuity - likewise for terms 4 and 8 when $(v\,\partial/\partial y)$ is factored out. Replace *one* $(\partial u/\partial x)$ in term 1 by $(-\partial v/\partial y)$, and replace *one* $(\partial v/\partial y)$ in term 7 by $(-\partial u/\partial x)$, thus making terms 1 and 7 equal. Terms 3 and 5 are already equal. The final result is Eq. (3-256):

$$\nabla^2 p=2\rho\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\qquad\textbf{(Ans.)}$$

3. Kinematic Decomposition of flow fields

Previously, we discussed the decomposition of fluid motion into translation, rotation, and deformation. This was done locally for a fluid element. Now we shall see that a global decomposition is possible.

Helmholtz's Decomposition: any continuous and finite vector field can be expressed as the sum of the gradient of a scalar function ϕ plus the curl of a zero-divergence vector A. The vector A vanishes identically if the original vector field is irrotational.

$$\underline{V} = \underline{V}^{\omega} + \underline{V}^{\phi}$$

Where
$$\underline{\omega} = \nabla \times \underline{V}^{\omega}$$
$$0 = \nabla \times \underline{V}^{\phi}$$

The irrotational part of the velocity field can be expressed as the gradient of a scalar

$$\rightarrow \quad \underline{V}^{\phi} = \nabla \phi$$

If $\quad \nabla \cdot \underline{V} = \nabla \cdot \underline{V}^{\omega} + \nabla \cdot \underline{V}^{\phi} = 0$

Then $\quad \nabla^2 \phi = 0 \qquad$ *The GDE for ϕ is the Laplace Eq.*

And $\quad \underline{V}^{\omega} = \nabla \times \underline{A} \qquad$ *Since* $\nabla \cdot (\nabla \times \underline{A}) = 0$

$$\nabla \times \underline{V}^{\omega} = \underline{\omega} = \nabla \times \nabla \times \underline{A} \qquad \text{Again, by vector identity}$$

$$= -\nabla^2 \underline{A} + \nabla(\nabla \cdot \underline{A})$$

i.e $\quad \nabla^2 \underline{A} = -\underline{\omega}$

The solution of this equation is $\underline{A} = \frac{1}{4\pi} \int \frac{\underline{\omega}}{|\underline{R}|} \, d\forall$

Thus $\quad \underline{V}^{\omega} = -\frac{1}{4\pi} \int \frac{\underline{R} \times \underline{\omega}}{|\underline{R}|^3} \, d\forall$

Which is known as the Biot-Savart law.

The Biot-Savart law can be used to compute the velocity field induced by a known vorticity field. It has many useful applications, including in ideal flow theory (e.g., when applied to line vortices and vortex sheets it forms the basis of computing the velocity field in vortex-lattice and vortex-sheet lifting-surface methods).

The important conclusion from the Helmholtz decomposition is that any incompressible flow can be thought of as the vector sum of rotational and irrotational components. Thus, a solution for irrotational part $\underline{V}^{\phi}$ represents at least part of an exact solution. Under certain conditions, high Re flow about slender bodies with attached thin boundary layer and wake, $\underline{V}^{\omega}$ is small over much of the flow field such that $\underline{V}^{\phi}$ is a good approximation to $\underline{V}$. This is probably the strongest justification for ideal-flow theory. (*incompressible, inviscid, and irrotational flow)*.