INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN BIOMEDICAL ENGINEERING
*Int. J. Numer. Meth. Biomed. Engng.* 2011; **27**:1652–1659
Published online 26 March 2010 in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/cnm.1381
COMMUNICATIONS IN NUMERICAL METHODS IN ENGINEERING

LETTER TO THE EDITOR

# Comments on 'High-dimensional model representation for structural reliability analysis'

by R. Chowdhury, B. N. Rao and A. M. Prasad, *Communications in Numerical Methods in Engineering* 2009; **25**:301–337

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SUMMARY

The authors of the paper published in *Commun. Numer. Methods Eng.* (2009; **25**:301–337) have claimed to present a new computational tool entailing high-dimensional model representation for structural reliability analysis. This letter demonstrates that the underlying reliability method already exists and was originally developed in the Xu and Rahman paper, published in *Probabilist. Eng. Mech.* (2005; **20**:239–250). The authors do not acknowledge the contribution of Xu and Rahman and convey the impression that they are developing the method for the very first time. Instead, the authors have reinvented the wheel. 

## 1. INTRODUCTION

The authors, Chowdhury *et al.* [1], of the paper 'High-dimensional model representation for structural reliability analysis' published in *Communications in Numerical Methods in Engineering* [1] have claimed to present a new computational tool entailing high-dimensional model representation for structural reliability analysis. The objective of writing the letter is to refute this claim by showing that the underlying reliability method already existed and was originally developed by Xu and Rahman, as previously published in *Probabilistic Engineering Mechanics* [2]. Chowdhury *et al.* do not acknowledge any contribution by Xu and Rahman and convey the impression that they are presenting a new methodology and original contribution to the literature.

Section 2 provides a brief overview of dimensional decomposition of a multivariate function, followed by specific decomposition methods developed or employed by Xu and Rahman [2] and Chowdhury *et al.* [1] and discusses the selection of reference and sample points and subsequent Monte Carlo simulation for estimating the failure probability in both papers. Section 3 discusses numerical results, textual contents, and linguistic styles in both papers. Section 4 presents our conclusions.

## 2. DIMENSIONAL DECOMPOSITION

Dimensional decomposition of a multivariate function is a finite sum of simpler component functions of input variables with increasing dimensions. This decomposition, first presented by

Hoeffding [3] in relation to his seminal work on $U$-statistics, has been applied by many other researchers [4]: Sobol [5] used it in the study of quadrature methods, calling it the 'decomposition into summands of different dimensions' and also for analysis of variance (ANOVA) [6]; Efron and Stein [7] used it to prove their famous lemma on jackknife variances; Owen [8] presented a continuous space version of the nested ANOVA; and Hickernell [9] developed a reproducing kernel Hilbert space version. This decomposition has been lately examined by Rabitz and Alis [10] for high-dimensional model representation (HDMR), resulting in notable contributions to function approximations, and more recently, by Xu and Rahman [2] for reliability analysis. Takemura [11] provides a historical account, which reveals that the decomposition existed as early as the 1940s and, therefore, the original development of the decomposition predates the timeline specified in the authors' response (Chowdhury *et al.*, 2009).

Consider a continuous, differentiable, real-valued, and multivariate function $y(\boldsymbol{x})$ that depends on $\boldsymbol{x}=\{x_1,\ldots,x_N\}^{\mathrm{T}}\in\mathbb{R}^N$, where $\mathbb{R}^N$ is an $N$-dimensional real vector space. The dimensional decomposition represents a finite, hierarchical, and convergent expansion of a multivariate output function [2, 3, 6, 7, 10]

$$y(\boldsymbol{x}) = y_0+\sum_{i=1}^{N} y_i(x_i)+\sum_{i_1,i_2=1;i_1<i_2}^{N} y_{i_1 i_2}(x_{i_1},x_{i_2})+\sum_{i_1,i_2,i_3=1;i_1<i_2<i_3}^{N} y_{i_1 i_2 i_3}(x_{i_1},x_{i_2},x_{i_3}) + \cdots + \sum_{i_1,\ldots,i_S=1;i_1<\cdots<i_S}^{N} y_{i_1\ldots i_S}(x_{i_1},\ldots,x_{i_S})+\cdots+y_{12\ldots N}(x_1,\ldots,x_N) \quad (1)$$

in terms of input $\boldsymbol{x}$ with increasing dimensions, where $y_0$ is a constant and $y_{i_1\ldots i_S}:\mathbb{R}^S\to\mathbb{R}$, $1\leqslant S\leqslant N$ is an $S$-variate component function quantifying the cooperative effects of $S$ input variables $x_{i_1},\ldots,x_{i_S}$. Such a decomposition is also called the Hoeffding decomposition, the ANOVA decomposition, HDMR, and possibly others. If

$$\hat{y}_S(\boldsymbol{x}) = y_0+\sum_{i=1}^{N} y_i(x_i)+\sum_{i_1,i_2=1;i_1<i_2}^{N} y_{i_1 i_2}(x_{i_1},x_{i_2})+\sum_{i_1,i_2,i_3=1;i_1<i_2<i_3}^{N} y_{i_1 i_2 i_3}(x_{i_1},x_{i_2},x_{i_3}) + \cdots + \sum_{i_1,\ldots,i_S=1;i_1<\cdots<i_S}^{N} y_{i_1\ldots i_S}(x_{i_1},\ldots,x_{i_S}) \quad (2)$$

represents a general $S$-variate approximation of $y(\boldsymbol{x})$, the univariate $(S=1)$ and bivariate $(S=2)$ approximations, $\hat{y}_1(\boldsymbol{x})$ and $\hat{y}_2(\boldsymbol{x})$, respectively, provide two- and three-term approximants of the finite decomposition in Equation (1). Similarly, trivariate and other higher-variate approximations can be derived by appropriately selecting the value of $S$. The decomposition is useful only when the component functions exhibit insignificant $S$-variate effects cooperatively when $S\to N$.

If the input is random, say, an $N$-dimensional random vector $\boldsymbol{X}=\{X_1,\ldots,X_N\}^{\mathrm{T}}\in\mathbb{R}^N$, then the response $y(\boldsymbol{X})$ is also random. Structural reliability analysis involving $y(\boldsymbol{X})$ as a performance function and employing its surrogate $\hat{y}_S(\boldsymbol{X})$ from Equation (2) has been conducted by Xu and Rahman [2] in 2005 and Chowdhury *et al.* [1] in 2009. The following discussion demonstrates that the decomposition employed by Chowdhury *et al.* [1] is exactly the same as that previously developed by Xu and Rahman [2].

### 2.1. *Decomposition method of Xu and Rahman* [2]

Let $\boldsymbol{c}=\{c_1,\ldots,c_N\}^{\mathrm{T}}$ be a reference point of input $\boldsymbol{x}$ and $y(c_1,\ldots,c_{k_1-1},x_{k_1},c_{k_1+1},\ldots,c_{k_{S-i}-1},x_{k_{S-i}},c_{k_{S-i}+1},\ldots,c_N)$ represents an $(S-i)$th dimensional component function of $y(\boldsymbol{x})$, where $S<N$, $i=0,\ldots,S$, and $1\leqslant k_1<\cdots<k_{S-i}\leqslant N$. For example, when $S=1$, the zero-dimensional

component function, which is a constant, is $y(\boldsymbol{c})$ and the one-dimensional component functions are $y(x_1, c_2, \ldots, c_N)$, $y(c_1, x_2, \ldots, c_N)$, $\cdots$, $y(c_1, c_2, \ldots, x_N)$. In 2005, Xu and Rahman [2] developed the $S$-variate approximation

$$\hat{y}_{S,\mathrm{DD}}(\boldsymbol{x}) := \sum_{i=0}^{S} (-1)^i \binom{N-S+i-1}{i} \times \sum_{k_1,\ldots,k_{S-i}=1;k_1<\cdots<k_{S-i}}^{N} y(c_1, \ldots, c_{k_1-1}, x_{k_1}, c_{k_1+1}, \ldots, c_{k_{S-i}-1}, x_{k_{S-i}}, c_{k_{S-i}+1}, \ldots, c_N) \tag{3}$$

of $y(\boldsymbol{x})$, which appears as Equation 8 in the original paper [2]. The subscript 'DD' represents Xu and Rahman's [2] decomposition method, where the approximation $\hat{y}_{S,\mathrm{DD}}(\boldsymbol{x})$ in Equation (3) follows the same structure, as shown in Equation (2). Using a multivariate function decomposition theorem, Xu and Rahman [12] proved that the right side of Equation (3) consists of all terms of the Taylor series of $y(\boldsymbol{x})$ that have less than or equal to $S$ variables. The expanded form of Equation (3), when compared with the Taylor expansion of $y(\boldsymbol{x})$, indicates that the residual error in $\hat{y}_{S,\mathrm{DD}}(\boldsymbol{x})$ includes only terms of dimensions $S+1$ and higher. All higher-order $S$- and lower-variate terms of $y(\boldsymbol{x})$ are included in Equation (3), which should, therefore, generally provide a higher-order approximation of a multivariate function than equations derived from first- or second-order Taylor expansions.

When $S=1$ and $S=2$, Equation (3), respectively, degenerates to the univariate approximation

$$\hat{y}_{1,\mathrm{DD}}(\boldsymbol{x}) = \sum_{i=1}^{N} y(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_N) - (N-1)y(\boldsymbol{c}) \tag{4}$$

and the bivariate approximation

$$\hat{y}_{2,\mathrm{DD}}(\boldsymbol{x}) = \sum_{i_1,i_2=1;i_1<i_2}^{N} y(c_1, \ldots, c_{i_1-1}, x_{i_1}, c_{i_1+1}, \ldots, c_{i_2-1}, x_{i_2}, c_{i_2+1}, \ldots, c_N) - (N-2)\sum_{i=1}^{N} y(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_N) + \frac{(N-1)(N-2)}{2} y(\boldsymbol{c}), \tag{5}$$

presented as Equations (4) and (6) in the original paper [2]. Similarly, trivariate and other higher-variate approximations can be derived by appropriately selecting the value of $S$. Equation (3) generates a hierarchical and convergent sequence of approximations of $y(\boldsymbol{x})$.

To obtain explicit forms of Equations (4) and (5), Xu and Rahman [2] generated response surface approximations of univariate functions $y(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_N)$, $i=1, \ldots, N$ and bivariate functions $y(c_1, \ldots, c_{i_1-1}, x_{i_1}, c_{i_1+1}, \ldots, c_{i_2-1}, x_{i_2}, c_{i_2+1}, \ldots, c_N)$, $i_1 = 1, \ldots, N-1$, $i_2 = i_1+1, \ldots, N$ by using their Lagrange interpolations. Subsequently, the probabilistic characteristics of $y(\boldsymbol{X})$ were estimated by direct Monte Carlo simulation of $\hat{y}_{1,\mathrm{DD}}(\boldsymbol{X})$ or $\hat{y}_{2,\mathrm{DD}}(\boldsymbol{X})$. The reliability analyses employing $\hat{y}_{1,\mathrm{DD}}(\boldsymbol{X})$ and $\hat{y}_{2,\mathrm{DD}}(\boldsymbol{X})$ as surrogates of $y(\boldsymbol{X})$ are explicitly termed as univariate and bivariate decomposition methods, respectively, in the Xu and Rahman [2] paper.

### 2.2. *Cut-HDMR method of Chowdhury et al.* [1]

An important feature of the approximation in Equation (2) is the selection of the constant $y_0$ and component functions $y_{i_1 \ldots i_S}(x_{i_1}, \ldots, x_{i_S})$, $1 \leqslant S < N$. By defining an error functional associated with a given $y(\boldsymbol{x})$ and an appropriate kernel function, an optimization problem can be formulated



DOI: 10.1002/cnm

and solved to obtain the desired component functions. In particular, a decomposition involving the Dirac measure $\Pi_{i=1}^{N}\delta(x_i-c_i)$ at the reference point $\boldsymbol{c}$ as the kernel function leads to [10]

$$
\begin{aligned}
y_0 &:= y(\boldsymbol{c})\\
y_i(x_i) &:= y(c_1,\ldots,c_{i-1},x_i,c_{i+1},\ldots,c_N)-y(\boldsymbol{c})\\
y_{i_1i_2}(x_{i_1},x_{i_2}) &:= y(c_1,\ldots,c_{i_1-1},x_{i_1},c_{i_1+1},\ldots,c_{i_2-1},x_{i_2},c_{i_2+1},\ldots,c_N)\\
&\quad -y_{i_1}(x_{i_1})-y_{i_2}(x_{i_2})-y(\boldsymbol{c})\\
&\ \vdots\\
y_{i_1\ldots i_S}(x_{i_1},\ldots,x_{i_S}) &:= y(c_1,\ldots,c_{i_1-1},x_{i_1},c_{i_1+1},\ldots,c_{i_S-1},x_{i_S},c_{i_S+1},\ldots,c_N)\\
&\quad -\sum_{j_1<\cdots<j_{S-1}\subset\{i_1,\ldots,i_S\}} y_{j_1\ldots j_{S-1}}(x_{j_1},\ldots,x_{j_{S-1}})\\
&\quad -\sum_{j_1<\cdots<j_{S-2}\subset\{i_1,\ldots,i_S\}} y_{j_1\ldots j_{S-2}}(x_{j_1},\ldots,x_{j_{S-2}})\\
&\quad -\cdots-\sum_{j\subset\{i_1,\ldots,i_S\}} y_j(x_j)-y(\boldsymbol{c}),
\end{aligned}
\tag{6}
$$

the first three lines of which appear as Equations (2), (3), and (4), respectively, in the Chowdhury *et al.* [1] paper. For $S=1$, the first two lines of Equation (6) substituted in Equation (2) yield the first-order cut-HDMR approximation

$$
\hat{y}_{1,\text{cut}-\text{HDMR}}(\boldsymbol{x})=\sum_{i=1}^{N} y(c_1,\ldots,c_{i-1},x_i,c_{i+1},\ldots,c_N)-(N-1)y(\boldsymbol{c}), \tag{7}
$$

whereas for $S=2$, the first three lines of Equation (6) applied to Equation (2) yield the second-order cut-HDMR approximation

$$
\begin{aligned}
\hat{y}_{2,\text{cut}-\text{HDMR}}(\boldsymbol{x}) = &\sum_{i_1,i_2=1;i_1<i_2}^{N} y(c_1,\ldots,c_{i_1-1},x_{i_1},c_{i_1+1},\ldots,c_{i_2-1},x_{i_2},c_{i_2+1},\ldots,c_N)\\
&-(N-2)\sum_{i=1}^{N} y(c_1,\ldots,c_{i-1},x_i,c_{i+1},\ldots,c_N)+\frac{(N-1)(N-2)}{2}y(\boldsymbol{c}),
\end{aligned}
\tag{8}
$$

which appear as Equations (9) and (10) in Chowdhury *et al.*'s [1] paper. The subscript 'cut-HDMR' represents the cut-HDMR approximation employed by Chowdhury *et al.* [1]. The reliability analyses using $\hat{y}_{1,\text{cut}-\text{HDMR}}(\boldsymbol{X})$ and $\hat{y}_{2,\text{cut}-\text{HDMR}}(\boldsymbol{X})$ as surrogates of $y(\boldsymbol{X})$ are called first-order cut-HDMR and second-order cut-HDMR methods, respectively, in the Chowdhury *et al.* [1] paper.

Comparing Equations (4), (5) and (7), (8) reveals that the $S$th ($S=1$ or 2) order cut-HDMR approximation employed by Chowdhury *et al.* [1] is identical to the $S$-variate dimensional decomposition developed by Xu and Rahman [2]. In other words, the first-order cut-HDMR and second-order cut-HDMR are exactly the same as Xu and Rahman's [2] univariate and bivariate decomposition methods, respectively. Therefore, the claim by Chowdhury *et al.* [1] of a new computational tool is unfounded. Instead, the authors have reinvented the wheel. As well, Chowdhury *et al.* [1] fail to acknowledge any contribution from Xu and Rahman [2], who have essentially derived the same multivariate function decomposition. The only difference between the Chowdhury*et al.* [1] and Xu and Rahman [2] papers is the use of moving least-squares approximation by Chowdhury *et al.* [1] as opposed to Lagrange interpolation by Xu and Rahman [2] to approximate component functions of the decomposition. Otherwise, the methods and results in Chowdhury *et al.* [1], which derive from Equations (7) and (8), are based on identical derivations previously developed by Xu and Rahman [2].

*2.3. Reference point, sample points, failure probability*

The univariate (or first-order HDMR) and bivariate (or second-order HDMR) approximations in Equations (4) (or (7)) and (5) (or (8)) require the constant $y(\boldsymbol{c})$, univariate component functions $y(c_1,\ldots,c_{i-1},x_i,c_{i+1},\ldots,c_N)$, $i=1,\ldots,N$, and bivariate component functions $y(c_1,\ldots,c_{i_1-1},x_{i_1},c_{i_1+1},\ldots,c_{i_2-1},x_{i_2},c_{i_2+1},\ldots,c_N)$, $i_1=1,\ldots,N-1$, $i_2=i_1+1,\ldots,N$, all of which depend on the reference point $\boldsymbol{c}$. Sobol [6] suggested selecting the reference point to be the realization $\boldsymbol{x}^*$ of the random input $\boldsymbol{X}$ that yields the corresponding response $y(\boldsymbol{x}^*)$ closest to the mean value of $y(\boldsymbol{X})$. However, Sobol's selection requires Monte Carlo simulation of $y(\boldsymbol{X})$, adding to the computational burden of the decomposition methods. Instead, Xu and Rahman [2] proposed selecting the mean value of the random input $\boldsymbol{X}$ as the reference point. Furthermore, Xu and Rahman [2] employed uniformly distributed sample points, centered at the mean point and distributed one standard deviation apart from each other, for response surface generations of $y(c_1,\ldots,c_{i-1},x_i,c_{i+1},\ldots,c_N)$ and $y(c_1,\ldots,c_{i_1-1},x_{i_1},c_{i_1+1},\ldots,c_{i_2-1},x_{i_2},c_{i_2+1},\ldots,c_N)$. It must be explicitly noted that Chowdhury *et al.* [1] also applied the mean point as the reference point and the uniform distribution of sample points, as did in Xu and Rahman [2], again, without appropriate acknowledgment of credit.

Once the explicit forms of component functions have been generated, Equations (4) (or (7)) and (5) (or (8)) furnish approximate but explicit maps $\hat{y}_{1,\mathrm{DD}}:\mathbb{R}^N\to\mathbb{R}$ [or $\hat{y}_{1,\mathrm{cut-HDMR}}:\mathbb{R}^N\to\mathbb{R}$] and $\hat{y}_{2,\mathrm{DD}}:\mathbb{R}^N\to\mathbb{R}$ [or $\hat{y}_{2,\mathrm{cut-HDMR}}:\mathbb{R}^N\to\mathbb{R}$] that can be viewed as surrogates of the exact map $y:\mathbb{R}^N\to\mathbb{R}$, describing the input–output relation from a complex numerical simulation. Therefore, any probabilistic characteristic of $y(\boldsymbol{X})$, including failure probabilities, can be easily estimated by performing Monte Carlo simulation of $\hat{y}_{1,\mathrm{DD}}(\boldsymbol{X})$ [or $\hat{y}_{1,\mathrm{cut-HDMR}}(\boldsymbol{X})$] and $\hat{y}_{2,\mathrm{DD}}(\boldsymbol{X})$ [or $\hat{y}_{2,\mathrm{cut-HDMR}}(\boldsymbol{X})$] rather than of $y(\boldsymbol{X})$, as previously proposed by Xu and Rahman [2] and later followed by Chowdhury *et al.* [1]. Indeed, Equation (31) in Chowdhury *et al.* [1] and Equations (17) and (18) in Xu and Rahman [2] for estimating the failure probability are identical, even though the latter work is not cited by Chowdhury *et al.* [1].

In the authors' response, Chowdhury *et al.* erroneously stated that the selection of the reference point as the mean point was originally suggested by Sobol [6]. In fact, the concept of 'mean point' does not appear anywhere in the entire Sobol paper. A simple word search of Sobol's paper on 'mean point' will verify our claim. Nor does Sobol [6] suggest utilization of the mean value of random input as the reference point. Instead, Section 10 of Sobol [6] proposes determination of the reference point by random sampling of input variables and then minimizing an objective function from the input samples. The last three equations of the Sobol [6] paper, which discuss the reference point, do not involve the mean value of random input at all, raising a serious question as to the credibility of the authors' response. It is worth noting that the authors failed to explain their choices of sample points and Monte Carlo simulation, which are identical to those employed by Xu and Rahman [2].

*2.4. Development and coinage of decomposition methods*

In the authors' response, Chowdhury *et al.* questioned our development of 'decomposition methods' in comparison with Rabitz and Alis's cut-HDMR method and coinage of terms 'univariate decomposition method' and 'bivariate decomposition method' in the Xu and Rahman [2] paper. Our response is as follows.

Although the dimensional decomposition and cut-HDMR methods are equivalent, see Sections 2.1 and 2.2, the former method was developed independently and from a completely different perspective. The decomposition method was formulated based on the Taylor series expansion described in detail in a prequel [12]. In contrast, the cut-HDMR was developed by minimizing an error functional and a Dirac measure as the kernel function [10]. In our decomposition methods, we provided rigorous proof of a multivariate function decomposition theorem and a generalized formulation [12] that cannot be found in the HDMR literature. Precisely because of this new perspective, we were able to develop a new generalized equation (Equation (3)), which appears as Equation (8) in the Xu and Rahman [2] paper, to yield univariate and bivariate



DOI: 10.1002/cnm

approximations of the response as special cases. The generalized equation and/or its special cases do not exist in the cut-HDMR method or any other HDMR-related work that we are familiar with. Readers should note that the authors' response contains an ill-considered attempt to deflect our arguments by repeatedly disparaging our decomposition methods, but avoids the most pressing issue in our letter: Equations (7) and (8) 'derived' in Chowdhury *et al.*'s (2009) paper are identical to Equations (4) and (5) originally developed in the Xu and Rahman [2] paper, refuting the claim of new reliability tool by Chowdhury *et al.*

At the time we developed our decomposition methods, we were not aware of the HDMR methods, due to the fact that the HDMR papers were published in different disciplines than ours. Consequently, we were not able to cite the cut-HDMR method in the Xu and Rahman [2] paper, which we regret. However, once we became cognizant of the HDMR methods and realized the similarity between these two methods, we acknowledged the HDMR methods in our subsequent papers [13, 14] and other related publications.

Since the right side of Equation (4) comprises only univariate functions, the interpolation or integration of $\hat{y}_{1,\mathrm{DD}}(\boldsymbol{X})$ is essentially univariate. Similarly, the right side of Equation (5), which contains at most bivariate functions, requires at most bivariate interpolation or integration of $\hat{y}_{2,\mathrm{DD}}(\boldsymbol{X})$. Therefore, we believe that appellation of the terms 'univariate decomposition method' and 'bivariate decomposition method' for approximations resulting from $\hat{y}_{1,\mathrm{DD}}(\boldsymbol{X})$ in Equation (4) and $\hat{y}_{2,\mathrm{DD}}(\boldsymbol{X})$ in Equation (5), respectively, makes sense and is more appropriate than referring to them as first-order and second-order methods. Note that the component functions $y(c_1,\ldots,c_{i-1},x_i,c_{i+1},\ldots,c_N)$ and $y(c_1,\ldots,c_{i_1-1},x_{i_1},c_{i_1+1},\ldots,c_{i_2-1},x_{i_2},c_{i_2+1},\ldots,c_N)$, embedded in the expressions of $\hat{y}_{1,\mathrm{DD}}(\boldsymbol{X})$ and/or $\hat{y}_{2,\mathrm{DD}}(\boldsymbol{X})$, are generally nonlinear. Moreover, a univariate or bivariate approximation may contain very high-order (i.e. higher than first- or second-order) terms, depending on the nonlinearity of the response. Therefore, characterizing these approximations by first- and second-order methods is confusing and possibly inaccurate based on the traditional definition of the order of a function.

## 3. NUMERICAL RESULTS, TEXTUAL CONTENTS, LINGUISTIC STYLES

Although Chowdhury *et al.* [1] include many examples, two problems examined in Examples 7 and 9 are the same or closely related to those studied by Xu and Rahman [2] and another related work [15]. For instance, Example 7 in Chowdhury *et al.* [1] and Example 4 in Xu and Rahman [2], which are based on reliability analysis of a 10-bar truss, are identical, even though the latter work is not cited by Chowdhury *et al.* [1]. Rather, Chowdhury *et al.* [1] cite a similar work by Penmetsa and Grandhi [16], which, however, does not apply the same performance function. The critical thresholds of displacement in the Chowdhury *et al.* [1] and Xu and Rahman [2] papers are both 18 in, whereas Penmetsa and Grandhi [16] stipulate a threshold of 1.8 in. Clearly, citation of Xu and Rahman [2] and a discussion of their numerical results would have been more appropriate. It must be explicitly noted that the first eight lines in Section 6.2.5, p. 326 of Chowdhury *et al.* [1] are almost an identical copy of the text contained in Example 4, p. 245 of Xu and Rahman's [2] paper. Therefore, by not acknowledging the work of Xu and Rahman [2], Chowdhury *et al.* [1] convey a misleading impression that they have developed a new reliability method and are presenting an original contribution to the literature.

From Xu and Rahman's [2] paper, Table I [Table 4 of the Xu and Rahman [2] paper] shows failure probability estimates of the truss structure using various approximate methods, including the univariate decomposition method. The results of the univariate method are quite similar to those presented in Table IX of Chowdhury *et al.*'s [1] paper. Again, it appears that Chowdhury *et al.* [1] have disregarded prior works, which could have been effectively employed to critically examine their HDMR methods. Indeed, Example 9 in Chowdhury *et al.* [1] is closely related to a problem solved in yet another paper by Wei and Rahman [15].

In addition to the fundamental similarities in technical aspects of the Chowdhury *et al.* [1] and Xu and Rahman [2] papers discussed earlier, there exist striking similarities in textual content and linguistic style in the Summary, Introduction, Concept of HDMR and Its Importance to Reliability



DOI: 10.1002/cnm

Table I. Failure probability estimates for 10-bar truss [2].

| Method | Failure probability | No. of function evaluations* |
|---|---|---|
| Univariate method | 0.1357 | 61† |
| FORM | 0.0863 | 127 |
| SORM (Breitung)‡ | 0.1286 | 506 |
| SORM (Hohenbichler)§ | 0.1524 | 506 |
| SORM (Cai & Elishakoff)¶ | 0.1467 | 506 |
| Direct Monte Carlo simulation | 0.1397 | 1 000 000 |

*Total number of times the original performance function is calculated.
†$(7-1)\times 10+1=61$.
‡See Ref. 4 of Xu and Rahman [2].
§See Ref. 5 of Xu and Rahman [2].
¶See Ref. 6 of Xu and Rahman [2].

Analysis, Failure Probability Estimation, and Numerical Examples sections. The specific places in Chowdhury *et al.*'s [1] paper, where close imitation of our language is detected, include (1) lines 1–5 and lines 10–12 in page 301 (2) lines 5–25 in page 302; (3) lines 6–9 in page 305; (4) lines 18–20 in page 309; (5) lines 5–7 and lines 13–17 in page 311; (6) lines 11–19 in page 326. We would be happy to provide a side-by-side comparison of text if desired.

## 4. CONCLUDING REMARKS

The paper by Chowdhury *et al.*, published in *Communications in Numerical Methods in Engineering* (2009; **25**:301–337), claims to present a new HDMR-based computational tool for structural reliability analysis. This letter demonstrates such a claim to be unfounded, given that the decomposition method previously developed by Xu and Rahman, as published in *Probabilistic Engineering Mechanics* 2005; **20**:239–250, is precisely the same as the cut-HDMR employed by Chowdhury *et al*. Therefore, Chowdhury *et al*. have reinvented the wheel. The only difference that exists between these two papers is the application of different approximation or interpolation schemes for approximating lower-variate component functions. Otherwise, the reliability method employing multivariate function decomposition is exactly the same as that developed four years prior by Xu and Rahman. Chowdhury *et al.* do not acknowledge Xu and Rahman, despite near-identical theoretical formulations presented in both works. Having thus presented our evidence, we are unable to reject the possibility that this failure to extend appropriate credit may be intentional, given the obvious similarities in theoretical development, numerical results, and even descriptive style and content.

The authors' response is replete with erroneous arguments and misleading statements, including incomplete historical account of the function decomposition, mistaken conclusion as to the reference point, total disregard of our distinct perspective in developing decomposition methods, and misinterpretation of our designation of the terms 'univariate' and 'bivariate' decomposition methods, to name a few.

### ACKNOWLEDGEMENTS

S. Rahman acknowledges financial support from the U.S. National Science Foundation under Grant Nos. DMI-0355487 and CMMI-0653279.

### REFERENCES

1. Chowdhury R, Rao BN, Prasad AM. High-dimensional model representation for structural reliability analysis. *Communications in Numerical Methods in Engineering* 2009; **25**:301–337.
2. Xu H, Rahman S. Decomposition methods for structural reliability analysis. *Probabilistic Engineering Mechanics* 2005; **20**:239–250.

3. Hoeffding W. A class of statistics with asymptotically normal distributions. *Annals of Mathematical Statistics* 1948; **19**:293–325.
4. Owen AB. The dimension distribution and quadrature test functions. *Statistica Sinica* 2003; **13**:1–17.
5. Sobol IM. *Multidimensional Quadrature Formulas and Haar functions*. Nauka: Moscow, 1969 (in Russian).
6. Sobol IM. Theorems and examples on high dimensional model representations. *Reliability Engineering and System Safety* 2003; **79**:187–193.
7. Efron B, Stein C. The Jackknife estimate of variance. *The Annals of Statistics* 1981; **9**:586–596.
8. Owen AB. Monte Carlo variance of scrambled net quadrature. *SIAM Journal of Numerical Analysis* 1997; **34**:1884–1910.
9. Hickernell FJ. Quadrature error bounds with applications to lattice rules. *SIAM Journal of Numerical Analysis* 1996; **33**:1995–2016. Corrected printing of Sections 3-6 in SIAM Journal on Numerical Analysis 1997; **34**: 853–866.
10. Rabitz H, Alis O. General foundations of high-dimensional model representations. *Journal of Mathematical Chemistry* 1999; **25**:197–233.
11. Takemura A. Tensor analysis of ANOVA decomposition. *Journal of American Statistical Association* 1983; **78**:894–900.
12. Xu H, Rahman S. A generalized dimension-reduction method for multi-dimensional integration in stochastic mechanics. *International Journal for Numerical Methods in Engineering* 2004; **61**:1992–2019.
13. Rahman S. A dimensional decomposition method for stochastic fracture mechanics. *Engineering Fracture Mechanics* 2006; **73**:2093–2109.
14. Rahman S. Stochastic dynamic systems with complex-valued eigensolutions. *International Journal for Numerical Methods in Engineering* 2007; **71**:963–986.
15. Wei D, Rahman S. Structural reliability analysis by univariate decomposition and numerical integration. *Probabilistic Engineering Mechanics* 2007; **22**:27–38.
16. Penmetsa R, Grandhi R. Efficient estimation of structural reliability for problems with uncertain intervals. *Computers and Structures* 2002; **80**:1103–1112.

 
DOI: 10.1002/cnm