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Decomposition methods for structural reliability analysis revisited

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ABSTRACT

This paper presents new theoretical results to demonstrate that the referential dimensional decomposition (RDD) and cut-high-dimensional model representation (cut-HDMR), each developed independently and from a distinct perspective, lead to identical function approximations. Therefore, the reliability method stemming from the cut-HDMR approximation is precisely the same as the reliability method rooted in the RDD approximation. However, a second-moment error analysis finds neither the RDD approximation nor the cut-HDMR approximation to be optimal, whereas the approximation derived from the analysis-of-variance dimensional decomposition (ADD) results in minimum error for an arbitrary truncation. The expected errors from the RDD approximations are at least four to eight times larger than the errors from the ADD approximations. Therefore, further enhancements of decomposition-based reliability methods are possible by switching from RDD to ADD approximations. For both approximations, the decomposition can be truncated by an effective superposition dimension linked to respective approximation errors.

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1. Introduction

Structural reliability analyses employing multivariate function approximations derived from dimensional decomposition [1] and cut-high-dimensional model representation (cut-HDMR) [2] have been reported by several researchers [1,3]. Both analyses conducted exploit the dimensional hierarchy lurking behind a probabilistic response, leading to accurate and efficient reliability estimates. However, there exist a few fundamental questions that remain unresolved: (1) Does there exist a relationship between the function approximations resulting from dimensional decomposition and cut-HDMR? If such a relationship exists, what is it? (2) What are the approximation qualities resulting from dimensional decomposition and cut-HDMR? Are they better or worse than approximations generated from alternative decompositions? How does one truncate such decompositions? Answering the first set of questions should illuminate the theoretical foundations of the reliability methods stemming from dimensional decomposition and cut-HDMR. Exploring the second set of questions should lead to error estimates and effective dimensions in calculating the probabilistic response characteristics by various decomposition methods.

This paper presents new theoretical analyses and results pertaining to dimensional decomposition that have important implications for structural reliability analysis. Section 2 provides a brief overview of dimensional decomposition of a multivariate function, followed by its three variants: referential dimension decomposition (RDD), cut-HDMR, and analysis-of-variance (ANOVA) dimensional decomposition (ADD). It presents new theorems and a lemma, revealing the relationship between the RDD and cut-HDMR approximations. Section 3 provides a new theorem for estimating approximation errors by various decomposition methods. Section 4 describes the effective dimension for truncating dimensional decomposition. Finally, Section 5 presents conclusions.

2. Dimensional decomposition

Dimensional decomposition of a multivariate function is a finite sum of simpler component functions of input variables with increasing dimensions. This decomposition, first presented by Hoeffding [4] in relation to his seminal work on U-statistics, has been applied by many other researchers [5]: Sobol [6] used it in the study of quadrature methods, calling it the "decomposition into summands of different dimensions" and also for ANOVA [7]; Efron and Stein [8] used it to prove their famous lemma on jackknife variances; Owen [9] presented a continuous space version of the nested ANOVA; and Hickernell [10] developed a reproducing kernel Hilbert space version. This decomposition has also been examined by Rabitz and Alis [2] for HDMR, resulting in notable contributions to function approximations [11,12], and recently, by Xu and Rahman [1] for reliability analysis. Takemura [13] provides a historical account, which reveals that the decomposition existed as early as the 1940s.

Consider a continuous, differentiable, real-valued, multivariate function $y(\boldsymbol{x})$ that depends on $\boldsymbol{x} = \{x_1, \ldots, x_N\}^T \in \mathbb{R}^N$, where $\mathbb{R}^N$ is an N-dimensional, real vector space. The dimensional decomposition represents a finite, hierarchical, convergent expansion of a

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multivariate output function [1,2,4,7,8]

$$y(\boldsymbol{x}) = y_0 + \sum_{i=1}^{N} y_i(x_i) + \sum_{i_1=1}^{N-1}\sum_{i_2=i_1+1}^{N} y_{i_1 i_2}(x_{i_1}, x_{i_2}) + \cdots + \sum_{i_1=1}^{N-s+1}\cdots\sum_{i_s=i_{s-1}+1}^{N} y_{i_1\cdots i_s}(x_{i_1}, \ldots, x_{i_s}) + \cdots + y_{12\cdots N}(x_1, \ldots, x_N) \tag{1}$$

in terms of input $\boldsymbol{x}$ with increasing dimensions, where y_0 is a constant and $y_{i_1\cdots i_s} : \mathbb{R}^s \to \mathbb{R}$, $1 \le s \le N$, $1 \le i_1 < \cdots < i_s \le N$, is an s-variate component function quantifying the cooperative effects of s input variables $x_{i_1}, \ldots, x_{i_s}$ on y. If

$$\hat{y}_S(\boldsymbol{x}) = y_0 + \sum_{i=1}^{N} y_i(x_i) + \sum_{i_1=1}^{N-1}\sum_{i_2=i_1+1}^{N} y_{i_1 i_2}(x_{i_1}, x_{i_2}) + \cdots + \sum_{i_1=1}^{N-S+1}\cdots\sum_{i_S=i_{S-1}+1}^{N} y_{i_1\cdots i_S}(x_{i_1}, \ldots, x_{i_S}) \tag{2}$$

represents a general S-variate approximation of $y(\boldsymbol{x})$, then the univariate ($S = 1$) and bivariate ($S = 2$) approximations, $\hat{y}_1(\boldsymbol{x})$ and $\hat{y}_2(\boldsymbol{x})$, respectively, provide two- and three-term approximants of the finite decomposition in Eq. (1). Similarly, trivariate and other higher-variate approximations can be derived by appropriately selecting the value of $S \le N$. The decomposition is useful only when the component functions exhibit insignificant s-variate effects cooperatively as $s \to N$.

2.1. RDD approximation

Let $\boldsymbol{c} = \{c_1, \ldots, c_N\}^T$ be a reference point of input $\boldsymbol{x}$ and $y(c_1, \ldots, c_{i_1-1}, x_{i_1}, c_{i_1+1}, \ldots, c_{i_{S-k}-1}, x_{i_{S-k}}, c_{i_{S-k}+1}, \ldots, c_N)$ represent an $(S-k)$th dimensional component function of $y(\boldsymbol{x})$, where $S < N$, $k = 0, \ldots, S$, and $1 \le i_1 < \cdots < i_{S-k} \le N$. For example, when $S = 1$, the zero-dimensional component function, which is a constant, is $y(\boldsymbol{c})$ and the one-dimensional component functions are $y(x_1, c_2, \ldots, c_N), y(c_1, x_2, \ldots, c_N), \ldots, y(c_1, c_2, \ldots, x_N)$.

Theorem 1. *For a multivariate function $y(\boldsymbol{x}) = y(x_1, \ldots, x_N)$, if*

$$\hat{y}_{S,R}(\boldsymbol{x}) := \sum_{k=0}^{S}(-1)^k \binom{N-S+k-1}{k} \times \sum_{i_1,\ldots,i_{S-k}=1; i_1<\cdots<i_{S-k}}^{N} y(c_1, \ldots, c_{i_1-1}, x_{i_1}, c_{i_1+1}, \ldots, c_{i_{S-k}-1}, x_{i_{S-k}}, c_{i_{S-k}+1}, \ldots, c_N) \tag{3}$$

represents an S-variate RDD approximation of $y(\boldsymbol{x})$, then $\hat{y}_{S,R}(\boldsymbol{x})$ consists of all terms of the Taylor series expansion of $y(\boldsymbol{x})$ at $\boldsymbol{c}$ that have less than or equal to S variables, i.e.,

$$\hat{y}_{S,R}(\boldsymbol{x}) = \sum_{k=0}^{S} t_k, \tag{4}$$

where

$$t_0 = y(\boldsymbol{c}),$$

$$t_k = \sum_{j_1,\ldots,j_k} \frac{1}{j_1!\cdots j_k!} \sum_{i_1,\ldots,i_k=1; i_1<\cdots<i_k}^{N} \frac{\partial^{j_1+\cdots+j_k} y}{\partial x_{i_1}^{j_1}\cdots\partial x_{i_k}^{j_k}}(\boldsymbol{c}) \times (x_{i_1} - c_{i_1})^{j_1}\cdots(x_{i_k} - c_{i_k})^{j_k}; \quad 1 \le k \le S. \tag{5}$$

Proof. See Xu and Rahman's paper ([14], pp. 1996–2000), which provides a proof when $\boldsymbol{c} = \boldsymbol{0}$ without any loss of generality. The method associated with the RDD approximation was simply called "decomposition method" by Xu and Rahman [1]. □

Remark 1. Theorem 1, defined as the multivariate function theorem by Xu and Rahman [14], implies that the RDD approximation $\hat{y}_{S,R}(\boldsymbol{x})$ in Eq. (3), when compared with the Taylor series expansion of $y(\boldsymbol{x})$, yields a residual error that includes only terms of dimensions $S+1$ and higher. All higher-order S- and lower-variate terms of $y(\boldsymbol{x})$ are included in Eq. (3), which should therefore generally provide a higher-order approximation of a multivariate function than equations derived from first- or second-order Taylor expansions. See the works of Xu and Rahman [1,14] for further details.

2.2. Cut-HDMR approximation

An important feature of the approximation in Eq. (2) is the selection of the constant y_0 and component functions $y_{i_1\cdots i_s}(x_{i_1}, \ldots, x_{i_s})$, $1 \le s \le S$. By defining an error functional associated with a given $y(\boldsymbol{x})$ and an appropriate kernel function, an optimization problem can be formulated and solved to obtain the desired component functions. In particular, a decomposition involving the Dirac measure $\prod_{i=1}^{N}\delta(x_i - c_i)$ at the reference point $\boldsymbol{c}$ as the kernel function leads to the S-order cut-HDMR approximation [2]

$$\hat{y}_{S,\text{cut-HDMR}}(\boldsymbol{x}) = y_0^R + \sum_{i=1}^{N} y_i^R(x_i) + \sum_{i_1=1}^{N-1}\sum_{i_2=i_1+1}^{N} y_{i_1 i_2}^R(x_{i_1}, x_{i_2}) + \cdots + \sum_{i_1=1}^{N-S+1}\cdots\sum_{i_S=i_{S-1}+1}^{N} y_{i_1\cdots i_S}^R(x_{i_1}, \ldots, x_{i_S}) \tag{6}$$

of $y(\boldsymbol{x})$, where

$$\begin{aligned}
&y_0^R := y(\boldsymbol{c})\\
&y_i^R(x_i) := y(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_N) - y_0^R\\
&y_{i_1 i_2}^R(x_{i_1}, x_{i_2})\\
&\quad := y(c_1, \ldots, c_{i_1-1}, x_{i_1}, c_{i_1+1}, \ldots, c_{i_2-1}, x_{i_2}, c_{i_2+1}, \ldots, c_N)\\
&\qquad - y_{i_1}^R(x_{i_1}) - y_{i_2}^R(x_{i_2}) - y_0^R\\
&\vdots\\
&y_{i_1\cdots i_S}^R(x_{i_1}, \ldots, x_{i_S})\\
&\quad := y(c_1, \ldots, c_{i_1-1}, x_{i_1}, c_{i_1+1}, \ldots, c_{i_S-1}, x_{i_S}, c_{i_S+1}, \ldots, c_N)\\
&\qquad - \sum_{j_1<\cdots<j_{S-1}\subset\{i_1,\ldots,i_S\}} y_{j_1\cdots j_{S-1}}^R(x_{j_1}, \ldots, x_{j_{S-1}})\\
&\qquad - \sum_{j_1<\cdots<j_{S-2}\subset\{i_1,\ldots,i_S\}} y_{j_1\cdots j_{S-2}}^R(x_{j_1}, \ldots, x_{j_{S-2}})\\
&\qquad - \cdots - \sum_{j\subset\{i_1,\ldots,i_S\}} y_j^R(x_j) - y_0^R
\end{aligned} \tag{7}$$

appears as Eq. (49) in the paper by Rabitz and Alis [2]. Note that $\hat{y}_{S,R}(\boldsymbol{x})$ from the RDD approximation and $\hat{y}_{S,\text{cut-HDMR}}(\boldsymbol{x})$ from the cut-HDMR approximation both follow the same structure, as shown in Eq. (2). However, due to distinct perspectives involved in developing the RDD and cut-HDMR approximations, it is unclear if a relationship exists between Eq. (3) and Eqs. (6), (7). Finding this relationship – a principal objective of this study – should provide important insights into the reliability methods rooted in RDD and cut-HDMR.

2.3. Relationship between RDD and cut-HDMR approximations

In this subsection, a new theorem and lemma, also proven herein, are presented to provide rigorous proof that indeed Eq. (3) and Eqs. (6), (7) lead to identical function approximations, as follows.

Lemma. *For any positive integers N, $S < N$, and $0 \le k \le S$,*

$$\sum_{k=0}^{S}(-1)^k \binom{N}{k} = (-1)^S \binom{N-1}{S}. \tag{8}$$

Proof. Setting $S = 1$, the left and right sides of Eq. (8) are both equal to $1 - N$, satisfying the basic step. For the inductive step, replace S with $S + 1$ in the left side of Eq. (8), yielding

$$\begin{aligned}\sum_{k=0}^{S+1}(-1)^k\binom{N}{k} &= \sum_{k=0}^{S}(-1)^k\binom{N}{k} + (-1)^{S+1}\binom{N}{S+1}\\ &= (-1)^S\binom{N-1}{S} + (-1)^{S+1}\binom{N}{S+1}\\ &= (-1)^{S+1}\left[-\binom{N-1}{S} + \binom{N}{S+1}\right]\\ &= (-1)^{S+1}\binom{N-1}{S+1}, \end{aligned} \tag{9}$$

where the last line follows from the Pascal's rule. Therefore, the lemma is proved by induction. □

Theorem 2. *For any positive integers N, $S < N$, the S-order cut-HDMR approximation $\hat{y}_{S,\text{cut-HDMR}}(\boldsymbol{x})$ of $y(\boldsymbol{x})$, where the component functions are defined in Eq. (7), is the same as the S-variate RDD approximation $\hat{y}_{S,R}(\boldsymbol{x})$, i.e., $\hat{y}_{S,\text{cut-HDMR}}(\boldsymbol{x}) = \hat{y}_{S,R}(\boldsymbol{x})$.*

Proof. Substituting the component functions from Eq. (7) into (6) yields

$$\begin{aligned}&\hat{y}_{S,\text{cut-HDMR}}(\boldsymbol{x})\\ &= y(\boldsymbol{c}) + \sum_{i=1}^{N}[y(c_1,\ldots,c_{i-1},x_i,c_{i+1},\ldots,c_N) - y(\boldsymbol{c})]\\ &\quad + \sum_{i_1=1}^{N-1}\sum_{i_2=i_1+1}^{N}\Bigg[y(c_1,\ldots,c_{i_1-1},x_{i_1},c_{i_1+1},\ldots,c_{i_2-1},x_{i_2},\\ &\quad c_{i_2+1},\ldots,c_N) - y^R_{i_1}(x_{i_1}) - y^R_{i_2}(x_{i_2}) - y(\boldsymbol{c})\Bigg] + \cdots\\ &\quad + \sum_{i_1=1}^{N-S+1}\cdots\sum_{i_S=i_{S-1}+1}^{N}\Bigg[y(c_1,\ldots,c_{i_1-1},x_{i_1},c_{i_1+1},\ldots,\\ &\quad c_{i_S-1},x_{i_S},c_{i_S+1},\ldots,c_N)\\ &\quad - \sum_{j_1<\cdots<j_{S-1}\subset\{i_1,\ldots,i_S\}} y^R_{j_1\cdots j_{S-1}}(x_{j_1},\ldots,x_{j_{S-1}})\\ &\quad - \sum_{j_1<\cdots<j_{S-2}\subset\{i_1,\ldots,i_S\}} y^R_{j_1\cdots j_{S-2}}(x_{j_1},\ldots,x_{j_{S-2}}) - \cdots\\ &\quad - \sum_{j\subset\{i_1,\ldots,i_S\}} y^R_j(x_j) - y(\boldsymbol{c})\Bigg], \end{aligned} \tag{10}$$

which, after rearrangement of terms, results in

$$\begin{aligned}\hat{y}_{S,\text{cut-HDMR}}(\boldsymbol{x}) &= \sum_{i_1=1}^{N-S+1}\cdots\sum_{i_S=i_{S-1}+1}^{N} y(c_1,\ldots,c_{i_1-1},x_{i_1},c_{i_1+1},\ldots,\\ &\quad c_{i_S-1},x_{i_S},c_{i_S+1},\ldots,c_N) + [1-(N-S+1)]\\ &\quad\times\sum_{i_1=1}^{N-S+2}\cdots\sum_{i_{S-1}=i_{S-2}+1}^{N} y(c_1,\ldots,c_{i_1-1},x_{i_1},\\ &\quad c_{i_1+1},\ldots,c_{i_{S-1}-1},x_{i_{S-1}},c_{i_{S-1}+1},\ldots,c_N) + \cdots\\ &\quad + \left[1 - N + \frac{N(N-1)}{2} - \cdots\right.\\ &\quad \left. + (-1)^S\frac{N(N-1)\cdots(N-S+1)}{S!}\right]y(\boldsymbol{c})\\ &= (-1)^0\sum_{i_1=1}^{N-S+1}\cdots\sum_{i_S=i_{S-1}+1}^{N} y(c_1,\ldots,c_{i_1-1},x_{i_1},\\ &\quad c_{i_1+1},\ldots,c_{i_S-1},x_{i_S},c_{i_S+1},\ldots,c_N)\\ &\quad + (-1)^1\binom{N-S}{1}\times\sum_{i_1=1}^{N-S+2}\cdots\sum_{i_{S-1}=i_{S-2}+1}^{N} y(c_1,\ldots,\\ &\quad c_{i_1-1},x_{i_1},c_{i_1+1},\ldots,c_{i_{S-1}-1},x_{i_{S-1}},c_{i_{S-1}+1},\ldots,c_N)\\ &\quad + \cdots + \sum_{k=0}^{S}(-1)^k\binom{N}{k}y(\boldsymbol{c}). \end{aligned} \tag{11}$$

Using the lemma, *i.e.*, Eq. (8),

$$\begin{aligned}\hat{y}_{S,\text{cut-HDMR}}(\boldsymbol{x}) &= (-1)^0\sum_{i_1=1}^{N-S+1}\cdots\sum_{i_S=i_{S-1}+1}^{N} y(c_1,\ldots,c_{i_1-1},x_{i_1},\\ &\quad c_{i_1+1},\ldots,c_{i_S-1},x_{i_S},c_{i_S+1},\ldots,c_N)\\ &\quad + (-1)^1\binom{N-S}{1}\times\sum_{i_1=1}^{N-S+2}\cdots\sum_{i_{S-1}=i_{S-2}+1}^{N} y(c_1,\ldots,\\ &\quad c_{i_1-1},x_{i_1},c_{i_1+1},\ldots,c_{i_{S-1}-1},x_{i_{S-1}},c_{i_{S-1}+1},\ldots,c_N)\\ &\quad + \cdots + (-1)^S\binom{N-1}{S}y(\boldsymbol{c})\\ &= \sum_{k=0}^{S}(-1)^k\binom{N-S+k-1}{k}\\ &\quad\times\sum_{i_1,\ldots,i_{S-k}=1;i_1<\cdots<i_{S-k}}^{N} y(c_1,\ldots,c_{i_1-1},x_{i_1},\\ &\quad c_{i_1+1},\ldots,c_{i_{S-k}-1},x_{i_{S-k}},c_{i_{S-k}+1},\ldots,c_N)\\ &=: \hat{y}_{S,R}(\boldsymbol{x}), \end{aligned} \tag{12}$$

which proves the theorem for an arbitrary $S < N$. □

Remark 2. Although the RDD and cut-HDMR approximations are equivalent, the former was developed independently and from a completely different perspective. Xu and Rahman [14] provided rigorous proof of the multivariate function decomposition theorem (Theorem 1) and a generalized formulation that cannot be found in the HDMR literature.

Remark 3. If the input is random, say, an N-dimensional random vector $\boldsymbol{X} = \{X_1,\ldots,X_N\}^T \in \mathbb{R}^N$, then the output response $y(\boldsymbol{X})$ is also random. When $S = 1$ and $S = 2$, Eq. (3) or (6) degenerates to the univariate RDD approximation

$$\begin{aligned}\hat{y}_{1,R}(\boldsymbol{X}) &= \hat{y}_{1,\text{cut-HDMR}}(\boldsymbol{X})\\ &= \sum_{i=1}^{N} y(c_1,\ldots,c_{i-1},X_i,c_{i+1},\ldots,c_N) - (N-1)y(\boldsymbol{c}) \end{aligned} \tag{13}$$

and the bivariate RDD approximation

$$\begin{aligned}\hat{y}_{2,R}(\boldsymbol{X}) &= \hat{y}_{2,\text{cut-HDMR}}(\boldsymbol{X})\\ &= \sum_{i_1=1}^{N-1}\sum_{i_2=i_1+1}^{N} y(c_1,\ldots,c_{i_1-1},X_{i_1},c_{i_1+1},\ldots,c_{i_2-1},\\ &\quad X_{i_2},c_{i_2+1},\ldots,c_N) - (N-2)\sum_{i=1}^{N} y(c_1,\ldots,c_{i-1},\\ &\quad X_i,c_{i+1},\ldots,c_N) + \frac{(N-1)(N-2)}{2}y(\boldsymbol{c}), \end{aligned} \tag{14}$$

respectively. Structural reliability analyses involving $y(\boldsymbol{X})$ as a performance function and employing its surrogates $\hat{y}_{1,R}(\boldsymbol{X})$ or

$\hat{y}_{2,R}(\boldsymbol{X})$ and $\hat{y}_{1,\text{cut-HDMR}}(\boldsymbol{X})$ or $\hat{y}_{2,\text{cut-HDMR}}(\boldsymbol{X})$, as conducted by Xu and Rahman [1] and Chowdhury and Rao [3], respectively, are nearly identical. The only difference is the use of moving least-squares approximation [3] as opposed to Lagrange interpolation [1] to approximate the component functions of the decomposition. Otherwise, the reliability method stemming from the cut-HDMR approximation is precisely the same as the reliability method rooted in the RDD approximation.

Remark 4. Since the right side of Eq. (13) comprises only univariate functions, the interpolation or integration of $\hat{y}_{1,R}(\boldsymbol{X})$ is essentially univariate. Similarly, the right side of Eq. (14), which contains at most bivariate functions, requires at most bivariate interpolation or integration of $\hat{y}_{2,R}(\boldsymbol{X})$. Therefore, appellation of the terms "univariate decomposition method" and "bivariate decomposition method" for approximations resulting from $\hat{y}_{1,R}(\boldsymbol{X})$ in Eq. (13) and $\hat{y}_{2,R}(\boldsymbol{X})$ in Eq. (14), respectively, is more appropriate than referring to them as first-order and second-order methods. Note that the component functions embedded in the expressions of $\hat{y}_{1,R}(\boldsymbol{X})$ and/or $\hat{y}_{2,R}(\boldsymbol{X})$ are generally nonlinear. Moreover, a univariate or bivariate approximation may contain very high-order (*i.e.*, higher than first- or second-order) terms, depending on the nonlinearity of the response. Therefore, characterizing these approximations by first- and second-order methods is confusing and possibly inaccurate based on the traditional definition of the order of a function.

3. Error analysis

3.1. ADD approximation

If $\boldsymbol{X}$ comprises independent components X_j with marginal probability density functions $f_j(x_j), j = 1, \ldots, N$, then by defining its joint probability density function $f_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{j=1}^{j=N} f_j(x_j)$ as the kernel of error minimization results in the S-variate ADD approximation [7]

$$\hat{y}_{S,A}(\boldsymbol{x}) = y_0^A + \sum_{i=1}^{N} y_i^A(x_i) + \sum_{i_1=1}^{N-1}\sum_{i_2=i_1+1}^{N} y_{i_1 i_2}^A(x_{i_1}, x_{i_2}) + \cdots + \sum_{i_1=1}^{N-S+1}\cdots\sum_{i_S=i_{S-1}+1}^{N} y_{i_1\cdots i_S}^A(x_{i_1}, \ldots, x_{i_S}) \tag{15}$$

of $y(\boldsymbol{x})$, where the constant y_0^A and component functions $y_{i_1\cdots i_S}^A(x_{i_1}, \ldots, x_{i_S}), S = 1, \ldots, N$, defined by

$$\begin{aligned}
y_0^A &:= \int_{\mathbb{R}^N} y(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\\
y_i^A(x_i) &:= \int_{\mathbb{R}^{N-1}} y(\boldsymbol{x}) \prod_{j\neq i} f_j(x_j)\mathrm{d}x_j - y_0^A\\
y_{i_1 i_2}^A(x_{i_1}, x_{i_2}) &:= \int_{\mathbb{R}^{N-2}} y(\boldsymbol{x}) \prod_{j\neq[i_1,i_2]} f_j(x_j)\mathrm{d}x_j - y_{i_1}^A(x_{i_1}) - y_{i_2}^A(x_{i_2}) - y_0^A\\
&\vdots\\
y_{i_1\cdots i_S}^A(x_{i_1}, \ldots, x_{i_S}) &:= \int_{\mathbb{R}^{N-S}} y(\boldsymbol{x}) \prod_{j\neq[i_1,\ldots,i_S]} f_j(x_j)\mathrm{d}x_j\\
&\quad - \sum_{j_1<\cdots<j_{S-1}\subset\{i_1,\ldots,i_S\}} y_{j_1\cdots j_{S-1}}^A(x_{j_1}, \ldots, x_{j_{S-1}})\\
&\quad - \sum_{j_1<\cdots<j_{S-2}\subset\{i_1,\ldots,i_S\}} y_{j_1\cdots j_{S-2}}^A(x_{j_1}, \ldots, x_{j_{S-2}})\\
&\quad - \cdots - \sum_{j\subset\{i_1,\ldots,i_S\}} y_j^A(x_j) - y_0^A,
\end{aligned} \tag{16}$$

also form the same structure of Eq. (2). If $\mathbb{E}$ is the expectation operator with respect to $f_{\boldsymbol{X}}(\boldsymbol{x})$, then two important properties of the ANOVA decomposition required for uniquely defining its component functions are as follows.

Property 1. *The ANOVA component functions* $y_{i_1\cdots i_s}^A(X_{i_1}, \ldots, X_{i_s})$, $s = 1, \ldots, N$, *have zero means, i.e.,*

$$\mathbb{E}\left[y_{i_1\cdots i_s}^A(X_{i_1}, \ldots, X_{i_s})\right] := \int_{\mathbb{R}^N} y_{i_1\cdots i_s}^A(x_{i_1}, \ldots, x_{i_s}) f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0. \tag{17}$$

Property 2. *Two distinct ANOVA component functions* $y_{i_1\cdots i_s}^A(X_{i_1}, \ldots, X_{i_s})$ *and* $y_{i_1\cdots i_t}^A(X_{i_1}, \ldots, X_{i_t})$, *where* $1 \le s \le N$ *and* $1 \le t \le N$ *are two distinct integers, are uncorrelated, i.e., they satisfy the orthogonality property*

$$\begin{aligned}
&\mathbb{E}\left[y_{i_1\cdots i_s}^A(X_{i_1}, \ldots, X_{i_s}) y_{i_1\cdots i_t}^A(X_{i_1}, \ldots, X_{i_t})\right]\\
&\quad := \int_{\mathbb{R}^N} y_{i_1\cdots i_s}^A(x_{i_1}, \ldots, x_{i_s}) y_{i_1\cdots i_t}^A(x_{i_1}, \ldots, x_{i_t}) f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0.
\end{aligned} \tag{18}$$

Traditionally, Eqs. (15) and (16) with $X_j, j = 1, \ldots, N$, following independent, standard uniform distributions, have been identified as the ANOVA decomposition [7] or ANOVA-HDMR [2]; however, the author's recent work [15] reveals no fundamental requirement for a specific probability measure of $\boldsymbol{X}$, provided that the resultant integrals in Eq. (16) exist and are finite. It is important to note that no orthogonality property exists in the RDD component functions for an arbitrary distribution of $\boldsymbol{X}$.

3.2. Second-moment errors

The S-variate ($S < N$) approximations from RDD and ADD both represent sums of lower-dimensional component functions of $y(\boldsymbol{X})$. An obvious question is how to quantify the errors in calculating the statistical moments (*e.g.*, variance) of $y(\boldsymbol{X})$ by employing these two approximations. The ADD has desirable orthogonal properties, but the ANOVA component functions are difficult to compute due to the high-dimensional integrals involved. The RDD is theoretically less attractive, but the component functions are much easier to obtain. Nonetheless, one must be able to estimate the errors from both approximations and determine if one is better than the other. For RDD, an additional question arises regarding the reference point, which, if improperly selected, can spoil the approximation.

3.2.1. ADD approximation error

Applying the expectation operator on $y(\boldsymbol{X})$ and $\hat{y}_{S,A}(\boldsymbol{X})$ from Eqs. (1) and (15), respectively, and noting Property 1, the mean $\mathbb{E}\left[\hat{y}_{S,A}(\boldsymbol{X})\right] = y_0^A$ of the S-variate ADD approximation matches the exact mean $\mathbb{E}[y(\boldsymbol{X})] := \int_{\mathbb{R}^N} y(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = y_0^A$, regardless of S. Applying the expectation operator again, this time on $\left(\hat{y}_{S,A}(\boldsymbol{X}) - y_0^A\right)^2$, and recognizing Property 2 results in the variance

$$\begin{aligned}
\hat{\sigma}_{S,A}^2 &:= \mathbb{E}\left[\left(\hat{y}_{S,A}(\boldsymbol{X}) - y_0^A\right)^2\right]\\
&= \sum_{i=1}^{N} \sigma_i^2 + \sum_{i_1=1}^{N-1}\sum_{i_2=i_1+1}^{N} \sigma_{i_1 i_2}^2 + \cdots + \sum_{i_1=1}^{N-S+1}\cdots\sum_{i_S=i_{S-1}+1}^{N} \sigma_{i_1\cdots i_S}^2
\end{aligned} \tag{19}$$

of the S-variate ADD approximation, where $\sigma_{i_1\cdots i_S}^2 = \mathbb{E}\left[\left(y_{i_1\cdots i_S}^A(X_{i_1}, \ldots, X_{i_S})\right)^2\right]$ represents the corresponding contribution to the

partial sum of variance terms. Clearly, the approximate variance in Eq. (19) approaches the exact variance

$$\sigma^2 := \mathbb{E}\left[\left(y(\boldsymbol{X}) - y_0^A\right)^2\right] = \sum_{i=1}^{N}\sigma_i^2 + \sum_{i_1=1}^{N-1}\sum_{i_2=i_1+1}^{N}\sigma_{i_1 i_2}^2 + \cdots + \sum_{i_1=1}^{N-s+1}\cdots\sum_{i_s=i_{s-1}+1}^{N}\sigma_{i_1\cdots i_s}^2 + \cdots + \sigma_{12\cdots N}^2, \tag{20}$$

the total sum of variance terms, when $S \to N$. The mean-square convergence of $\hat{y}_{S,A}$ is guaranteed as y and its component functions are all members of the associated Hilbert spaces.

Define an error

$$e_{S,A} := \mathbb{E}\left[\left(y(\boldsymbol{X}) - \hat{y}_{S,A}(\boldsymbol{X})\right)^2\right] = \int_{\mathbb{R}^N}\left[y(\boldsymbol{x}) - \hat{y}_{S,A}(\boldsymbol{x})\right]^2 f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \tag{21}$$

representing the second moment of the difference between $y(\boldsymbol{X})$ and $\hat{y}_{S,A}(\boldsymbol{X})$. Replacing y and $\hat{y}_{S,A}$ in Eq. (21) with the right sides of Eqs. (1) and (15), respectively, and then recognizing both Properties 1 and 2 yields the ADD error

$$e_{S,A} = \sum_{i_1=1}^{N-S}\cdots\sum_{i_{S+1}=i_S+1}^{N}\sigma_{i_1\cdots i_{S+1}}^2 + \cdots + \sigma_{12\cdots N}^2, \tag{22}$$

which completely eliminates the variance terms associated with S- and all lower-variate contributions, an attractive property of the ANOVA decomposition. By setting $S = 1, 2, \ldots$, the error can be easily determined for any truncation of ADD. For instance, when $N = 3$, the errors from the univariate ($S = 1$) and bivariate ($S = 2$) ADD approximations become

$$e_{1,A} = \sum_{i_1=1}^{2}\sum_{i_2=i_1+1}^{3}\sigma_{i_1 i_2}^2 + \sigma_{123}^2 = \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2 + \sigma_{123}^2 \tag{23}$$

and

$$e_{2,A} = \sigma_{123}^2, \tag{24}$$

respectively.

Consider a generic S-variate approximation $\hat{y}_S(\boldsymbol{x})$ of $y(\boldsymbol{x})$ other than the ANOVA approximation $\hat{y}_{S,A}(\boldsymbol{x})$. Since $y(\boldsymbol{x}) - \hat{y}_{S,A}(\boldsymbol{x})$ only contains higher than S-variate terms and $\hat{y}_{S,A}(\boldsymbol{x}) - \hat{y}_S(\boldsymbol{x})$ contains at most S-variate terms, the random variables $y(\boldsymbol{X}) - \hat{y}_{S,A}(\boldsymbol{X})$ and $\hat{y}_{S,A}(\boldsymbol{X}) - \hat{y}_S(\boldsymbol{X})$ are uncorrelated, *i.e.*, $\mathbb{E}[(y(\boldsymbol{X}) - \hat{y}_{S,A}(\boldsymbol{X}))(\hat{y}_{S,A}(\boldsymbol{X}) - \hat{y}_S(\boldsymbol{X}))] = 0$. Therefore, the second-moment error from any S-variate approximation is

$$e_S := \mathbb{E}\left[\left(y(\boldsymbol{X}) - \hat{y}_S(\boldsymbol{X})\right)^2\right] = \mathbb{E}\left[\left(y(\boldsymbol{X}) - \hat{y}_{S,A}(\boldsymbol{X})\right)^2\right] + \mathbb{E}\left[\left(\hat{y}_{S,A}(\boldsymbol{X}) - \hat{y}_S(\boldsymbol{X})\right)^2\right] = e_{S,A} + \mathbb{E}\left[\left(\hat{y}_{S,A}(\boldsymbol{X}) - \hat{y}_S(\boldsymbol{X})\right)^2\right] \geq e_{S,A}, \tag{25}$$

i.e., the S-variate ADD approximation is optimal. In other words, the approximation error derived from an RDD, regardless of how the reference point is selected, will always be equal to or greater than the ANOVA approximation error. Further details of the RDD approximation error are described as follows.

3.2.2. RDD approximation error

Following similar consideration, define another error

$$e_{S,R} := \mathbb{E}\left[\left(y(\boldsymbol{X}) - \hat{y}_{S,R}(\boldsymbol{X})\right)^2\right] = \int_{\mathbb{R}^N}\left[y(\boldsymbol{x}) - \hat{y}_{S,R}(\boldsymbol{x})\right]^2 f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{26}$$

associated with the S-variate RDD approximation $\hat{y}_{S,R}(\boldsymbol{X})$. Employing Theorem 1, *i.e.*, Eq. (3) to replace $\hat{y}_{S,R}$ in Eq. (26), produces

$$e_{S,R} = \int_{\mathbb{R}^N}\left[y(\boldsymbol{x}) - \sum_{k=0}^{S}(-1)^k\binom{N-S+k-1}{k}\times\sum_{i_1,\ldots,i_{S-k}=1;i_1<\cdots<i_{S-k}}^{N} y(c_1,\ldots,c_{i_1-1},x_{i_1},c_{i_1+1},\ldots, c_{i_{S-k}-1},x_{i_{S-k}},c_{i_{S-k}+1},\ldots,c_N)\right]^2 f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \tag{27}$$

which depends on the reference point $\boldsymbol{c}$. Wang [16] suggested choosing a random reference point uniformly distributed over $[0, 1]^N$ and then calculating the error on average. But, $\boldsymbol{X}$ defined here may follow an arbitrary probability distribution $f_{\boldsymbol{X}}(\boldsymbol{x})$; therefore, selecting the reference point characterized by the probability density $f_{\boldsymbol{X}}(\boldsymbol{c})$ is more appropriate, which leads to

$$\mathbb{E}\left[e_{S,R}\right] = \int_{\mathbb{R}^N} e_{S,R} f_{\boldsymbol{X}}(\boldsymbol{c})\mathrm{d}\boldsymbol{c} = \int_{\mathbb{R}^{2N}}\left[y(\boldsymbol{x}) - \sum_{k=0}^{S}(-1)^k\binom{N-S+k-1}{k}\times\sum_{i_1,\ldots,i_{S-k}=1;i_1<\cdots<i_{S-k}}^{N} y(c_1,\ldots,c_{i_1-1},x_{i_1},c_{i_1+1},\ldots, c_{i_{S-k}-1},x_{i_{S-k}},c_{i_{S-k}+1},\ldots,c_N)\right]^2 f_{\boldsymbol{X}}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{c})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{c}, \tag{28}$$

as the expected value of the RDD error. Simplifying Eq. (28) in terms of the variance components for an arbitrary S and/or N is a formidable task. Instead, an adequately small value of $N = 3$ was selected to derive explicit expressions of the expected errors from the univariate ($S = 1$) and bivariate ($S = 2$) RDD approximations as follows.

Theorem 3. *For $N = 3$, $S = 1$ or 2, if the joint probability density of the reference point is $f_{\boldsymbol{X}}(\boldsymbol{c}) = \prod_{j=1}^{j=3} f_j(c_j)$, where $f_j(c_j)$ is the marginal density of its jth coordinate, then the expected errors from the univariate and bivariate RDD approximations are*

$$\mathbb{E}\left[e_{1,R}\right] = 4\left(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2\right) + 8\sigma_{123}^2 \tag{29}$$

and

$$\mathbb{E}\left[e_{2,R}\right] = 8\sigma_{123}^2, \tag{30}$$

respectively, where $\sigma_{i_1 i_2}^2 = \mathbb{E}\left[\left(y_{i_1 i_2}^A(X_{i_1}, X_{i_2})\right)^2\right]$, $i_1 = 1, 2, 3$, $i_2 = i_1 + 1, \ldots, 3$, and $\sigma_{123}^2 = \mathbb{E}\left[\left(y_{123}^A(X_1, X_2, X_3)\right)^2\right]$.

Proof. Setting $N = 3$ in Eq. (26) and noting independent coordinates of $\boldsymbol{X}$,

$$e_{S,R} = \int_{\mathbb{R}^3}\left[y(x_1,x_2,x_3) - \hat{y}_{S,R}(x_1,x_2,x_3)\right]^2\prod_{i=1}^{3} f_i(x_i)\mathrm{d}x_i. \tag{31}$$

Expanding the square inside the integral of Eq. (31) and then integrating both sides with respect to the reference point with

density $f_{\boldsymbol{X}}(\boldsymbol{c}) = \prod_{j=1}^{j=3} f_j(c_j)$ over $\mathbb{R}^3$,

$$\begin{aligned}\mathbb{E}\left[e_{S,R}\right] = &\int_{\mathbb{R}^6} y^2(x_1,x_2,x_3)\prod_{i=1}^{3} f_i(x_i)\mathrm{d}x_i\prod_{j=1}^{3} f_j(c_j)\mathrm{d}c_j\\ &+\int_{\mathbb{R}^6}\hat{y}^2_{S,R}(x_1,x_2,x_3)\prod_{i=1}^{3} f_i(x_i)\mathrm{d}x_i\prod_{j=1}^{3} f_j(c_j)\mathrm{d}c_j\\ &-2\int_{\mathbb{R}^6} y(x_1,x_2,x_3)\hat{y}_{S,R}(x_1,x_2,x_3)\\ &\times\prod_{i=1}^{3} f_i(x_i)\mathrm{d}x_i\prod_{j=1}^{3} f_j(c_j)\mathrm{d}c_j. \end{aligned}\tag{32}$$

The first term on the right side of Eq. (32) expressed in terms of the variance components,

$$\begin{aligned}&\int_{\mathbb{R}^6} y^2(x_1,x_2,x_3)\prod_{i=1}^{3} f_i(x_i)\mathrm{d}x_i\prod_{j=1}^{3} f_j(c_j)\mathrm{d}c_j = \sigma^2+\left(y_0^A\right)^2\\ &=\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_{12}^2+\sigma_{13}^2+\sigma_{23}^2+\sigma_{123}^2+\left(y_0^A\right)^2,\end{aligned}\tag{33}$$

is independent of S. However, similar expressions of the remaining terms depend on S and require exploiting Sobol's formula [17],

$$\begin{aligned}\sum_{v\subseteq u}\sigma_v^2 = &\int_{\mathbb{R}^{6-|u|}} y(x_1,x_2,x_3)y(\boldsymbol{x}_u,\boldsymbol{c}_{-u})\\ &\times\prod_{i=1}^{3} f_i(x_i)\mathrm{d}x_i\prod_{j\subseteq -u} f_j(c_j)\mathrm{d}c_j-\left(y_0^A\right)^2,\end{aligned}\tag{34}$$

for their derivations, where $\emptyset\neq u\subseteq\{1,\ldots,3\}$ is an arbitrary subset with the complementary set $-u$ and cardinality $1\leq|u|\leq 3$ and $(\boldsymbol{x}_u,\boldsymbol{c}_{-u})$ denotes a three-dimensional vector whose ith component is x_i if $i\in u$ and is c_i if $i\notin u$. Setting $S=1$ (or $S=2$), replacing $\hat{y}_{1,R}$ (or $\hat{y}_{2,R}$) with the right side of Eq. (13) (or Eq. (14)) for $N=3$, and then employing Sobol's formula (Eq. (34)) for $u=\{j\}$, $u=-\{j\}$, and $u=-\{i,j\}$, where $i=1,2,3$ and $j=1,2,3$, the second and third terms on the right side of Eq. (32) respectively become

$$\begin{aligned}&\int_{\mathbb{R}^6}\hat{y}^2_{1,R}(x_1,x_2,x_3)\prod_{i=1}^{3} f_i(x_i)\mathrm{d}x_i\prod_{j=1}^{3} f_j(c_j)\mathrm{d}c_j\\ &=\sigma_1^2+\sigma_2^2+\sigma_3^2+3\left(\sigma_{12}^2+\sigma_{13}^2+\sigma_{23}^2\right)+7\sigma_{123}^2+\left(y_0^A\right)^2\end{aligned}\tag{35}$$

and

$$\begin{aligned}&-2\int_{\mathbb{R}^6} y(x_1,x_2,x_3)\hat{y}_{1,R}(x_1,x_2,x_3)\prod_{i=1}^{3} f_i(x_i)\mathrm{d}x_i\prod_{j=1}^{3} f_j(c_j)\mathrm{d}c_j\\ &=-2\left[\sigma_1^2+\sigma_2^2+\sigma_3^2+\left(y_0^A\right)^2\right]\end{aligned}\tag{36}$$

for $S=1$ and

$$\begin{aligned}&\int_{\mathbb{R}^6}\hat{y}^2_{2,R}(x_1,x_2,x_3)\prod_{i=1}^{3} f_i(x_i)\mathrm{d}x_i\prod_{j=1}^{3} f_j(c_j)\mathrm{d}c_j\\ &=\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_{12}^2+\sigma_{13}^2+\sigma_{23}^2+7\sigma_{123}^2+\left(y_0^A\right)^2\end{aligned}\tag{37}$$

and

$$\begin{aligned}&-2\int_{\mathbb{R}^6} y(x_1,x_2,x_3)\hat{y}_{2,R}(x_1,x_2,x_3)\prod_{i=1}^{3} f_i(x_i)\mathrm{d}x_i\prod_{j=1}^{3} f_j(c_j)\mathrm{d}c_j\\ &=-2\left[\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_{12}^2+\sigma_{13}^2+\sigma_{23}^2+\left(y_0^A\right)^2\right]\end{aligned}\tag{38}$$

for $S=2$. Adding all variance terms in Eqs. (33), (35) and (36) and Eqs. (33), (37) and (38) yields Eqs. (29) and (30), respectively. Hence, the theorem is proven. □

Remark 5. Theorem 3 demonstrates that on average the error from the S-variate RDD approximation eliminates the variance terms associated with the S- and all lower-variate contributions. The S-variate ADD approximation also satisfies this important property. However, the coefficients of higher-variate terms in the RDD error are larger than unity, implying greater errors from the RDD approximations than from the ADD approximations.

Remark 6. Comparing Eqs. (23) and (29), the expected error from the univariate RDD approximation is at least four times larger than the error from the univariate ADD approximation. In contrast, Eqs. (24) and (30) reveal the expected error from the bivariate RDD approximation to be eight times larger than the error from the bivariate ADD approximation. The factor by which the expected RDD error is larger than the ADD error rises quickly with respect to the truncation S.

Remark 7. Theorem 3 and Remarks 1 and 2 are strictly valid when $N=3$. For $N>3$, the errors from the univariate and bivariate RDD or ADD approximations will have additional terms, including higher-variate contributions. Nonetheless, the error analysis presented here is adequate for obtaining the lower bounds, *i.e.*,

$$\begin{aligned}&\mathbb{E}\left[e_{1,R}\right]\geq 4e_{1,A}\quad\text{(univariate)},\\ &\mathbb{E}\left[e_{2,R}\right]\geq 8e_{2,A}\quad\text{(bivariate)},\end{aligned}\tag{39}$$

which are useful for weighing the RDD approximations with respect to the ADD approximations for an arbitrarily large number of random variables.

For a general stochastic response with arbitrary N and $S<N$, no simple expression exists for the expected error from the S-variate RDD approximation. However, if the response function is separable, admitting a multiplicative decomposition of functions of random variables, then a simple expression for the expected error is still possible [16]. Unfortunately, it is of little value as most responses cannot be described by separable functions.

4. Truncation of dimensional decomposition

When employing dimensional decomposition for stochastic analysis, an important decision is how to select the truncation parameter S for approximating the performance function $y(\boldsymbol{X})$. For square-integrable performance functions, the truncation can be achieved by employing the notion of an effective dimension. Caflisch et al. [18] were the first to formally define two such effective dimensions, exploiting a low effective dimension to explain why the quasi-Monte Carlo method beats the crude Monte Carlo algorithm for evaluating a certain class of high-dimensional integrals.

4.1. Effective dimension

Let $\emptyset\neq u\subseteq\{1,\ldots,N\}$ be a subset with the complementary set $-u=\{1,\ldots,N\}-u$ and cardinality $1\leq|u|\leq N$. If $y_u^A(\boldsymbol{X}_u)$ is a $|u|$-variate, zero-mean ANOVA component function describing the cooperative influence of $\boldsymbol{X}_u$, a subvector of $\boldsymbol{X}$, on y, then its component variance is $\sigma_u^2=\mathbb{E}\left[\left(y_u^A(\boldsymbol{X}_u)\right)^2\right]$. The following two notions of effective dimension derived from ADD were introduced by Caflisch et al. [18].

Definition 1. The function y has effective dimension $S_s\leq N$ in the superposition sense, henceforth denoted as the superposition

dimension, if

$$S_s := \min\left\{S : 1 \le S \le N \text{ such that } \sum_{1\le|u|\le S} \sigma_u^2 \ge p\sigma^2\right\} \tag{40}$$

and the function y has effective dimension $S_t \le N$ in the truncation sense, henceforth denoted as the truncation dimension, if

$$S_t := \min\left\{S : 1 \le S \le N \text{ such that } \sum_{u\subseteq\{1,\ldots,S\}} \sigma_u^2 \ge p\sigma^2\right\}, \tag{41}$$

where $\sigma^2 := \mathbb{E}\left[\left(y(\boldsymbol{X}) - y_0^A\right)^2\right]$ is the total variance and $0 \le p \le 1$ is a percentile threshold close to *one*.

Both definitions capture the notion in which y is almost S_s- or S_t-dimensional. Caflisch et al. [18] used the 99th percentile for p, but it can be treated as a threshold parameter linked to the desired accuracy of a stochastic solution. The relevance of the truncation or superposition dimension depends on the nature of the performance function. The former signifies the number of important random variables and is appropriate when some variables are more important than others in an ordered set. In contrast, the latter determines whether the low-variate component functions of dimensional decomposition dominate the function and is appropriate when all variables are equally important. For truly high-dimensional problems, all variables contribute to a function value; therefore, the superposition dimension is more useful than the truncation dimension.

4.2. Superposition dimension

Note that definitions of effective dimensions in Eqs. (40) and (41) can be generalized for any dimensional decomposition, including RDD. Furthermore, it is possible to link the effective superposition dimension to the second-moment errors introduced in the preceding section. To do so, consider the errors $e_{S,A}$ and $\mathbb{E}\left[e_{S,R}\right]$ associated with the S-variate ADD and RDD approximations, respectively, of $y(\boldsymbol{X})$. An alternative set of definitions of the effective superposition dimension is as follows.

Definition 2. The function y represented by ADD has effective superposition dimension $S_{s,A} \le N$ if

$$S_{s,A} := \min\left\{S : 1 \le S \le N \text{ such that } e_{S,A} \le (1-p)\sigma^2\right\} \tag{42}$$

and the function y represented by RDD has effective superposition dimension $S_{s,R} \le N$ if

$$S_{s,R} := \min\left\{S : 1 \le S \le N \text{ such that } \mathbb{E}\left[e_{S,R}\right] \le (1-p)\sigma^2\right\}. \tag{43}$$

Remark 8. For the ADD approximation, the ANOVA component functions satisfy desirable theoretical properties (Properties 1 and 2). In that case, the variance terms simplify, for instance, $\sum_{1\le|u|\le S} \sigma_u^2 = \hat{\sigma}_{S,A}^2$, which is the S-variate ADD approximation of the variance in Eq. (19). Comparing Eqs. (19), (20) and (22), $e_{S,A} = \sigma^2 - \hat{\sigma}_{S,A}^2$. Therefore, the definitions of the superposition dimension expressed by Eqs. (40) and (42) are equivalent. No such analogy exists for RDD or other variants of dimensional decomposition.

The error analysis and effective dimensions presented in this paper pertain to only second-moment characteristics of $y(\boldsymbol{X})$. Similar analyses or definitions aimed at higher-order moments or probability distribution of y can be envisioned, but no closed-form solutions and simple expressions are possible. However, if y satisfies the requirements of the Chebyshev inequality or its descendants – a condition fulfilled by many realistic performance functions – then the results and findings from Sections 3 and 4 provide useful information that can be effectively exploited for decomposition-based reliability analysis.

5. Conclusions

This paper presents a new theorem and lemma, also proven herein, to demonstrate that the truncations of RDD and cut-HDMR, which can be selected arbitrarily, lead to identical function approximations. Although RDD and cut-HDMR approximations are equivalent, the former was developed independently and from a completely different perspective, including a generalized formulation that cannot be found in the HDMR literature. The importance of this finding is that the reliability method stemming from the cut-HDMR approximation is precisely the same as the reliability method rooted in the RDD approximation.

A second-moment error analysis, generating an additional new theorem and its proof, was conducted to compare the RDD and ADD approximations. The analysis finds the ADD approximation to be optimal, resulting in minimum error and, therefore, revealing the superiority of ADD over RDD approximations for an arbitrary truncation of the decomposition. On average the error from the univariate RDD approximation is at least four times larger than the error from the univariate ADD approximation. In contrast, the expected error from the bivariate RDD approximation is at least eight times larger than the error from the bivariate ADD approximation. Therefore, the accuracy of existing decomposition-based reliability methods should improve significantly by replacing RDD with ADD approximations. For both approximations, the decomposition can be truncated by an effective superposition dimension linked to respective approximation errors.

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