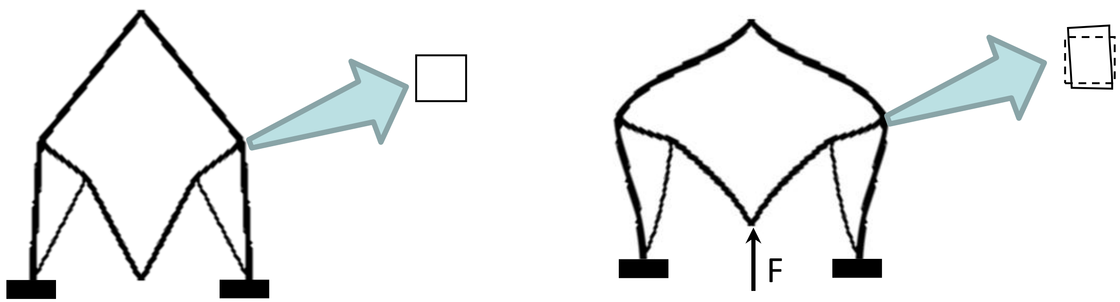
Period #5: Strain

A. Context

Structural mechanics deals with the forces placed upon mechanical systems and the resulting deformations of the system.

Solid mechanics, on the smaller scale, relates stresses in the material to the strains in the materials that comprise the structure.



(a) (b)

Fig. 5.1. (a) unloaded, undeformed structure; and (b) loaded structure with visible deformation.

In this period, we'll introduce the concept of strain and the strain tensor.

B. Normal Strain

Normal strain measures the change in length per unit length of an infinitesimally small "fiber" of material.

Consider the deformation of a rubber ball that is being compressed as shown.

If we track the infinitesimal element dA_0 of dimensions dx_o by dy_0, it maps to the deformed infinitesimal element dA of dimensions dx by dy in the deformed body.

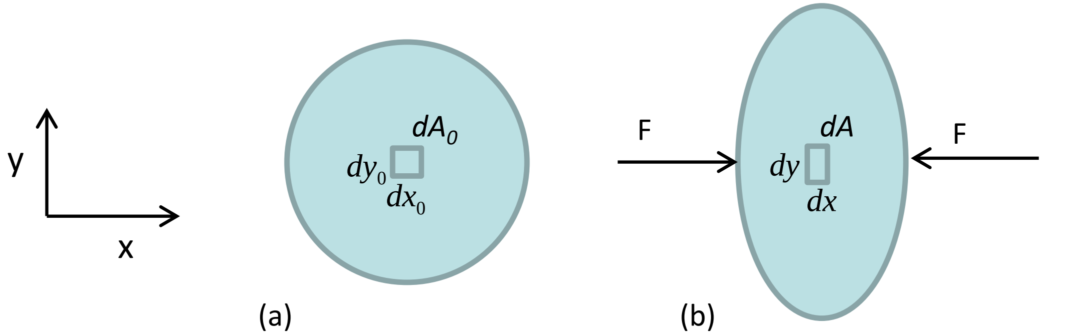


Fig. 5.1. (a) undeformed sphere; and (b) deformed sphere.

The fiber dx_0 deforms to dx and its change in length per unit length is:

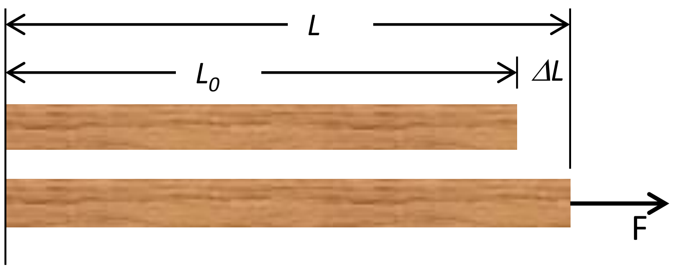
$$\varepsilon_x = \frac{dx - dx_0}{dx_0}$$: the normal strain in the x-direction

Similarly, the normal strain in the y-direction is: $\varepsilon_y = \dfrac{dy - dy_0}{dy_0}$

$$\begin{array}{ll} \text{If } \varepsilon < 0 & \rightarrow \text{shortening of the material} \\ \text{If } \varepsilon = 0 & \rightarrow \text{no change in length} \\ \text{If } \varepsilon > 0 & \rightarrow \text{lengthening of the material} \end{array}$$

While normal strain was defined above over infinitesimal zones, the concept can be extended to longer and larger members as well.

For example, in Fig. 5.2 below, the average axial strain in a structural member of initial length L_o, is $\varepsilon = \Delta L / L_0$



$$\bar{\varepsilon} = \frac{\Delta L}{L_o} \quad \text{average axial strain}$$

Fig. 5.2. Extension of a structural member leading to average axial strain.

C. Shear Strains

Shear strain represents the reduction in angle (radians) during deformation between two infinitesimal fibers that were initially perpendicular.

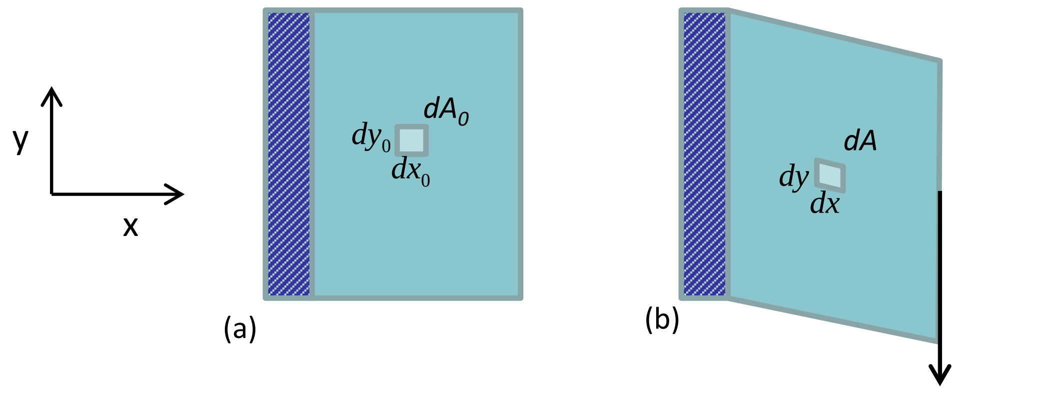
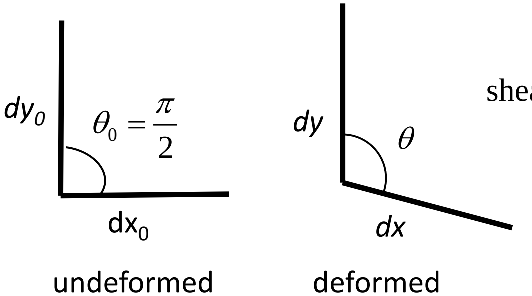


Fig. 5.3. (a) undeformed system; and (b) deformed system.

When the infinitesimal element dA_0 experiences shear deformation, the two fibers dx_0 and dy_0 that were originally orthogonal lose that orthogonality when they deform to dx and dy.



shear strain : $\gamma_{xy} = \theta_0 - \theta = \frac{\pi}{2} - \theta$

D. The Strain Tensor

In solid mechanics, the way to describe how a body moves and deforms in response to applied loads is through a vector displacement field. For each point in a body defined by initial coordinates (x,y,z) there is a displacement vector:

$$\mathbf{u}(x,y,z) = u_x(x,y,z)\mathbf{e}_x + u_y(x,y,z)\mathbf{e}_y + u_z(x,y,z)\mathbf{e}_z$$

When neighboring points in the body move differently, this leads to deformation or distortion of the body.

To describe the state of deformation at a point in the body, a second rank strain tensor is used:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

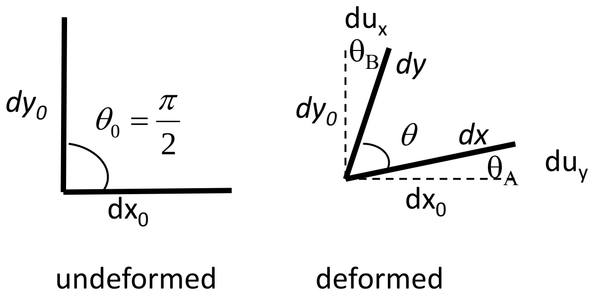
$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}; \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}; \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z};$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \tfrac{1}{2}\gamma_{xy} = \tfrac{1}{2}\gamma_{yx} = \tfrac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right);$$

$$\varepsilon_{xz} = \varepsilon_{zx} = \tfrac{1}{2}\gamma_{xz} = \tfrac{1}{2}\gamma_{zx} = \tfrac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right);$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \tfrac{1}{2}\gamma_{yz} = \tfrac{1}{2}\gamma_{zy} = \tfrac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right);$$

Note that γ values are used to represent engineering shear strains which have magnitudes twice those of the tensor shear strains.



shear strain : $\gamma_{xy} = \theta_0 - \theta = \frac{\pi}{2} - \theta$

$$\tan\theta_A \approx \theta_A = \frac{du_y}{dx_0}$$

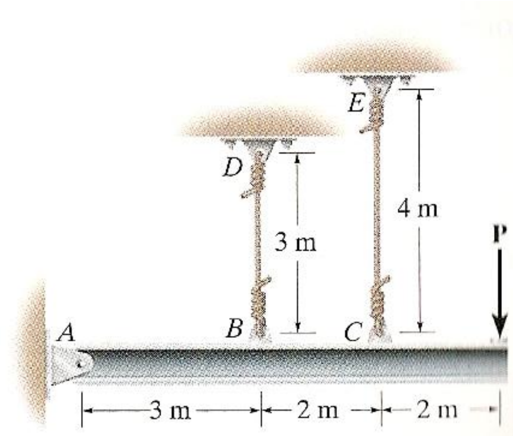
$$\tan\theta_B \approx \theta_B = \frac{du_x}{dy_0}$$

$$\gamma_{xy} = \theta_0 - \theta = \frac{\pi}{2} - \left(\frac{\pi}{2} - \theta_A - \theta_B\right)$$

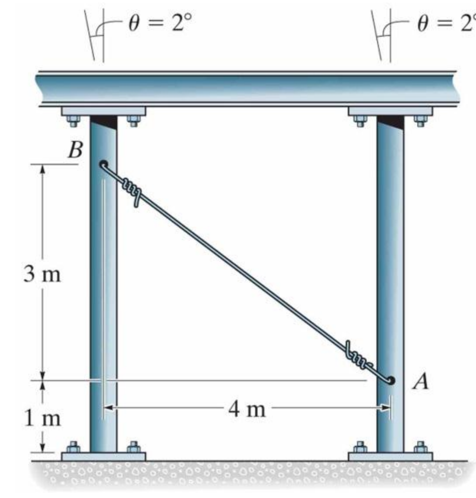
$$\gamma_{xy} = \theta_A + \theta_B = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}$$

E. Example Problems

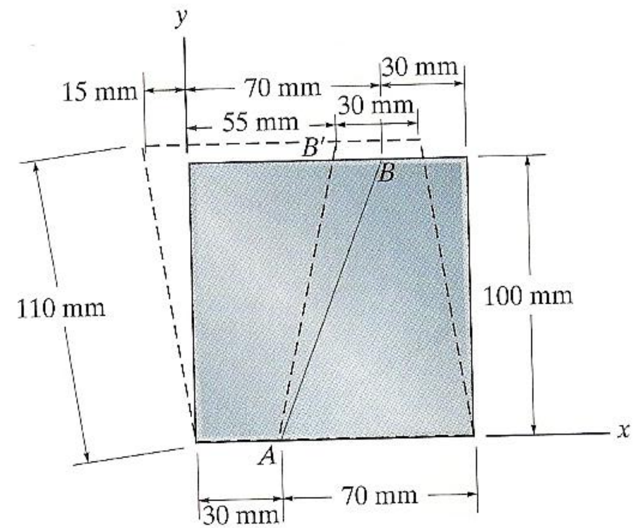
Example 5.1 The rigid beam is supported by a pin at *A* and wires *BD* and *CE*. If the allowable normal strain in each wire is ε_{max} = .002, determine the maximum vertical displacement of the load **P**.



Example 5.2 The guy wire AB of a building frame is originally unstretched. Due to an earthquake, the two columns of the frame tilt θ=2°. Determine the approximate normal strain in the wire when the frame is in this position. Assume the columns are rigid and rotate about their lower supports.

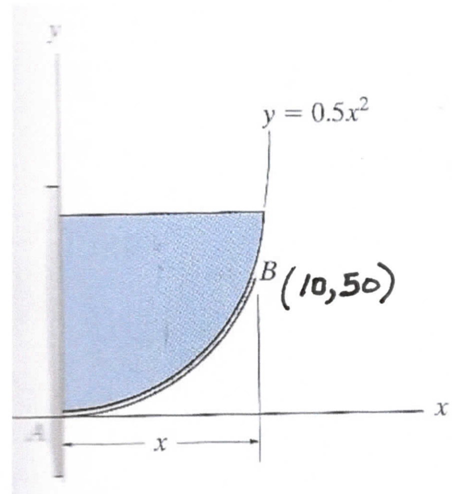


Example 5.3. The block is deformed into the position shown by the dashed lines. Determine: a) the average normal strain along line *AB*; and the shear strain γ_{xy}



Example 5.4. A thin wire is wrapped along a surface having the form $y=0.5x^2$, where x and y are in inches. Originally the end at B is at x=10in. If the wire undergoes a normal strain along its length of $\varepsilon=.005x$, determine the change in length of the wire. Hint: For the curve, y=f(x).

$$ds = \sqrt{1+(dy/dx)^2}\,dx$$



Original length of the wire:

$$L_o = \int_0^{10''} \sqrt{1+x^2}\,dx = \left[\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left|x+\sqrt{1+x^2}\right|\right]_0^{10''}$$

$$L_o = 51.748''$$

$$\Delta L = \int \epsilon\, ds = \int_0^{10''} .005x\sqrt{1+x^2}\,dx = \left.\frac{(1+x^2)^{3/2}}{600}\right|_0^{10''}$$

$$= \frac{(101)^{3/2} - 1^{3/2}}{600} = 1.69$$

$$\boxed{\Delta L = 1.69''}$$