Period #6: Material Properties

A. Context

We have introduced two concepts thus far:

Loads applied to structures result in internal <u>stresses</u>.
i.e. the stress tensor (Fig. 6.1)

Deformation on the material scale is quantified by <u>strain</u>.
i.e. the strain tensor (Fig. 6.2)

Material properties and constitutive relations provide the critically important connection between stress and strain in materials.

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

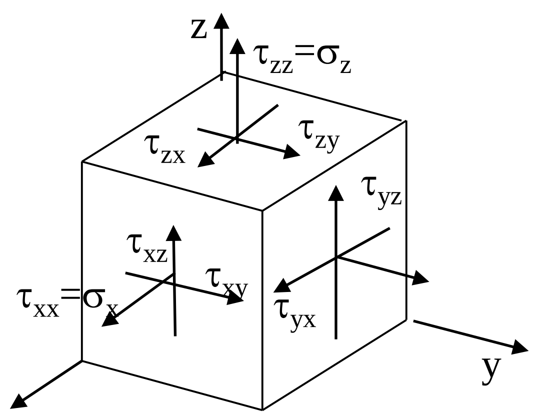


Fig. 6.1. Meaning of the stress tensor components

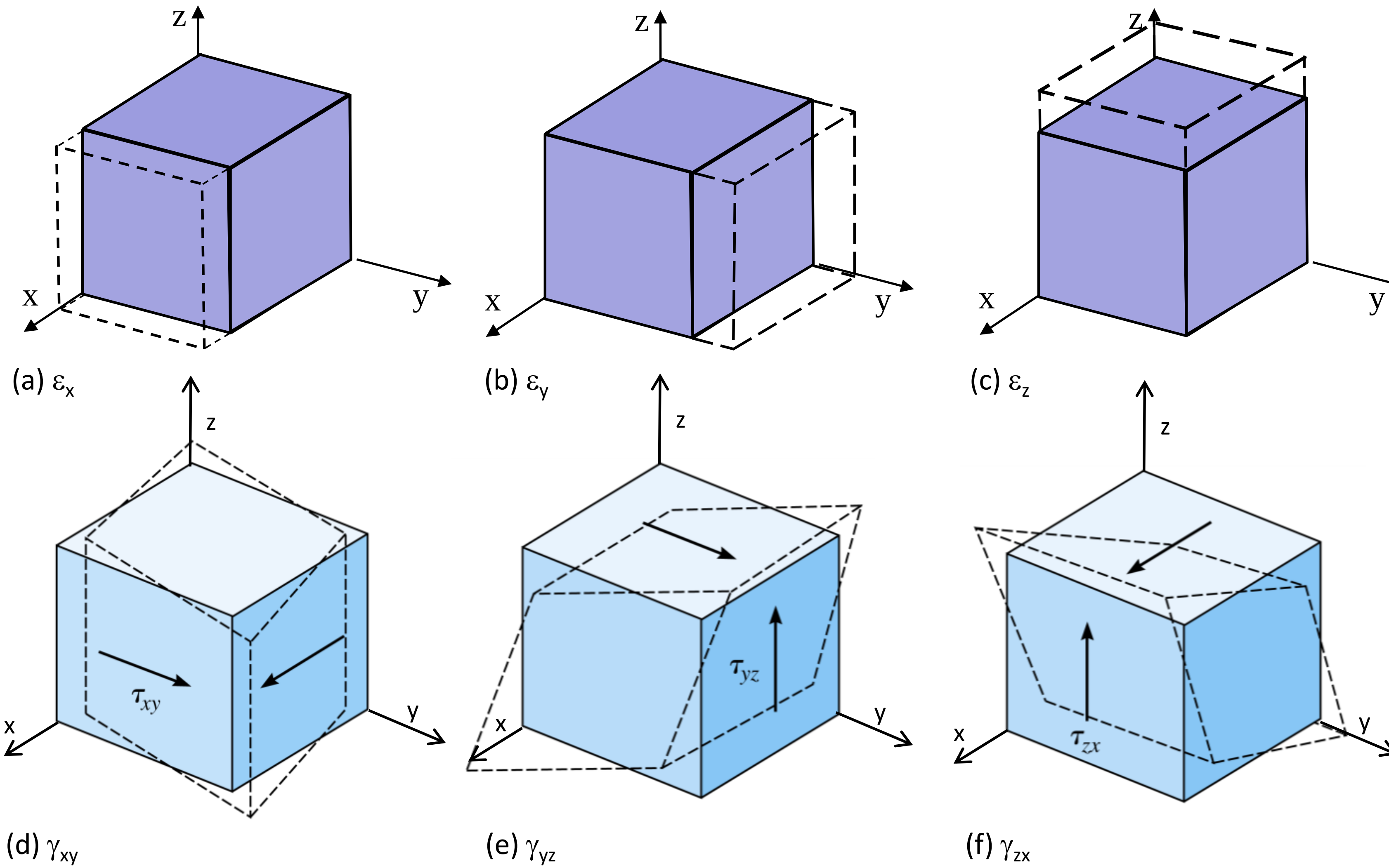


Fig. 6.2. Meaning of the strain tensor components

B. General Material Properties Relating Stress and Strain

Common Material Tests

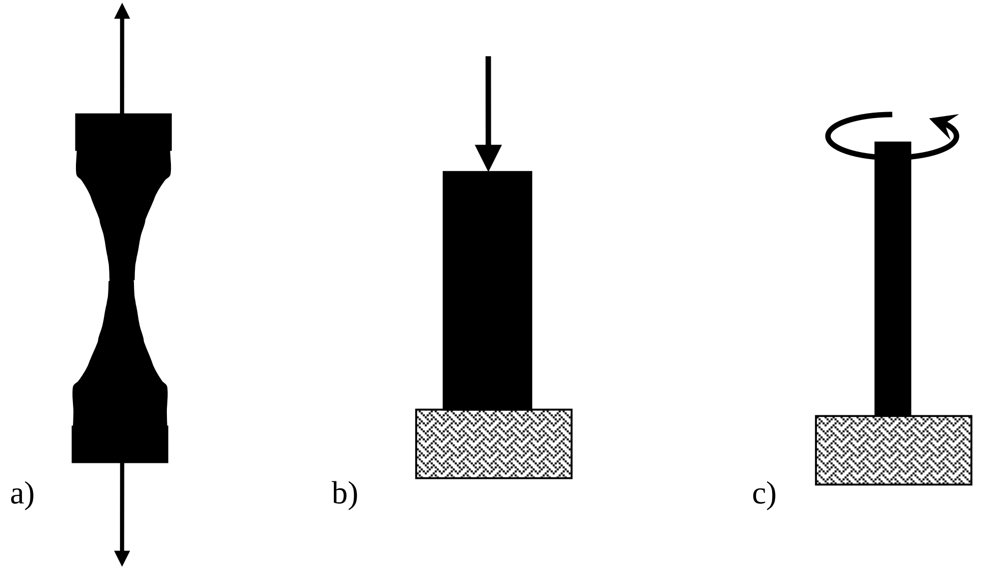
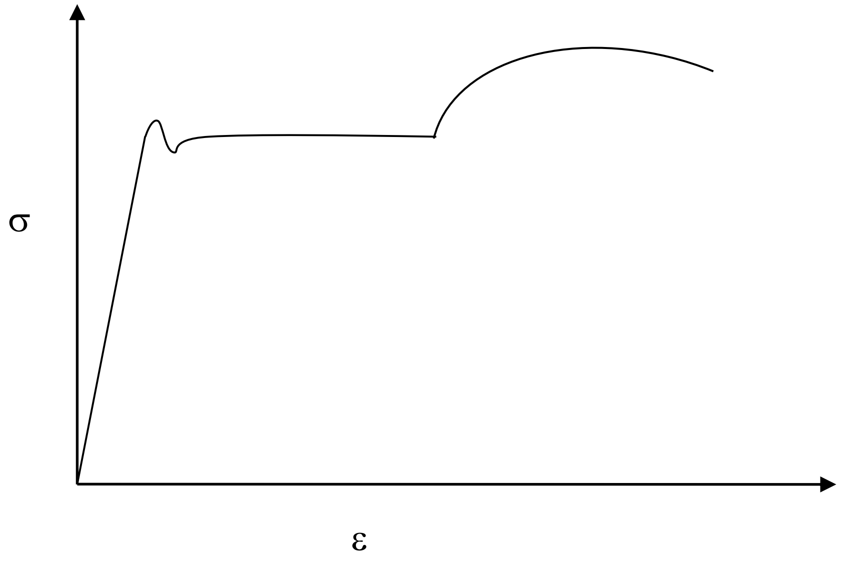


Fig. 6.3 Material properties are measured in a variety of mechanical tests. Some of the common mechanical tests performed on materials are: a) the tension test; b) the compression test; and c) the torsion test.

When a monotonic tension test to failure is performed on a mild untreated steel specimen, one typically sees the type of result shown below.

The result would look quite different for different material types (aluminum, FRP composites, bone, muscle, concrete, wood, etc.)



Features of the stress-strain diagram

- Elastic modulus:
- Yield stress:
- Ultimate stress
- Fracture or Breaking stress
- Ductility
- Modulus of toughness
- Modulus of resilience
- Strain energy

Types of material behavior:

- Elastic behavior
- Linear behavior
- Plastic behavior
- Hardening behavior

C. Models for Linear Elastic Material Behavior

Hooke's Law for linear elastic materials (not necessarily isotropic)

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&C_{14}&C_{15}&C_{16}\\ C_{21}&C_{22}&C_{23}&C_{24}&C_{25}&C_{26}\\ C_{31}&C_{32}&C_{33}&C_{34}&C_{35}&C_{36}\\ C_{41}&C_{42}&C_{43}&C_{44}&C_{45}&C_{46}\\ C_{51}&C_{52}&C_{53}&C_{54}&C_{55}&C_{56}\\ C_{61}&C_{62}&C_{63}&C_{64}&C_{65}&C_{66}\end{bmatrix}\cdot\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{bmatrix}$$

$$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{bmatrix}=\begin{bmatrix}S_{11}&S_{12}&S_{13}&S_{14}&S_{15}&S_{16}\\ S_{21}&S_{22}&S_{23}&S_{24}&S_{25}&S_{26}\\ S_{31}&S_{32}&S_{33}&S_{34}&S_{35}&S_{36}\\ S_{41}&S_{42}&S_{43}&S_{44}&S_{45}&S_{46}\\ S_{51}&S_{52}&S_{53}&S_{54}&S_{55}&S_{56}\\ S_{61}&S_{62}&S_{63}&S_{64}&S_{65}&S_{66}\end{bmatrix}\cdot\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{bmatrix}$$

Hooke's Law for isotropic, linear elastic materials

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{bmatrix}=\frac{E}{1+\nu}\begin{bmatrix}\left(\frac{1-\nu}{1-2\nu}\right)&\left(\frac{\nu}{1-2\nu}\right)&\left(\frac{\nu}{1-2\nu}\right)&0&0&0\\ \left(\frac{\nu}{1-2\nu}\right)&\left(\frac{1-\nu}{1-2\nu}\right)&\left(\frac{\nu}{1-2\nu}\right)&0&0&0\\ \left(\frac{\nu}{1-2\nu}\right)&\left(\frac{\nu}{1-2\nu}\right)&\left(\frac{1-\nu}{1-2\nu}\right)&0&0&0\\ 0&0&0&\frac{1}{2}&0&0\\ 0&0&0&0&\frac{1}{2}&0\\ 0&0&0&0&0&\frac{1}{2}\end{bmatrix}\cdot\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{bmatrix}$$

$$Note:\quad G=\frac{E}{2(1+\nu)}\ \text{(Shear Modulus)}$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \cdot \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

Special cases of Hooke's Law (for linear elastic, isotropic materials)

Uniaxial Stress State (in the x-direction). All other stresses vanish

$$\sigma_{xx} = E\varepsilon_{xx} \quad \leftrightarrow \varepsilon_{xx} = E/\sigma_{xx}$$

$$\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx}$$

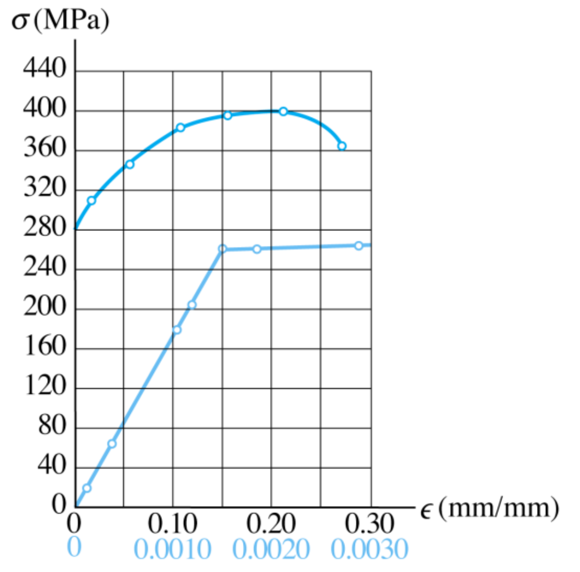
Here, is known as the *Poisson's ratio*.

Pure Shear

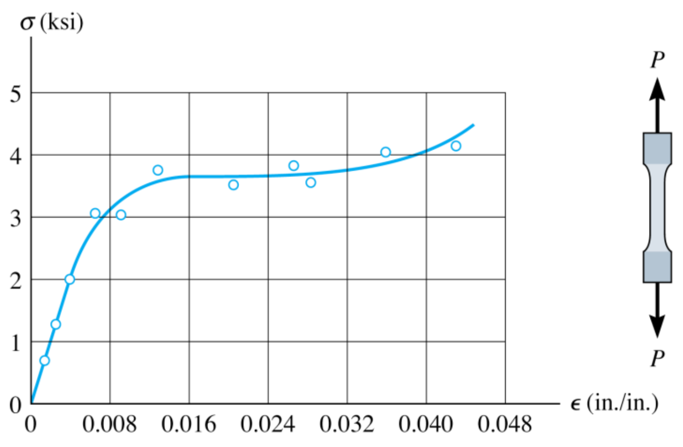
$\tau_{xy} = G\gamma_{xy}$ where $G = \dfrac{E}{2(1+\nu)}$ is known as the *shear modulus*

D. Example Problems

Example #6.1. The stress-strain diagram for a steel alloy having an initial diameter of 12.5 mm and an initial gauge length of 50mm is provided in the figure. Determine the approximate modulus of elasticity for the material, the load on the specimen that causes yielding, and the ultimate load that the specimen will support



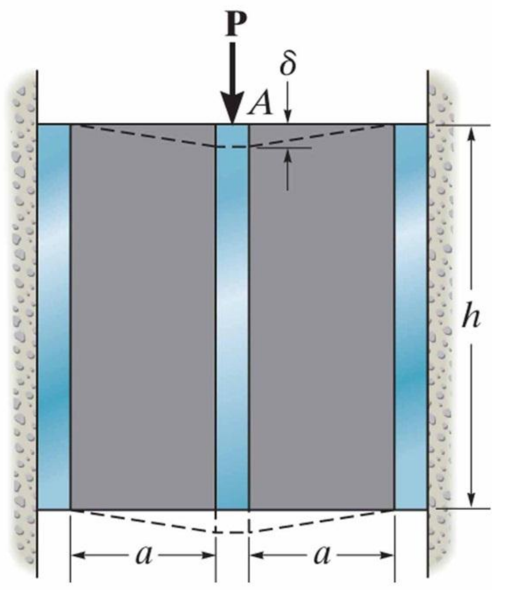
Example #6.2. The stress-strain diagram for polyethylene, which is used to sheath coaxial cables, is determined from testing a specimen that has a gauge length of 10 in. If a load *P* on the specimen develops a strain of $\varepsilon = 0.024$, determine the approximate length of the specimen, measured between the gauge points, when the load is removed. Assume the specimen recovers elastically.



Example #6.3. A plug has a diameter of 30mm and fits within a rigid sleeve having an inner diameter of 32mm. Both the plug and the sleeve are 50mm long. Determine the axial stress σ that must be applied to the plug so that it just makes contact with the sleeve. Also, how far must the plug be compressed downward in order to do this? The plug is made of a material for which E=5 MPa, and ν = 0.45.

Example #6.4: A shear spring is made from two blocks of rubber, each having a height h, width b, and thickness a. The blocks are bonded to three plates as shown. If the plates are rigid and the shear modulus of the rubber is G, determine the displacement of plate A if a vertical load P is applied to this plate. Assume that the displacement is small so that

$$\delta = a\tan\gamma \approx a\gamma$$



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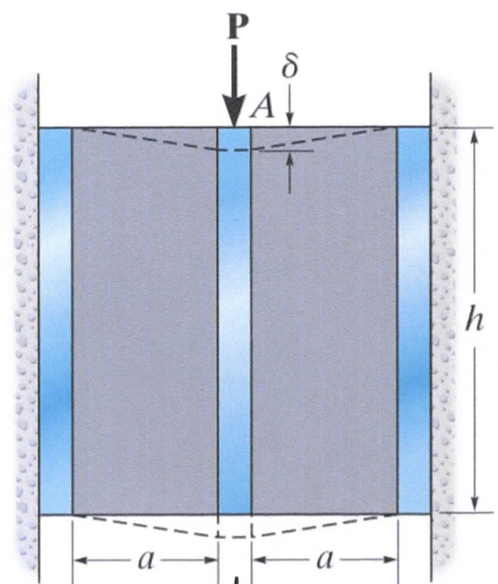

$\bar{\tau} = \dfrac{P}{2bh}$ = average shear stress on a vertical section through the rubber

$\gamma = \bar{\tau}/G = \dfrac{P}{2Gbh} = \delta/a$

$$\boxed{\delta = \gamma a = \frac{Pa}{2Gbh}}$$