Period #8: Axial Load/Deformation in Indeterminate Members

A. Review

We are considering axial members in tension or compression in the linear, elastic regime of behavior. Thus the magnitudes of axial stresses in the members are less than the material yield stresses, and the magnitudes of axial strains are less than the material yield strains.

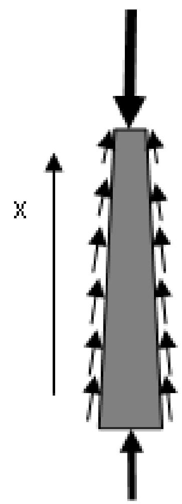
Constant force, area

$$\sigma_{axial} = F/A$$

$$\varepsilon_{axial} = \frac{\sigma_{axial}}{E} = \frac{F}{AE} = \frac{\Delta L}{L_0}$$

$$\therefore \Delta L = \frac{FL_0}{AE}$$

Variable force, area



$$\sigma(x) = \frac{F(x)}{A(x)}$$

$$\varepsilon(x) = \frac{\sigma(x)}{E(x)} = \frac{F(x)}{A(x)E(x)}$$

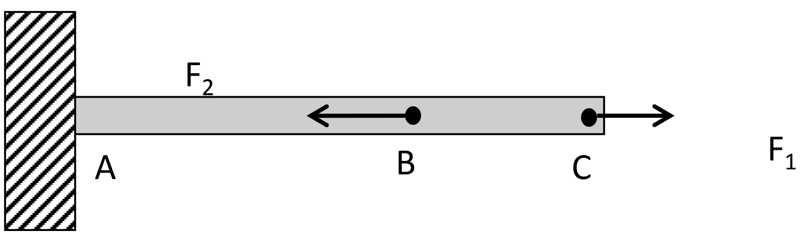
$$\Delta L = \int_0^{L_0} \varepsilon(x)dx = \int_0^{L_0} \frac{F(x)}{A(x)E(x)}dx$$

B. Principle of linear superposition

In solid mechanics and structural mechanics, it is sometimes very convenient to consider multiple loads acting on a system one at a time, and to then add their effects. This is known as superposition.

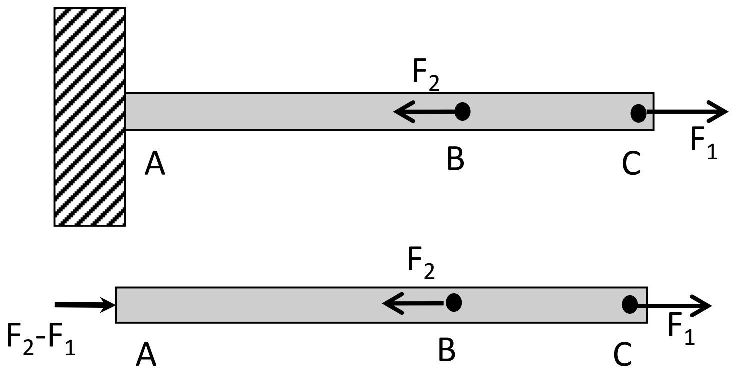
This can generally be done only when the structure behaves linearly (i.e. features a mathematically linear relationship between applied forces and resulting displacements). Two necessary conditions for the principle of linear superposition to apply are:

1. the material in the structure must be in the linear elastic regime of behavior; and;
2. the deformations in the structure (displacements, rotations, and strains) must be small.



Consider this structure using and not using superposition

Analysis w/o Superposition



from A to B:

$$F(x) = F_2 - F_1 \text{ (compressive)}$$

$$\varepsilon(x) = \frac{F}{AE} = -\frac{F_2 - F_1}{AE}$$

$$\delta_{B/A} = -\frac{F_2 - F_1}{AE} L_{AB}$$

from B to C:

$$F(x) = F_1 \text{ (tensile)}$$

$$\varepsilon(x) = \frac{F_1}{AE}$$

$$\delta_{C/B} = \frac{F_1}{AE} L_{BC}$$

from A to C:

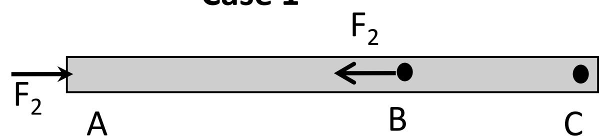
$$\delta_C = \delta_A + \delta_{B/A} + \delta_{C/B}$$

$$= 0 + -\frac{F_2 - F_1}{AE} L_{AB} + \frac{F_1}{AE} L_{BC}$$

$$\delta_C = \frac{}{AE}\left[F_1 L_{AC} - F_2 L_{AB}\right]$$

Analysis w/ Superposition

Case 1



from A to B:

$$F(x) = -F_2$$

$$\varepsilon(x) = \frac{F}{AE} = \frac{-F_2}{AE}$$

$$\delta_{B/A} = \frac{-F_2 L_{AB}}{AE}$$

from B to C:

$$F(x) = 0$$

$$\varepsilon(x) = 0$$

$$\delta_{C/B} = 0$$

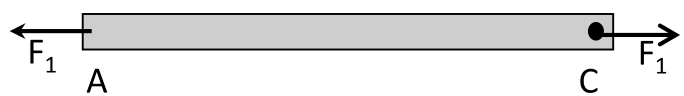
from A to C:

$$\delta_C = \delta_A + \delta_{B/A} + \delta_{C/B}$$

$$= 0 + -\frac{F_2 - F_1}{AE} L_{AB} + 0$$

$$\delta_C = \frac{-F_2 L_{AB}}{AE}$$

Case 2



from A to C:

$$F(x) = F_1$$

$$\varepsilon(x) = \frac{F_1}{AE}$$

$$\delta_{C/A} = \frac{F_1 L_{AC}}{AE}$$

Combining cases 1 & 2: $\delta_C = (\delta_C)_{case1} + (\delta_C)_{case2} = \frac{-F_2 L_{AB}}{AE} + \frac{F_1 L_{AC}}{AE}$

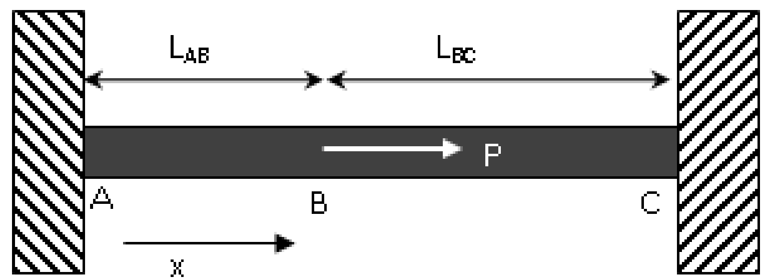
C. Application of linear superposition to analyze indeterminate axially loaded members.

Statically indeterminate structural systems are those where the number of unknown support reactions exceeds the number of global equilibrium equations. In such systems it is not possible to solve for the support reactions using only the equilibrium equations of statics.

To solve for the reaction forces at the redundant supports, we can use knowledge of how much the structure displaces at the support to solve for the unknown reaction.

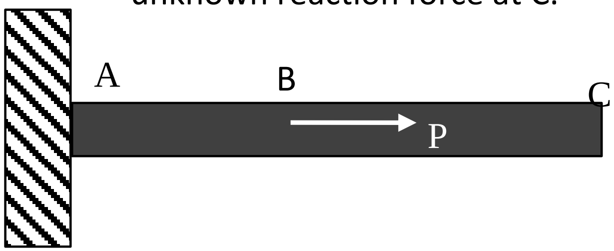
In this illustrative example, there is only one relevant equation of equilibrium but there are two support reactions at A and C to solve for. Thus the system is statically indeterminate.

To solve for both support reactions, we need an additional condition. Here, since the supports at both A and C are taken to be rigid, we can say that $\delta_{C/A}=0$ or that overall the axial member does not change its length.



$$\sum F_x = 0 \Rightarrow P = F_A + F_C$$

Now, letting A remain fixed, we'll consider the two loads acting on the system: (1) the known force of magnitude P acting at B; and (2) the unknown reaction force at C.



$$\text{Case 1: } \left(\delta_{C/A}\right)_1 = \frac{PL_{AB}}{AE}$$

$$\text{Case 2: } \left(\delta_{C/A}\right)_2 = -\frac{F_C L_{AC}}{AE}$$

Putting the two load cases together :

$$\left(\delta_{C/A}\right)_1 + \left(\delta_{C/A}\right)_2 = 0 \quad \Rightarrow \frac{PL_{AB}}{AE} - \frac{F_C L_{AC}}{AE} = 0$$

$$F_C = \frac{PL_{AB}}{L_{AC}}$$

From Statics, Solve for F_A :

$$P = F_A + F_C = F_A + \frac{PL_{AB}}{L_{AC}}$$

$$F_A = P\left(1 - \frac{L_{AB}}{L_{AC}}\right) = P\frac{L_{BC}}{L_{AC}}$$

Conclusion: Indeterminate problems can be solved!

D. Thermal expansions/contractions

Most materials undergo expansion when their temperature increases and contraction when the temperature decreases.

In unrestrained axial members the axial strain due to a temperature change ΔT is given by the expression:

$$\varepsilon_t = \alpha \Delta T$$

where α is the material's coefficient of thermal expansion.

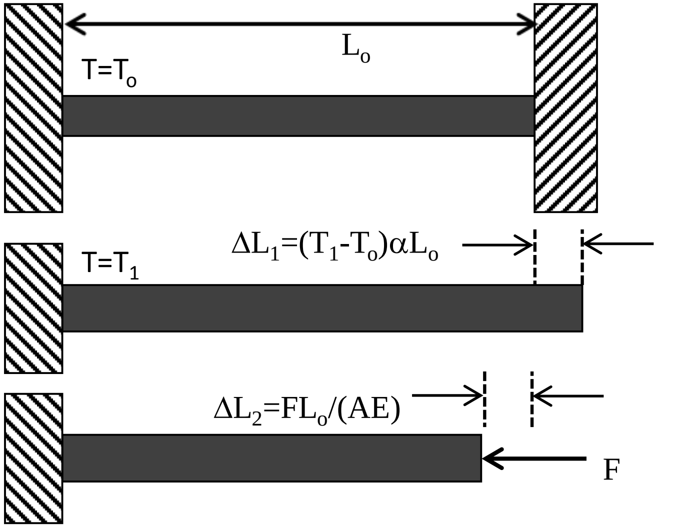
Often times members are restrained and cannot expand or contract when the temperature changes. In such cases, forces and associated stress build up in the members and these are called thermal stresses.

A rod with Youngs modulus E and coefficient of thermal expansion α is positioned between two rigid supports at initial temperature T_o.

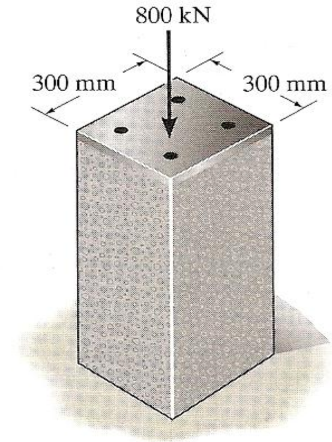
When the temperature increases to T_1 what is the axial force or stress in the rod?

$$\Delta L_1 + \Delta L_2 = 0$$

$$F = -\alpha \, \Delta T \, AE$$

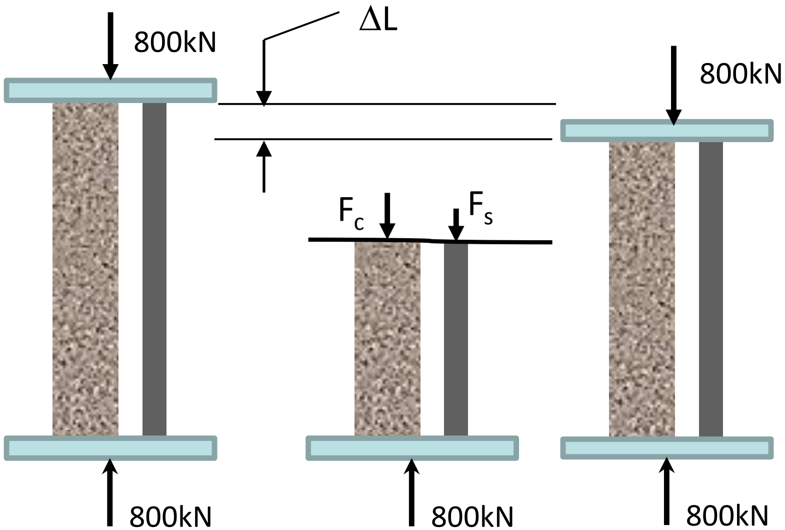


E. Example Problems



Example #8.1: The column is constructed from high-strength concrete and four A-36 steel reinforcing rods. If it is subjected to an axial force of 800kN, determine the required diameter of each rod so that one-fourth of the load is carried by the steel, and three-fourths by the concrete. E_{st}=200GPa, and E_c=25GPa.

Solution:



$$A = (0.3m)^2$$

$$A_s = 4\left(\frac{\pi}{4}d^2\right)$$

$$A_c = A - A_s$$

$$A_{rod} = \tfrac{1}{4}A_s = .0009m^2 = \frac{\pi}{4}d^2$$

$$d = \sqrt{\frac{4\cdot .009m^2}{\pi}} = .03385m = 33.85mm$$

$$\frac{1}{24} = \frac{A_s}{A_c} = \frac{A_s}{A - A_s} \rightarrow 24A_s = A - A_s$$

$$A_s = \frac{A}{25} = \frac{.09m^2}{25} = 0.0036m^2$$

$$F = 800kN$$

$$F_s = \tfrac{1}{4}F = 200kN$$

$$F_c = \tfrac{3}{4}F = 600kN$$

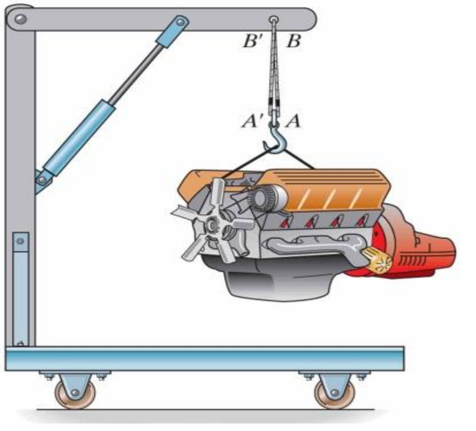
$$\Delta L = \Delta L_s = \Delta L_c$$

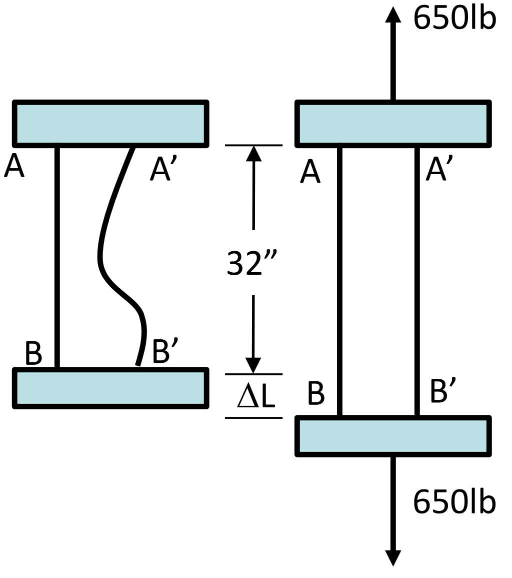
$$\varepsilon = \varepsilon_s = \varepsilon_c$$

$$\varepsilon_s = \frac{F_s}{A_s E_s} = \varepsilon_c = \frac{F_c}{A_c E_c}$$

$$\frac{F_s}{A_s E_s} = \frac{F_c}{A_c E_c} \rightarrow \frac{A_s}{A_c} = \frac{F_s E_c}{E_s F_c} = \frac{(200kN)(25GPa)}{(200GPa)(600kN)} = \frac{1}{24}$$

Example #8.2: Two A-36 steel wires are used to support the 650-lb engine. Originally, AB is 32 in. long and A'B' is 32.008 in. long. Determine the force supported by each wire when the engine is suspended from them. Each wire has a cross-sectional area of 0.01 in^2.





$$F_{AB} = E\varepsilon_{AB}A = EA\frac{\Delta L}{32''}$$

$$F_{A'B'} = E\varepsilon_{A'B'}A = EA\frac{\Delta L - 0.008''}{32''}$$

$$650\,lb = F_{AB} + F_{A'B'} = EA\left(\frac{\Delta L}{32''} + \frac{\Delta L - 0.008''}{32''}\right)$$

$$\frac{650\,lb}{EA} = \frac{0.65kip}{\left(29\cdot 10^3\,kip\cdot in^{-2}\right)\left(0.01in^2\right)} = \frac{2\Delta L}{32''} - \frac{.008''}{32''}$$

$$\Delta L = 0.03986''$$

$$\text{From Statics}: F = 650lb = F_{AB} + F_{A'B'}$$

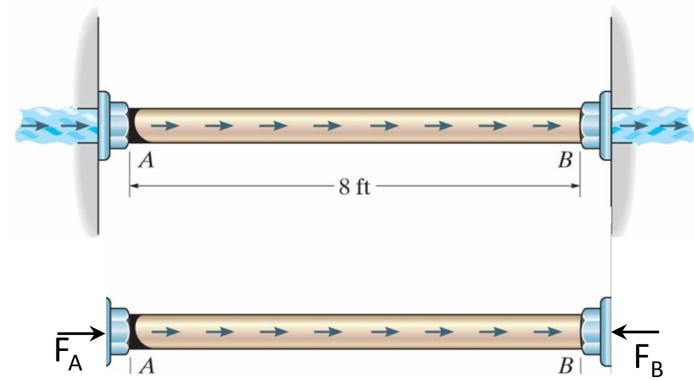
$$F_{AB} = \varepsilon_{AB}EA = \left(\frac{.03986''}{32''}\right)\left(29\cdot 10^3\,ksi\right)\left(.01in^2\right) = 361.3lb$$

$$\varepsilon_{AB} = \frac{\Delta L}{32''}$$

$$F_{A'B'} = \left(\frac{.03986''-.008''}{32''}\right)\left(29\cdot 10^3\,ksi\right)\left(.01in^2\right) = 288.7lb$$

$$\varepsilon_{A'B'} = \frac{\Delta L - 0.008''}{32.008''} \approx \frac{\Delta L - 0.008''}{32''}$$

Example #8.3: The bronze C86100 pipe has an inner radius of 0.5 in. and a wall thickness of 0.2 in. If the gas flowing through it changes the temperature of the pipe uniformly from T_A=200°F at A to T_B=60°F at B, determine the axial force it exerts on the walls. The pipe was fitted between the walls when T=60°F.



From statics: $F_A = F_B$

Solution:

Properties of C86100 bronze: $E=15\cdot10^3$ksi; $\alpha=9.6\cdot10^{-6}$/°F

Use superposition:

$$\Delta T(x) = 140^\circ F\left(1-\frac{x}{L}\right)$$

Case 1: expansion of the pipe due to thermal strains (assume pipe is free to extend)

$$\varepsilon_t(x) = \alpha\Delta T(x)$$

$$(\Delta L)_1 = \int_0^L \varepsilon_t(x)dx = \alpha(140^\circ F)\int_0^L\left(1-\frac{x}{L}\right)dx = \frac{\alpha L(140^\circ F)}{2} = 0.0645''$$

Case 2: compression of the pipe due to restraining wall forces $F_A=F_B$.

$$(\Delta L)_2 = \frac{FL}{AE} = -.0645''$$

$$F = -.0645''\frac{AE}{L} = -.0645''\frac{\pi(.7^2-.5^2)(15\cdot10^3 ksi)}{96''} = -7.60 kips$$