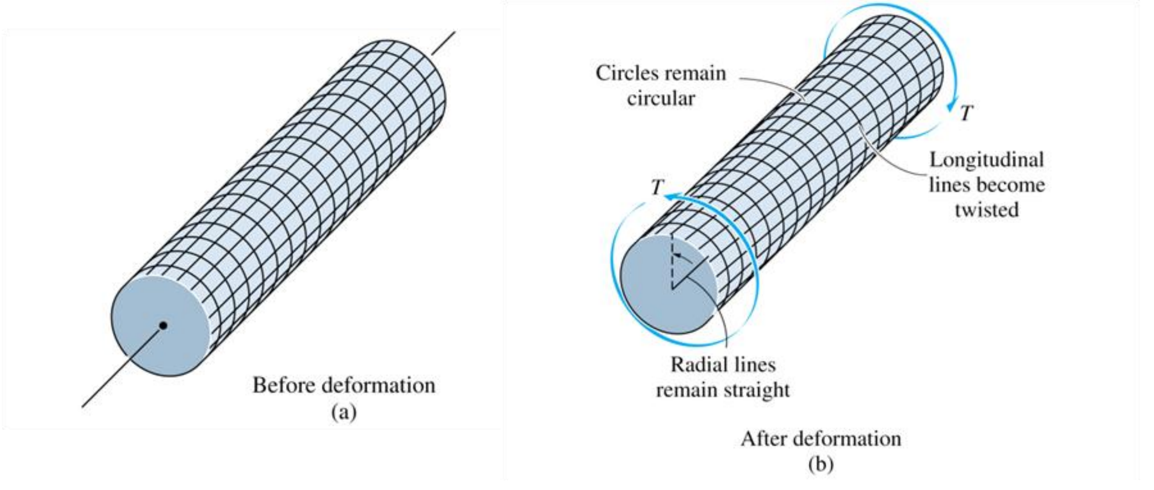
Period #9: Torsion

A. Context

A torque is a moment applied to mechanical system that causes it to twist or rotate.

In the present circumstances, we'll be concerned with torsional moments (torques) applied to long slender shafts as shown in the figure below.



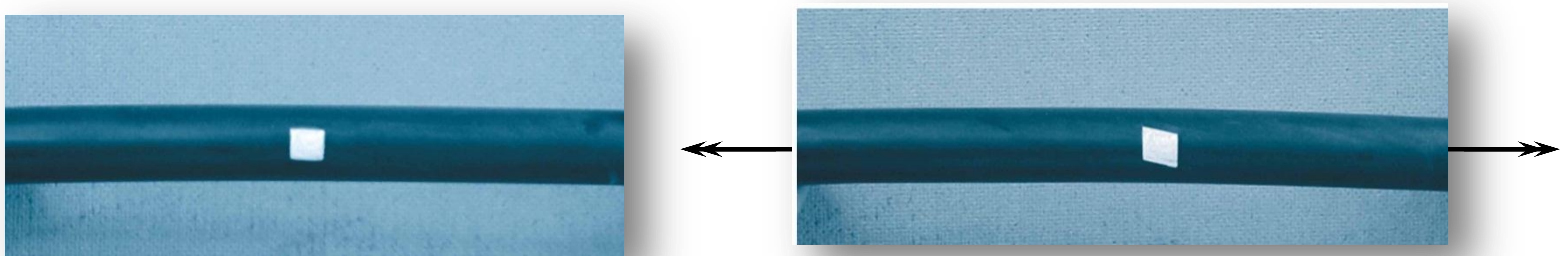
We'll especially want to determine what is the internal stress distribution in a shaft subjected to torques, and what is the resulting deformation (or twisting) of the shaft itself.

Initial assumptions:

- shaft has a circular cross-section;
- material behaves as a linear, elastic solid.

B. Kinematics of Torsion

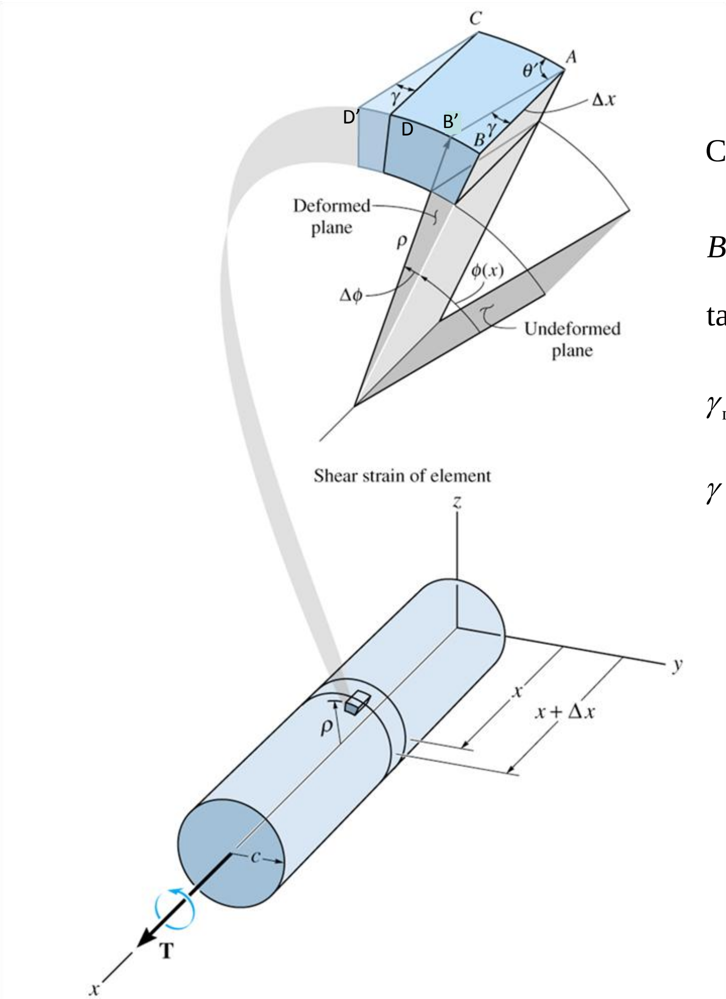
Kinematics of torsion: how the shaft deforms when subjected to a torque.



(a) (b)

Fig. 9.1 Torsional deformation in a rubber shaft. (a) unloaded, and (b) deformed under torque loading.

Observe that when the shaft is loaded, the rectangular patch undergoes essentially pure shear deformation.



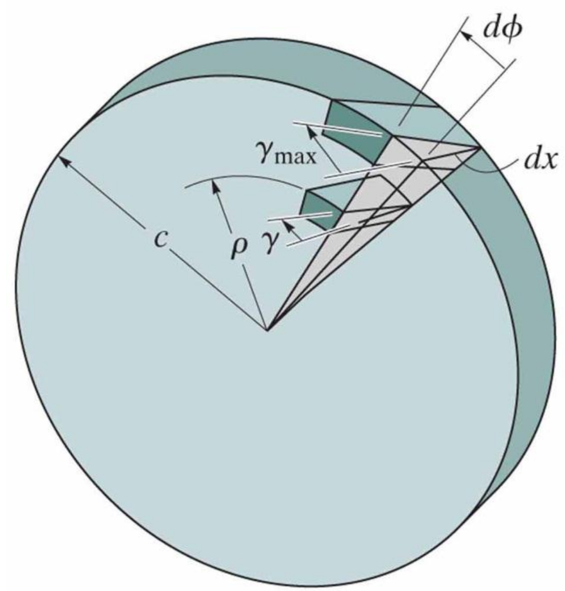
Consider the fibers AC and AB in the undeformed configuration :

$$BB' = \rho(\Delta\phi)$$

$$\tan\gamma \approx \gamma = \frac{BB'}{\Delta x} = \rho\frac{\Delta\phi}{\Delta x}$$

$$\gamma_{\max} = c\frac{\Delta\phi}{\Delta x}$$

$$\gamma = \frac{\rho}{c}\gamma_{\max}$$

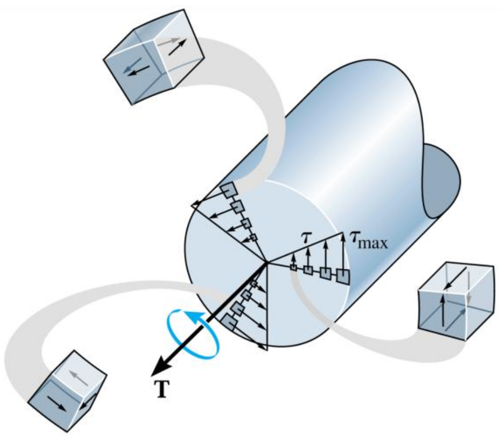


The shear strain at points on the cross section increases linearly with ρ, i.e., $\gamma = (\rho/c)\gamma_{\max}$.

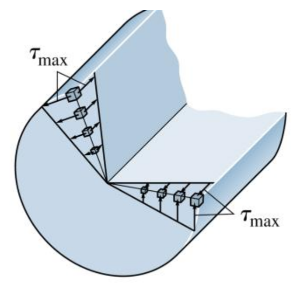
C. Shear Stresses

Kinematics of torsion established that $\gamma=(\rho/c)\gamma_{max}$.
Since by Hooke's Law, $\tau=G\gamma$ it follows that on the cross-section of the shaft: $\tau=(\rho/c)\tau_{max}$

$$\tau = G\gamma = G\left(\frac{\rho}{c}\right)\gamma_{max} = \left(\frac{\rho}{c}\right)\tau_{max}$$

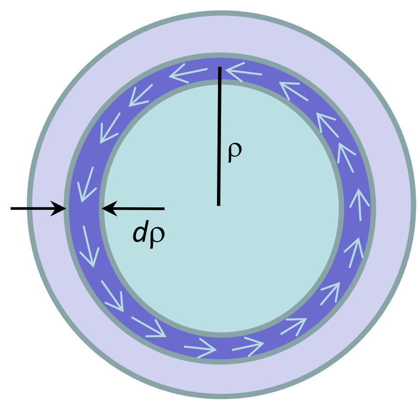


(a)



Shear stress varies linearly along each radial line of the cross section.
(b)

$$T = \int_A \rho\tau dA = \int_0^c 2\pi\,\rho^2\tau\,d\rho = \frac{\tau_{max}}{c}\int_0^c 2\pi\rho^3 d\rho = \frac{\tau_{max}}{c}2\pi\frac{c^4}{4} = \frac{\tau_{max}}{c}\frac{\pi c^4}{2}$$



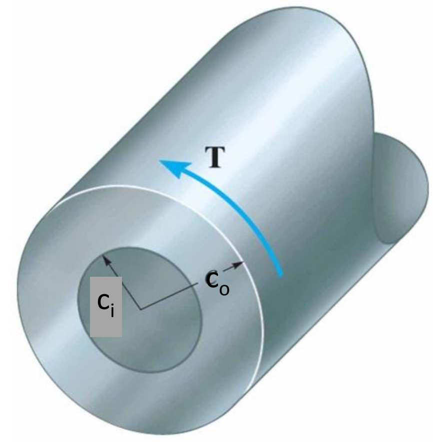
$$dA = 2\pi\rho\,d\rho$$

$$dT = \rho\tau\,dA = (\rho)\left(\frac{\rho}{c}\tau_{max}\right)(2\pi\rho\,d\rho) = \frac{\tau_{max}}{c}2\pi\rho^3 d\rho$$

$$T = \frac{\tau_{max}}{c}\int_0^c 2\pi\rho^3 d\rho = \frac{\tau_{max}}{c}J$$

D. The Polar Moment of Inertia

The polar moment of inertia: $J = \int \rho^2 dA$



circular cross-section (solid shaft)

$$J = \frac{\pi c^4}{2}$$

annular cross-section (tubular shaft)

$$J = \frac{\pi\left[c_o^{\,4} - c_i^{\,4}\right]}{2}$$

E. The Torsion Formula

$$\tau_{\max} = \frac{Tc}{J}; \qquad \tau = \frac{T\rho}{J}$$

F. Torque/Twist Relation:

$$\gamma = \frac{\tau}{G} = \frac{T\rho}{JG}$$

$$\rho \frac{d\phi}{dx} = \frac{T\rho}{JG}$$

$$\Delta\phi = \int \frac{\mathrm{T}}{\mathrm{GJ}} dx$$

G. Power Transmission

- It is often the case that a source of power such as an engine exists at one location, but work must be done at a different location.
- Power can be transmitted from the engine to the location where work needs to be done using drive shafts. The rate of power transmission by shaft is:

$$P = T\frac{d\theta}{dt} = T\omega = T * 2\pi f$$

where:

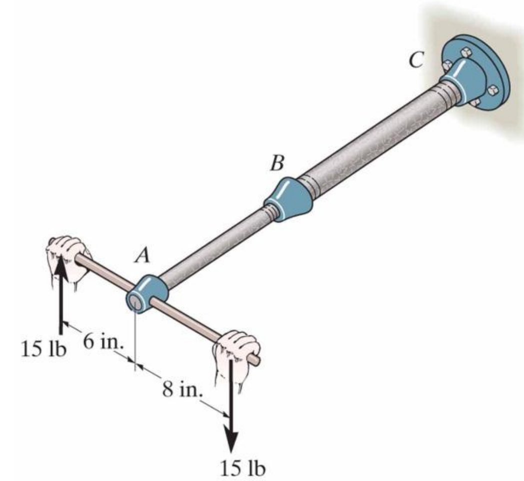
T is the torque in the shaft

ω is the rate at which the shaft turns (radians)

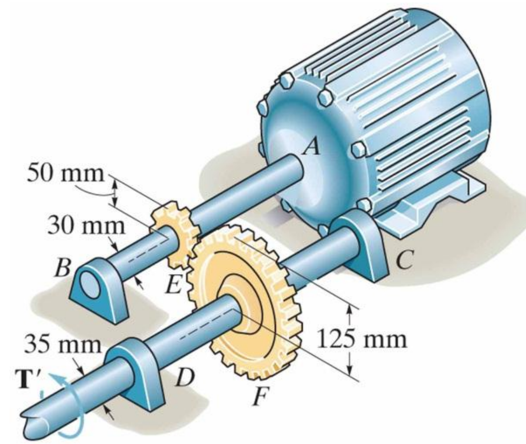
f is the rate at which the shaft turns (cycles)

H. Example Problems

Example 9.1: The assembly consists of two sections of galvanized steel pipe connected together using a reducing coupling at B. The smaller pipe has an outer diameter of 0.75 in. and an inner diameter of 0.68 in., whereas the larger pipe has an outer diameter of 1 in. and an inner wall diameter of 0.86 in. If the pipe is tightly secured to the wall at C, determine the maximum shear stress developed in each section of the pipe when the couple shown is applied to the handles of the wrench.



Example 9.2: If the applied torque on the shaft CD is T'=75N·m, determine the absolute maximum shear stress in each shaft. The bearings B, C, and D allow free rotations of the shafts, and the motor holds the shafts fixed from rotating



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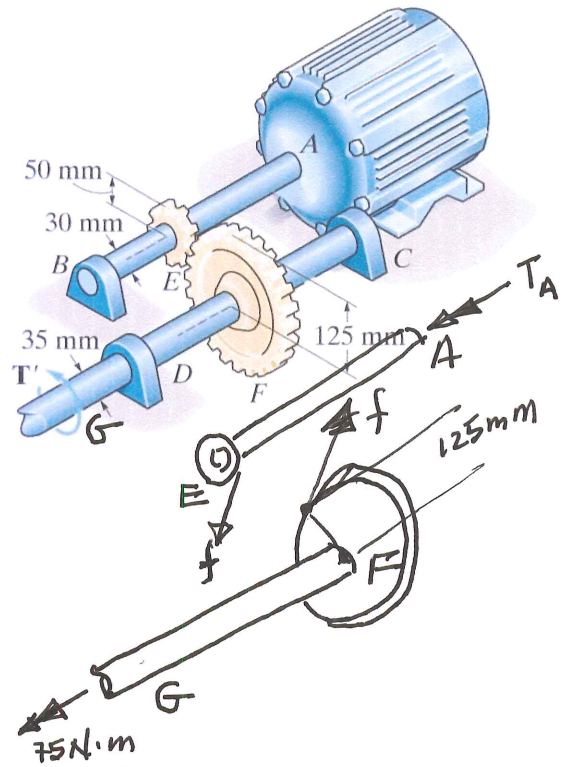


From G to F, $T_{FG} = 75\,N\cdot m$

From E to A, $T_{AE} = -\left(\frac{r_E}{r_E}\right)(75\,N\cdot m)$

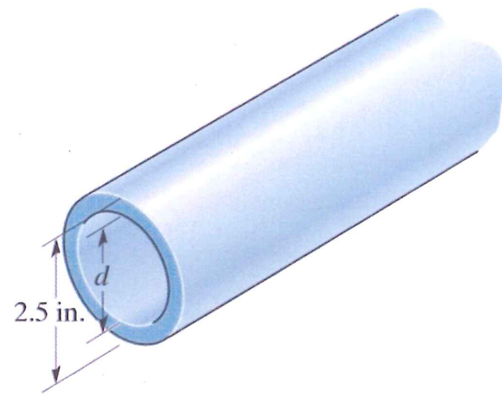
$$= -\left(\frac{50\,mm}{125\,mm}\right)(75\,N\cdot m) = -30\,N\cdot m$$

In shaft AEB, $\tau_{max} = \frac{T_{AE}\,C}{J} = \frac{(30\,N\cdot m)(.015m)}{\frac{\pi}{2}(.015m)^4}$

$$= 5.66\,MPa$$

In shaft GFC: $\tau_{max} = \frac{T_{FG}\,C}{J} = \frac{(75\,N\cdot m)(.0175m)}{\frac{\pi}{2}(.0175m)^4} = 8.91\,MPa$

Example 9.3: A steel tube having an outer diameter of 2.5 in. is used to transmit 35 hp when turning at 2700 rev/min. Determine the inner diameter d of the tube to the nearest 1/8 in. if the allowable shear stress is τ_{allow}=10 ksi.



Given: $d_o = 2.5$ inch; $P = 35$ hp; $f = 2700$ rpm

$\tau_{all} = 10$ ksi

Find: d_i to the nearest $1/8^{th}$ inch

$$T = \frac{P}{2\pi f} = \frac{(35\,hp)\left(\frac{550\,ft\cdot lb/s}{1\,hp}\right)(12\,in/ft)}{2\pi \times \frac{2700\,rev}{min} \times \frac{1\,min}{60\,s}}$$

$$T = 817\ lb\cdot in = 0.817\ kip\cdot in$$

$$\tau_{max} = \tau_{allow} = \frac{Tc}{J} \rightarrow J = \frac{Tc}{\tau_{allow}}$$

$$J_{req'd} = \frac{Tc}{\tau_{allow}} = \frac{(0.817\,kip\cdot in)(1.25\,in)}{10\,kip/in^2} = 0.1021\ in^4$$

$$J_{req'd} = .1021\ in^4 = \frac{\pi}{2}\left(c_o^4 - c_i^4\right)$$

$$c_i = \left[c_o^4 - \frac{2}{\pi}(.1021\,in^4)\right]^{1/4}$$

$$= 1.242\ in.$$

$$d_i = 2c_i = 2.483\ in \longrightarrow 2.375\ in.$$

$$\boxed{d_i = 2.375\ \text{inches}}$$