Period #11: Statically Indeterminate Shafts

A. Review

$$\tau_{max} = \frac{Tc}{J} \qquad \tau = \frac{T\rho}{J}$$

$$\gamma_{max} = \frac{Tc}{GJ} = c\frac{d\phi}{dx}$$

$$d\phi = \frac{T}{GJ}dx$$

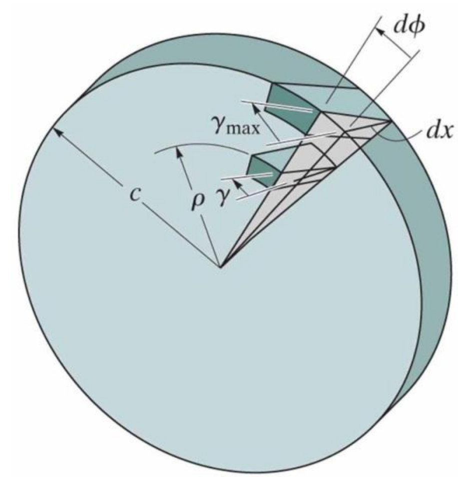


Fig. 11.1

B. Statically Indeterminate Shafts

Cannot simply use the equations of static equilibrium to calculate reactions or internal torques.

Must generally augment conditions of equilibrium with knowledge of how the indeterminate system deforms.

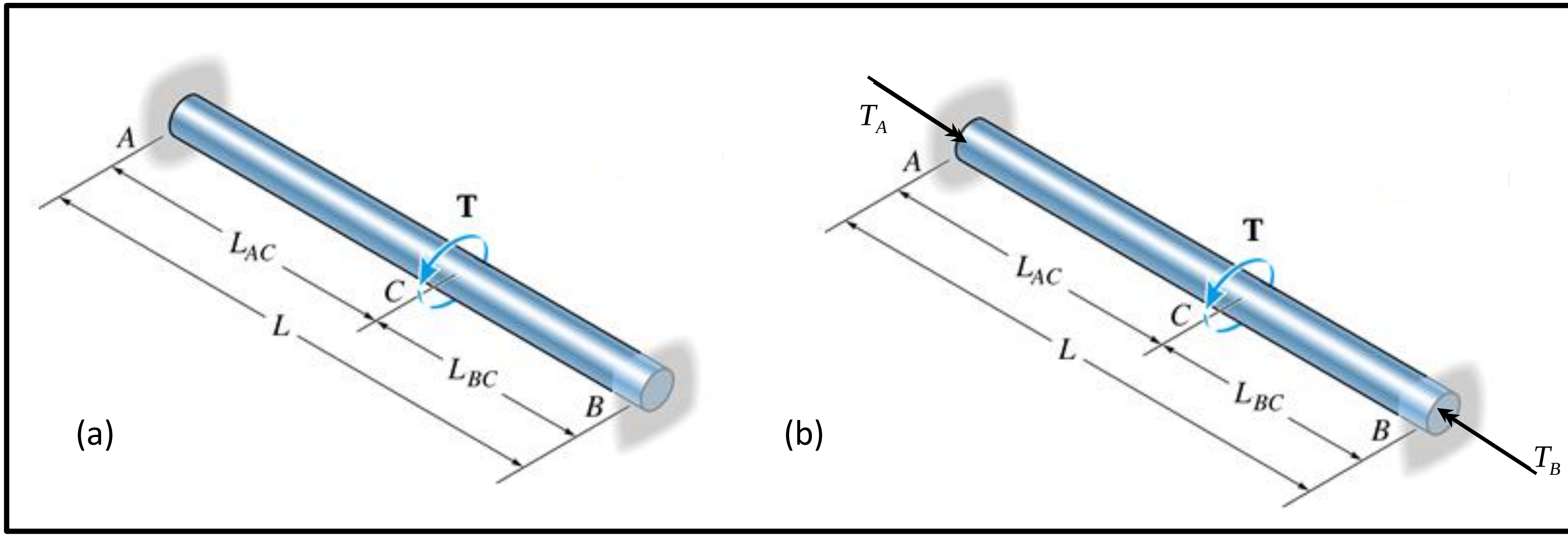


Fig. 11.2 (a) Indeterminate shaft fixed at both ends; and (b) the free-body diagram.

From statics: $T_A + T_B = T_C$

Additional constraint: $\phi_{B/A} = 0$

$$\phi_{B/A} = 0$$

$$= \int_A^C \frac{T_{AC}}{JG} dx + \int_C^B \frac{T_{CB}}{JG} dx$$

$$= \frac{T_A L_{AC}}{JG} - \frac{(T_C - T_A) L_{BC}}{JG}$$

$$T_A (L_{AC} + L_{CB}) = T_C L_{BC}$$

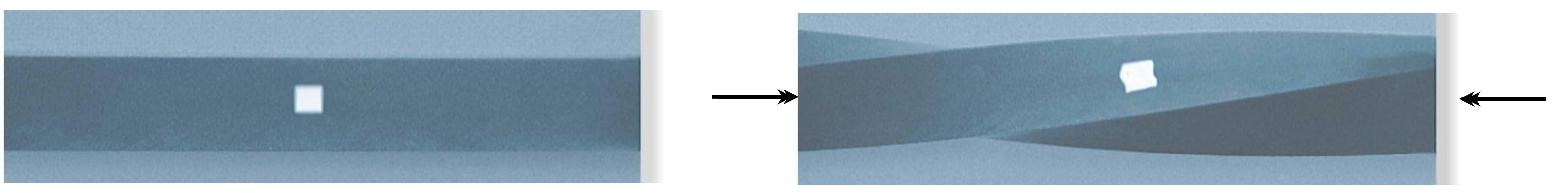
$$T_A = \frac{T_C L_{BC}}{L_{AB}}; \qquad T_B = \frac{T_C L_{AC}}{L_{AB}}$$

C. Shafts of Solid, Non-circular Cross-Sections (Textbook, Section 5.6)

To this point, we've been concerned with shafts that have circular cross-sections.

What about shafts with non-circular cross-sections?

Consider the torsion of a rubber shaft with a square cross-section shown below (Fig. 11.3):



(a) undeformed square shaft

(b) deformed square shaft when subjected to a torque

Figure 11.3 Twisting of rubber shaft with square cross-section

Note that the white square drawn on the square shaft "warps". The edges that were originally straight, become curved.

With non-circular cross-sections, the shear stress distribution over the cross-section cannot vary linearly with radial distance from the center (Fig. 11.4a).

This leads to warping of the shaft cross-section (Fig. 11.4b).

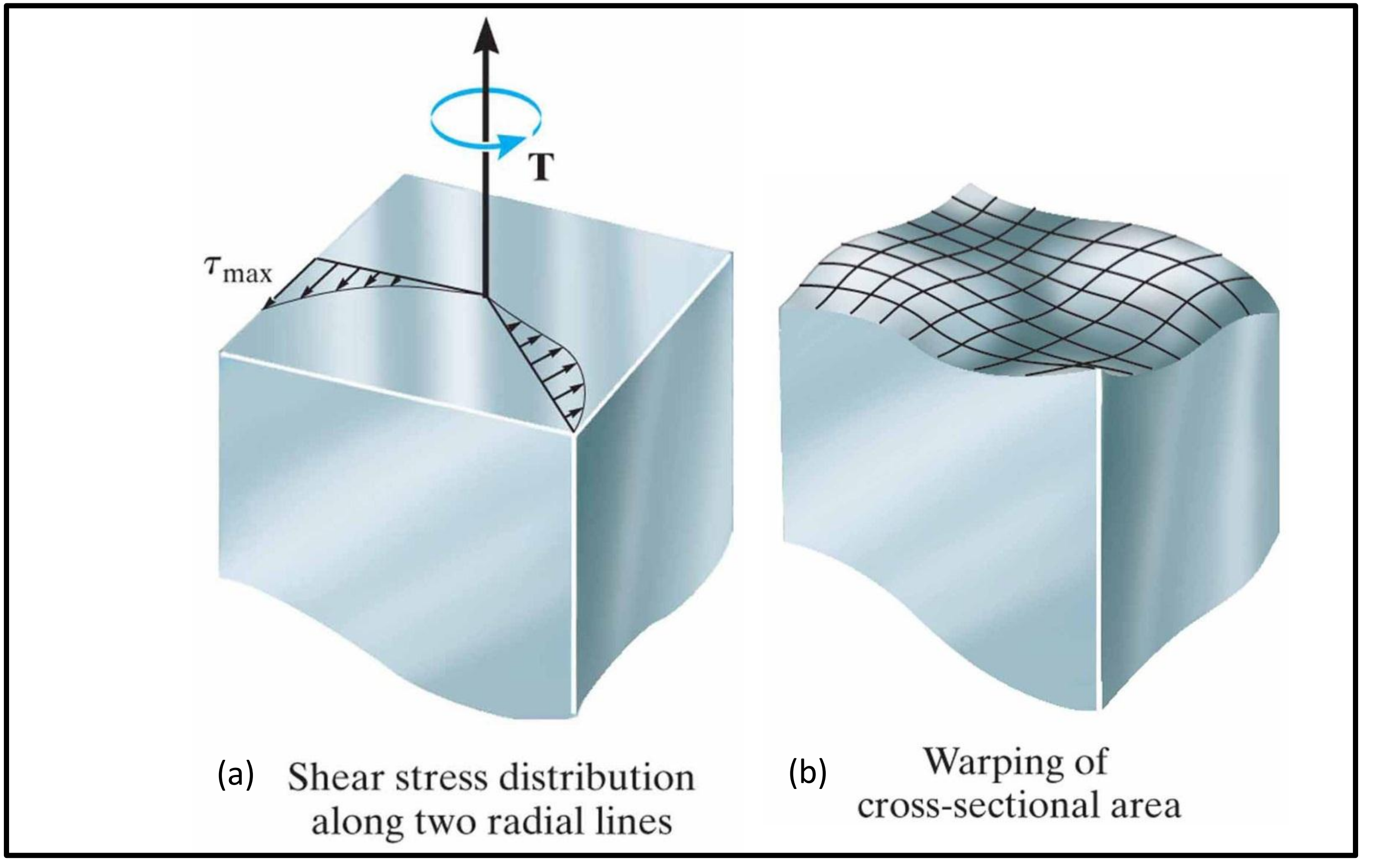
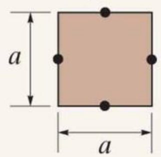
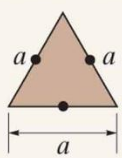
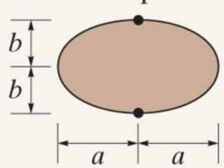


Fig. 11.4 Noncircular shaft response to torsion

Although the torsion formulae do not apply to shafts with non-circular cross-sections, the relevant properties of three sections are as indicated in the table below.

Table 11.1. Properties of non-circular sections.

Shape of cross section	τ_{max}	ϕ
Square 	$\frac{4.81\,T}{a^3}$	$\frac{7.10\,TL}{a^4 G}$
Equilateral triangle 	$\frac{20\,T}{a^3}$	$\frac{46\,TL}{a^4 G}$
Ellipse 	$\frac{2\,T}{\pi a b^2}$	$\frac{(a^2 + b^2)TL}{\pi a^3 b^3 G}$

D. Thin-walled Closed Section Shafts (Textbook 5.7)

Though you will not be responsible for this material, be aware that thin-walled, closed cross-sections can be treated using the concept of shear flow.

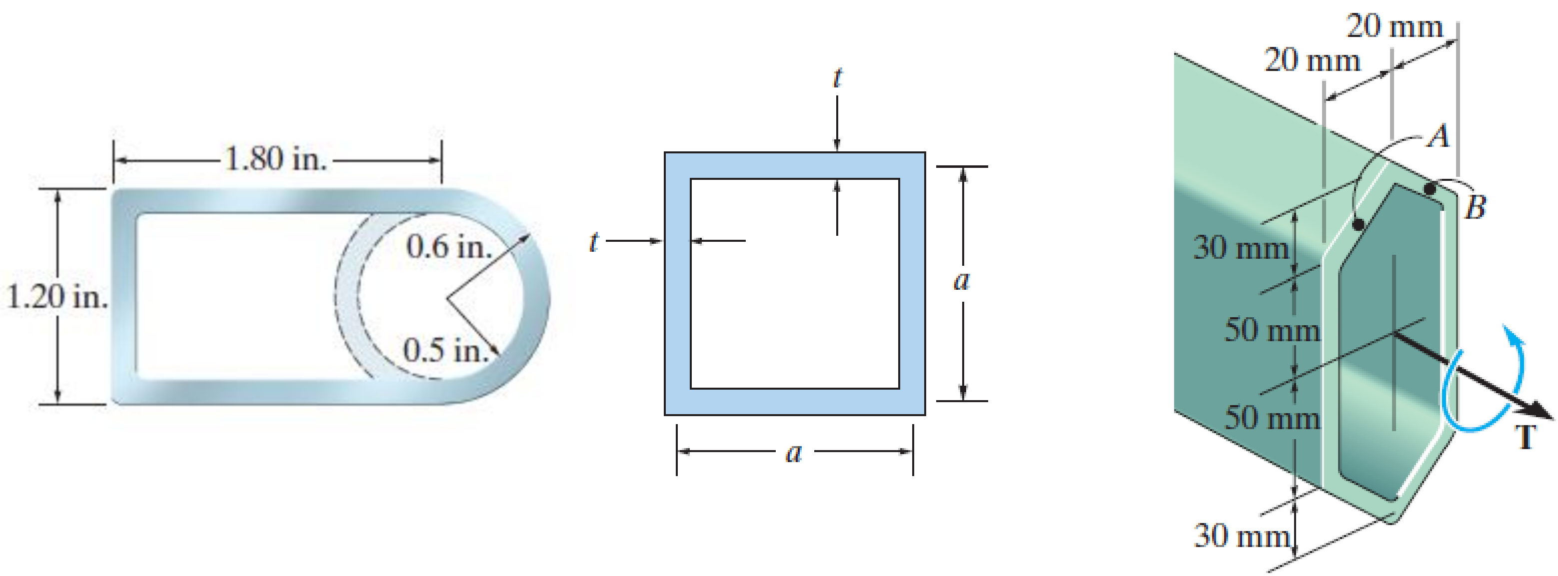
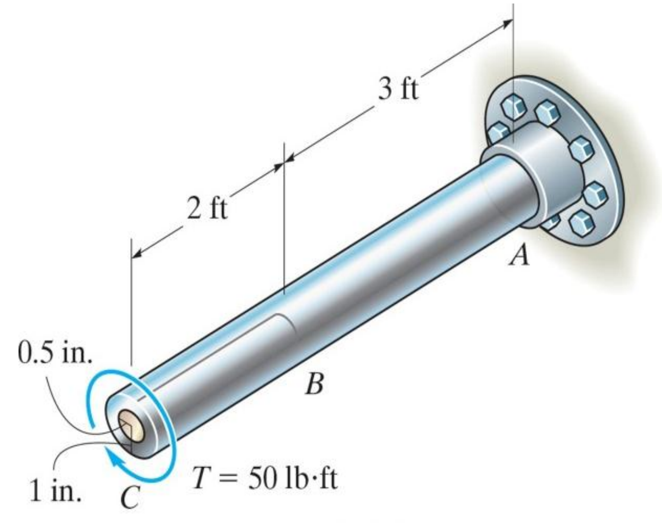


Fig. 11.5. Examples of thin-walled closed sections

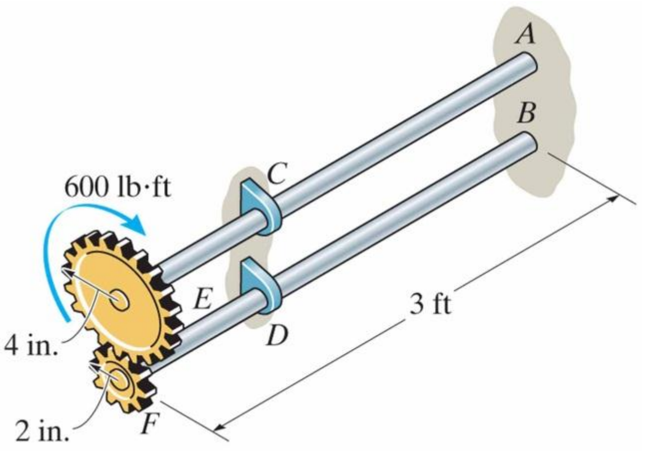
E. Examples

Example 11.1: The shaft is made from a solid steel section *AB* and a tubular portion made of steel and having a brass core. If it is fixed to a rigid support at *A*, and a torque of T=50lb.ft is applied to it at *C*, determine the angle of twist that occurs at *C* and compute the maximum shear and maximum shear strain in the brass and steel. Take

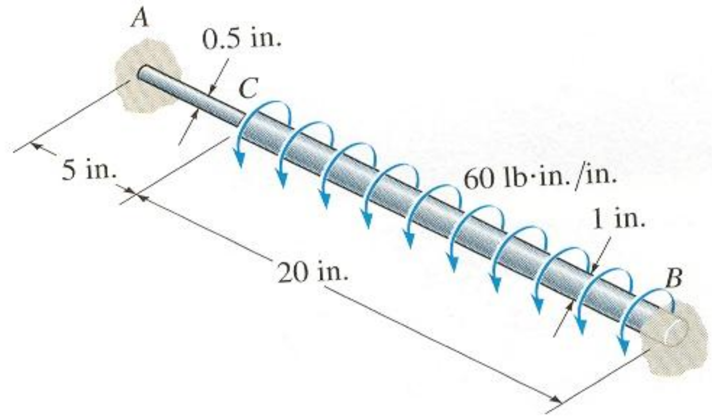
$G_{st} = 11.5(10^3)ksi \qquad G_{br} = 5.6(10^3)ksi$



Example 11.2. The two 3-ft-long shafts are made of 2014-T6 aluminum. Each has a diameter of 1.5in. and they are connected using the gears fixed to their ends. Their other ends are attached to fixed supports at A and B. They are also supported by bearings at C and D, which allow free rotation of the shafts about their axes. If a torque of 600 lb-ft is applied to the top gear as shown, determine the maximum shear stress in each shaft.



Example 11.3: The A-36 steel shaft is made from two segments: *AC* has a diameter of 0.5in and *CB* has a diameter of 1 in. If the shaft is fixed at its ends *A* and *B* and subjected to a uniform distributed torque of 60lb.in/in along segment *CB*, determine the absolute maximum shear stress in the shaft.



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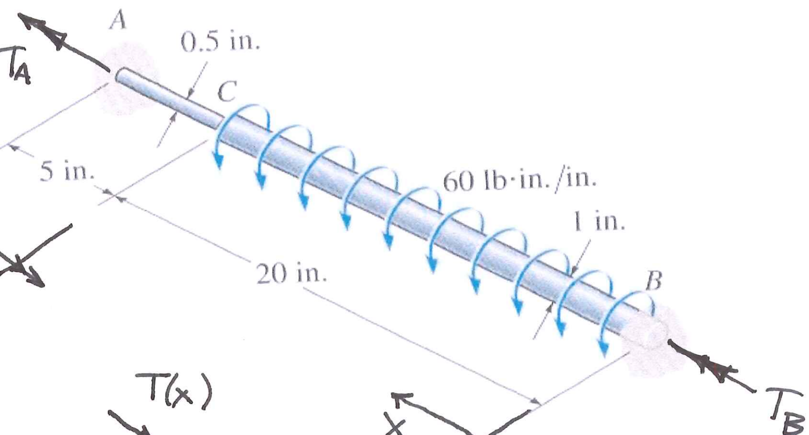
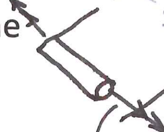
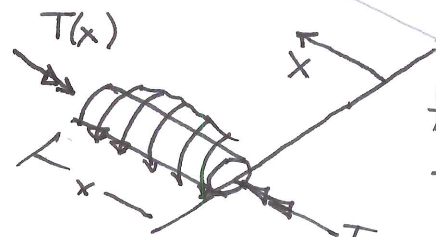




$T_A = T(x)$



From statics, $T_A + T_B = 1200\ \text{lb.in}$ (1)

Kinematic constraint $\phi_{A/B} = 0$

$$\phi_{A/B} = \int_B^C \frac{T}{JG}\,dx + \int_C^A \frac{T}{JG}\,dx$$

$$= \int_0^{20''} \frac{60x - T_B}{JG}\,dx + \int_{20''}^{25''} \frac{T_A}{JG}\,dx$$

$$= \frac{\left[\frac{60x^2}{2} - xT_B\right]_0^{20''}}{G\,\pi/32} + \left.\frac{T_A x}{G\,\pi/512}\right|_{20''}^{25''}$$

$$0 = \frac{12{,}000 - 20T_B}{\pi/32} + \frac{5T_A}{\pi/512}$$

$T_B - 4T_A = 600\ \text{lb.in}$ (2)

from B to C

$T(x) = (T_B - 60x)$

from C to A

$T(x) = T_A$

$$J_{BC} = \frac{\pi}{2}\left(\frac{1}{2}\text{in}\right)^4 = \frac{\pi}{32}\ \text{in}^4$$

$$J_{AC} = \frac{\pi}{2}\left(\frac{1}{4}\text{in}\right)^4 = \frac{\pi}{512}\ \text{in}^4$$

Combining (1) and (2):

$T_A = 120\ \text{lb.in}$; $T_B = 1080\ \text{lb.in}$

$$(\tau_{max})_{AC} = \frac{T_A \times (\frac{1}{4}'')}{(\pi/512)\ \text{in}^4} = 4.89\ \text{ksi}$$

$$(\tau_{max})_{BC} = \frac{(T_B)(\frac{1}{2}'')}{\pi/32\ \text{in}^4} = 5.50\ \text{ksi}$$